Consolidation due to Tangential Loads

Consolidation produite par des charges tangentielles

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SUMMARY

The paper studies the consolidation of a semi-infinite solid when a uniform tangential load is applied to the surface. The theory for a tangential surface load is developed in terms of three displacement functions. Relationships between stresses, displacements, excess pore pressures, and the displacement functions are given. The problem of a unit tangential load uniformly distributed over a circular area is solved, by Hankel-Laplace transform techniques. The excess pore pressures at a point are analysed for a pervious and an impervious surface.

SOMMAIRE

Ce rapport étudie la consolidation d'une masse semi-infinie soumise à une charge tangentielle uniformément appliquée sur la surface. La théorie d'une charge tangentielle de surface est développée par les trois fonctions de déplacement. On développe les équations entre les contraintes, les déplacements, les pressions de l'eau interstitielle et les fonctions de déplacement. On résoud le problème d'une charge tangentielle unitaire distribuée uniformément sur une surface circulaire au moyen de la transformation de Hankel-Laplace. Les pressions interstitielles en excès en un point sont analysées pour une surface perméable et imperméable.

THE THEORY OF CONSOLIDATION, as originally developed (Terzaghi, 1923; Terzaghi and Frohlich, 1936), concerned itself with the one-dimensional vertical compression of loaded clay layers. A general, mathematically exact, three-dimensional theory of consolidation was first developed by Biot (1935, 1941) and later independently by Florin (1937) and Mandel (1950). In addition to the general formulation, methods of stress and displacement functions have established means of solving boundary value problems in cases of plane strain and axial symmetry (Biot, 1956; Gibson and McNamee, 1957; de Josselin de Jong, 1957; Paria, 1958; McNamee and Gibson, 1960a).

Many problems of foundation engineering and vehicle mobility apply inclined loads to the surface of the ground. Furthermore, the application of a footing load is never shear free because of adhesion between the footing and the soil. All of these phenomena can be embraced in a consistent theory by analysing the consolidation of a half-space under the influence of tangential surface loads. This paper is concerned with this particular problem.

FORMULATION OF THEORY

The theory considered in this paper is the three-dimensional consolidation theory of Biot (1935, 1941), in which the soil has the following properties: (1) The soil skeleton is a linear isotropic, homogeneous, elastic material exhibiting small strains. (2) The soil is saturated. (3) The water is incompressible. (4) The flow of water in the medium is isotropic and follows Darcy's law for small velocity flow. (5) The stress increments are small. (6) The stress transfer between pore water and soil skeleton follows Terzaghi's effective stress equation.

The governing differential equations are the stress equations of equilibrium and the equation of continuity of pore water. In terms of displacements of the soil skeleton (u, v, w), in a cylindrical co-ordinate system (r, θ, z), and excess pore pressure (σ), the equilibrium equations are,

$$\nabla^2 u + (2\eta - 1)\frac{\partial e}{\partial r} - \frac{1}{r}\left[\frac{2}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right] + \frac{\partial \sigma}{G\partial r} = 0, \qquad (1a)$$

$$\nabla^2 v + (2\eta - 1)\frac{\partial e}{r\partial \theta} - \frac{1}{r^2}\left[v - 2\frac{\partial u}{\partial \theta}\right] + \frac{\partial \sigma}{Gr\partial \theta} = 0, \qquad (1b)$$

$$\nabla^2 w + (2\eta - 1)\partial e/\partial z + \partial \sigma / G\partial z = 0, \qquad (1c)$$

$$\eta = (1 - \nu)/(1 - 2\nu), \qquad (1d)$$

in which e is the dilatation, ν is Poisson's ratio of the soil skeleton, and G is the shear modulus of the soil skeleton.

The equation of continuity, in terms of the dilatation is,

$$c\,\nabla^2 e = \partial e/\partial t, \qquad (2a)$$

$$c = 2G\eta k/\gamma_w, \qquad (2b)$$

where c is the coefficient of consolidation, γ_w the unit weight of water, and k the coefficient of permeability.

