A Method for the Calculation of Settlements, Contact Pressures, and Bending Moments in a Foundation Including the Influence of the Flexural Rigidity of the Superstructure

Une Méthode pour le calcul des tassements, des pressions sur sol et des moments fléchissants dans une fondation tenant compte de l'influence de la rigidité de la superstructure

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SUMMARY

In this paper a method is developed for the analysis of foundation beams and foundation slabs, which are flexible in only one direction, the rigidity of the superstructure being taken into account. Structures with various degrees of rigidity including the perfectly flexible and the perfectly rigid cases on any subsoil are investigated, using unit settlements and expressed in matrix form. The method gives the contact pressures and the settlements of the structure as well as the resulting bending moments in the foundation and the superstructure.

A symmetrical superstructure, supported on four columns is considered as an example, and the relation between the rigidity of the foundation and that of the superstructure is determined and expressed in terms of the flexural rigidity of the whole structure. The conditions are shown under which the calculation of settlements and contact pressures can be simplified by assuming an imaginary foundation slab carrying no superstructure and having a flexural rigidity equal to that of the whole structure. Furthermore, the relationship of the bending moments in the foundation slab to the rigidity of structure and compressibility of subsoil is shown.

SOMMAIRE

Cet article démontre une méthode pour l'analyse des radiers et des poutres de fondations qui se déforment dans une seule direction, en tenant compte