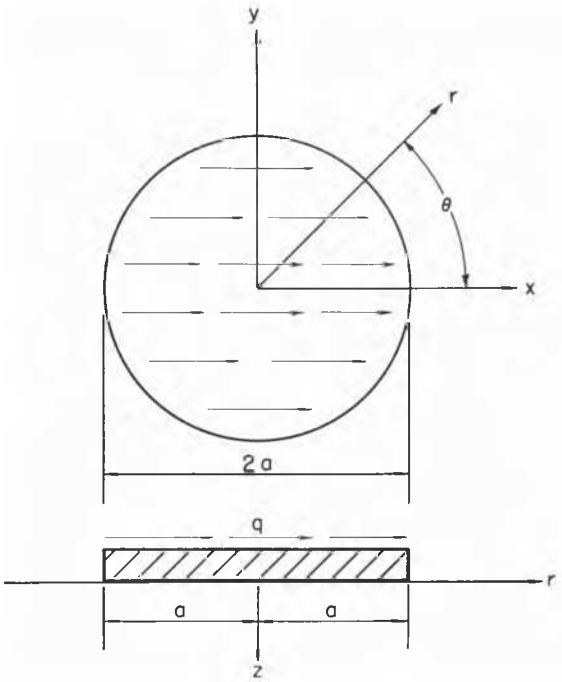
The specific boundary value situation to be considered is a tangential surface load of constant intensity q, uniformly distributed over a circular area of radius a, as shown in Fig. 1. For reasons of convenience, it is appropriate to normalize the system by dividing all stresses by q, all measures of length by a, and defining the time factor T as,

$$T = c_t/a^2. \qquad (3)$$

Assuming that the notation previously defined refers now only to dimensionless variables without change in symbol, Eqs 1 remain unchanged. The continuity equation becomes,

$$\nabla^2 e = \partial e/\partial T. \qquad (4)$$

The surface load is of unit intensity over a unit radius. The total stress boundary conditions, on the free surface, specify



1. Tangential surface loading.

the shear stress at that surface. The total normal stress on the free surface is assumed to be zero.

$$\sigma_{rz}(r, \theta, 0, T) = \begin{cases} \cos\theta & r \leqslant 1 \\ 0 & r > 1 \end{cases}, \tag{5a}$$

$$\sigma_{\theta z}(r, \theta, 0, T) = \begin{cases} -\sin\theta & r \leqslant 1 \\ 0 & r > 1 \end{cases}, \tag{5b}$$

$$\sigma_{zz}(r, \theta, 0, T) = 0 \qquad r \geqslant 0. \tag{5c}$$

In addition to the total stress components, the drainage conditions at the boundary must be specified. Two limiting conditions will be considered: that of free drainage in which the surface excess pore pressure (σ) is zero, and an impervious boundary in which the vertical excess pore pressure gradient at the boundary is zero. As infinite depth is approached, all stresses, displacements and excess pore pressures vanish. At the instant of loading, the volume change of the medium is zero ($e = 0$). Eqs 1 and 4, together with the appropriate boundary and initial conditions, as discussed, constitute a complete mathematical statement of the problem.

METHOD OF SOLUTION

The problem as formulated above can be solved by means of displacement functions. In axisymmetric problems, two such functions are necessary and sufficient to develop a complete solution (McNamee and Gibson, 1960a). In asymmetric problems, three functions are required. The three displacement functions E, S, and Q are related to the displacements as follows:

$$u(r, \theta, z, T) = \frac{\partial^2 E}{\partial r \partial z} - \frac{2}{r}\frac{\partial Q}{\partial \theta} - z\frac{\partial S}{\partial r}, \tag{6a}$$

$$v(r, \theta, z, T) = \frac{1}{r}\frac{\partial^2 E}{\partial \theta \partial z} + 2\frac{\partial Q}{\partial r} - \frac{z}{r}\frac{\partial S}{\partial \theta}, \tag{6b}$$

$$w(r, \theta, z, T) = \frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z} + S, \tag{6c}$$

$$e(r, \theta, z, T) = \frac{\partial}{\partial z}(\nabla^2 E). \tag{6d}$$

The displacements, and therefore strains, are related to the effective stresses by the stress-strain relationships of linear elasticity (Dougall, 1914). The total stresses in terms of the displacement functions are,

$$\frac{\sigma_{rr}}{2G} = \frac{\partial}{\partial z}\left\{\frac{\partial^2 E}{\partial r^2} - \nabla^2 E\right\} - z\frac{\partial^2 S}{\partial r^2} + \frac{\partial S}{\partial z} - \frac{2}{r}\frac{\partial}{\partial \theta}\left\{\frac{\partial Q}{\partial r} - \frac{Q}{r}\right\}, \tag{7a}$$

$$\frac{\sigma_{\theta\theta}}{2G} = \frac{\partial}{\partial z}\left\{\frac{1}{r}\frac{\partial E}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E}{\partial \theta^2} - \nabla^2 E\right\} + \frac{2}{r}\frac{\partial}{\partial \theta}\left\{\frac{\partial Q}{\partial r} - \frac{Q}{r}\right\} + \frac{\partial S}{\partial z} - \frac{z}{r}\frac{\partial S}{\partial r} - \frac{z}{r^2}\frac{\partial^2 S}{\partial \theta^2}, \tag{7b}$$

$$\frac{\sigma_{zz}}{2G} = \frac{\partial}{\partial z}\left\{\frac{\partial^2 E}{\partial z^2} - \nabla^2 E\right\} - z\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}, \tag{7c}$$

$$\frac{\sigma_{r\theta}}{2G} = \frac{\partial^2}{\partial \theta \partial z}\left\{\frac{\partial E}{\partial r} + \frac{1}{r^2}E\right\} + \frac{z}{r}\frac{\partial}{\partial \theta}\left\{\frac{1}{r}S - \frac{\partial S}{\partial r}\right\} + \frac{\partial^2 Q}{\partial r^2} - \frac{1}{r}\frac{\partial Q}{\partial r} - \frac{1}{r^2}\frac{\partial^2 Q}{\partial \theta^2}, \tag{7d}$$

$$\frac{\sigma_{\theta z}}{2G} = \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z}\right\} + \frac{\partial^2 Q}{\partial r \partial z}, \tag{7e}$$

$$\frac{\sigma_{rz}}{2G} = \frac{\partial}{\partial r}\left\{\frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z}\right\} - \frac{1}{r}\frac{\partial^2 Q}{\partial \theta \partial z}. \tag{7f}$$

The pore pressure (σ) is related to the displacement functions by,

$$\sigma(r, \theta, z, T) = 2G\left\{\frac{\partial S}{\partial z} - \eta\frac{\partial}{\partial z}(\nabla^2 E)\right\}. \tag{7g}$$

The differential equations governing E, S, and Q are,

$$\nabla^2 S = 0, \tag{8a}$$

$$\nabla^2 Q = 0, \tag{8b}$$

$$\nabla^4 E = \nabla^2(\partial E/\partial T). \tag{8c}$$

The boundary and initial conditions required are derived from the relations between stresses and displacement functions as previously given.

A useful means of solution is to separate variables by writing the displacement functions as a product of an axisymmetric function (noted by subscript one), and a trigonometric function (Muki, 1956, 1960). For the problem in question,

$$E(r, \theta, z, T) = E_1(r, z, T)\cos\theta, \tag{9a}$$

$$S(r, \theta, z, T) = S_1(r, z, T)\cos\theta, \tag{9b}$$

$$Q(r, \theta, z, T) = Q_1(r, z, T)\sin\theta. \tag{9c}$$

For general asymmetric functions, Eq 9 would take on the form of a series summed over m with the trigonometric argument being $m\theta$. Furthermore, there would be a separate axisymmetric displacement function for every value of m (Fungaroli, 1963).

The axisymmetric displacement functions will be governed by the same form of differential equations as in Eq 8 with the operators (^{2}) and (∇^4) being replaced by (∇_1^2) and (∇_1^4). The modified Laplacian operator (∇_1^2) is,

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \quad (10)$$

A solution can now be accomplished by using Hankel-Laplace transform techniques. The transformed axisymmetric displacement functions are:

$$\bar{\bar{E}}_1(\xi, z, p) = \int_0^\infty \int_0^\infty r E_1(r, z, T) J_1(\xi r) e^{-pT} dr dT \quad (11)$$

with ($\bar{\bar{S}}_1$) and ($\bar{\bar{Q}}_1$) taking on the same form as ($\bar{\bar{E}}_1$). By the Hankel-Laplace inversion theorem,

$$E_1(r, z, T) = \frac{1}{2\pi i} \int_0^\infty \int_{B_r} \xi \bar{\bar{E}}_1(\xi, z, p) \times J_1(\xi r) e^{pT} dp d\xi, \quad (12)$$

in which i is the imaginary unit and (B_r) denotes the Bromwich contour in the p-plane.

The transformation of Eq 8 results in three ordinary differential equations with general solution as follows:

$$\bar{\bar{E}}_1(\xi, z, p) = Ae^{-\xi z} + Be^{-z(\xi^2+p)^{\frac{1}{2}}}, \quad (13a)$$

$$\bar{\bar{S}}_1(\xi, z, p) = Ce^{-\xi z}, \quad (13b)$$

$$\bar{\bar{Q}}(\xi, z, p) = De^{-\xi z}. \quad (13c)$$

The coefficients A, B, C and D are determined from the transformed boundary conditions.

The displacement functions as found from Eqs 12 and 9 are for a free-draining surface,

$$\frac{E_1}{2G} = \int_0^\infty \frac{1}{\xi^3}\left[e^{-\xi z} + \frac{2e^{-\xi z}}{\pi\eta^2} I_1 + \frac{2I_2}{\pi\eta^2} + \left(\frac{2+\xi z}{\eta^2\alpha}\right)e^{-\xi z}\right] J_1(\xi) J_1(\xi r) d\xi, \quad (14a)$$

$$\frac{S_1}{2G} = \int_0^\infty \frac{1}{\xi}\left[\frac{2}{\pi\eta} I_3 e^{-\xi z} - \frac{2}{\eta\alpha} e^{-\xi z}\right] J_1(\xi) J_1(\xi r) d\xi, \quad (14b)$$

$$\frac{Q_1}{2G} = \int_0^\infty \frac{1}{\xi^2} e^{-\xi z} J_1(\xi) J_1(\xi r) d\xi, \quad (14c)$$

$$I_1 = \int_0^\infty \frac{u^2(1 - \eta[u^2+1])}{(u^2+1)^2(u^2+1-\alpha)} e^{-\xi^2(u^2+1)T} du, \quad (14d)$$

$$I_2 = \int_0^\infty \frac{u \sin \xi uz + \{1 - \eta(u^2+1)\} \cos \xi uz}{(u^2+1)^2(u^2+1-\alpha)} \times e^{-\xi^2(u^2+1)T} du, \quad (14e)$$

$$I_3 = \int_0^\infty \frac{u^2 e^{-\xi^2(u^2+1)T}}{(u^2+1)(u^2+1-\alpha)} du, \quad (14f)$$

$$\alpha = \frac{2\eta - 1}{\eta^2} \quad (14g)$$

In the case of an impervious surface the displacement functions are,

$$\frac{E_1}{2G} = \int_0^\infty \frac{1}{\xi^3}\left[e^{-\xi z} + \frac{2}{\pi\eta^2} e^{-\xi z} I_4 + \frac{2}{\pi\eta^2} I_5 + \left(\frac{2+\xi z}{\zeta\phi}\right) e^{-\xi z} + \frac{M}{\eta^2} e^{-\xi^2\zeta T - \xi z} - \frac{M}{(1-\zeta)\eta^2} e^{-\xi(1-\zeta)^{\frac{1}{2}} z - \xi^2 \zeta T}\right] J_1(\xi) J_1(\xi r) d\xi, \quad (15a)$$

$$\frac{S_1}{2G} = \int_0^\infty \frac{1}{\eta\xi}\left[\frac{2}{\pi} I_6 e^{-\xi z} - \frac{2e^{-\xi z}}{\zeta\phi} + N(1-\zeta)^{\frac{1}{2}} e^{-\xi z - \xi^2\zeta T}\right] J_1(\xi) J_1(\xi r) d\xi, \quad (15b)$$

$$\frac{Q_1}{2G} = \int_0^\infty \frac{1}{\xi^2} e^{-\xi z} J_1(\xi) J_1(\xi r) d\xi, \quad (15c)$$

$$I_4 = \int_0^\infty \frac{u^2 e^{-\xi^2(u^2+1)T}}{(u^2+1)^2(u^2+1-\zeta)(u^2+1+\phi)} du, \quad (15d)$$

$$I_5 = \int_0^\infty \frac{u(1 + \eta[u^2+1]) \sin \xi uz + \cos \xi uz}{(u^2+1)^2(u^2+1-\zeta)(u^2+1+\phi)} \times e^{-\xi^2(u^2+1)T} du, \quad (15e)$$

$$I_6 = \int_0^\infty \frac{u^2 e^{-\xi^2(u^2+1)T}}{(u^2+1)(u^2+1-\zeta)(u^2+1+\phi)}, \quad (15f)$$

$$M = \frac{(1-\zeta)^{\frac{1}{2}}[(1-\zeta)^{\frac{1}{2}}(\eta\zeta+1)+1]}{\zeta^2(\zeta+\phi)}, \quad (15g)$$

$$N = \frac{\zeta}{(1-\zeta)^{\frac{1}{2}}} M, \quad (15h)$$

$$\zeta = \frac{\sqrt{\eta^2 + 4\eta} - (2-\eta)}{2\eta}, \quad (15i)$$

$$\phi = \frac{2-\eta}{\eta} + \zeta. \quad (15j)$$

EXCESS PORE PRESSURES

For the two drainage cases in question the excess pore pressures can be computed by the evaluation of Eq 7g.

In the free-draining case, the excess pore pressure is,

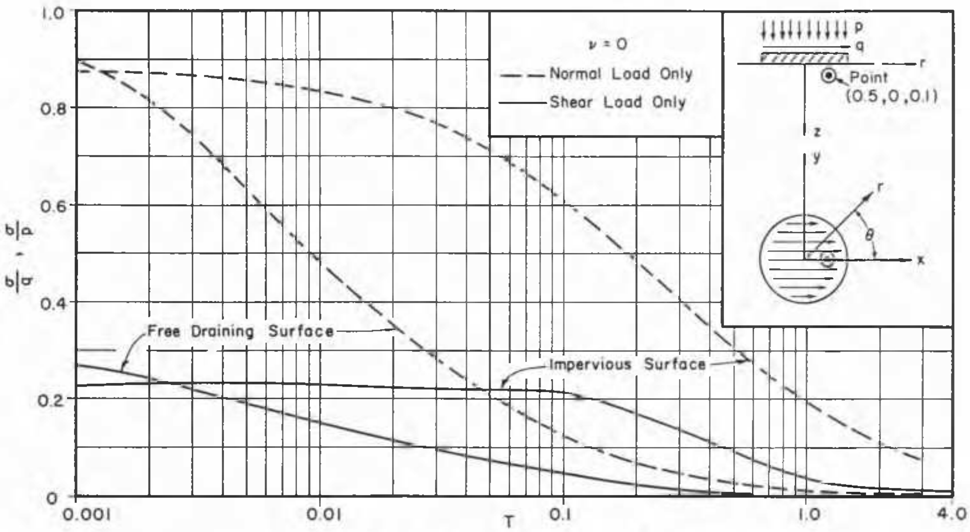
$$\sigma(r, \theta, z, T) = \frac{2\cos\theta}{\eta\pi} \int_0^\infty \int_0^\infty \times \left\{\frac{u^2 \cos \xi uz - u(1 - \eta[u^2+1]) \sin \xi uz - u^2 e^{-\xi z}}{(u^2+1)(u^2+1-\alpha)}\right\} \times J_1(\xi) J_1(\xi r) e^{-\xi^2(u^2+1)T} du d\xi. \quad (16)$$

while for the impervious boundary, the excess pore pressure at a point is,

$$\sigma(r, \theta, z, T) = \frac{2\cos\theta}{\eta\pi} \int_0^\infty \int_0^\infty \times \left\{\frac{u^2(1 + \eta[u^2+1]) \cos \xi uz - u \sin \xi uz - u^2 e^{-\xi z}}{(u^2+1)(u^2+1-\zeta)(u^2+1+\phi)}\right\}$$

$$\times\, J_1(\xi)J_1(\xi r)e^{-\xi^2(u^2+1)T}\,du\,d\xi + \frac{N\cos\theta}{\eta}\int_0^\infty [e^{-\xi(1-\zeta)^{\frac{1}{2}}z} - (1-\zeta)^{\frac{1}{2}}e^{-\xi z}]e^{-\xi^2\zeta T}\,J_1(\xi)J_1(\xi r)d\xi. \quad (17)$$

Computations were carried out to compare the excess pore pressure due to the tangential load as given by Eqs 16 and 17 with the excess pore pressures due to normal loads (McNamee and Gibson, 1960*b*). The excess pore pressure-time factor curves are presented in Fig. 2, for the case



2. Excess pore pressure—time relation.

where $\nu = 0$. The particular point in question was at the half radius of the area; at a depth of one-tenth the radius and for $(\theta = 0)$, a point along the direction of the surface force.

The initial normalized excess pore pressure σ/q, for the point in question is 0.181, independent of drainage conditions. As time progresses the excess pore pressure increases due to the steep stress gradients toward the edge of the load. The maximum value is reached earlier in the free-draining case. However, in both cases the maximum excess pore pressure is reached at relatively early times. Subsequent to this initial maximum, the pore pressure decreases with time. In the pervious case the decrease is relatively rapid due to the proximity of the drainage surface. On the other hand, the impervious case shows an almost negligible drop for two time decades.

In the free-draining case, the ratio of excess pore pressures (tangential to normal load) is almost constant, being approximately one-third, for over two time decades. This ratio in the impervious case starts out as one-quarter and at the end of two time decades builds up to one-third. Thus while the excess pore pressure in the free-draining case dissipates faster, the contribution, with respect to a normal load, is generally greater than in the impervious case.

The effect shown in Fig. 2 is, with respect to the angle θ, a maximum. Determination of the excess pore pressures at other points on the same radial arc are accomplished by multiplying the values given by the cosine of the angle measured from the positive x axis. Thus at the negative companion x axis point, the pore water would be in tension.

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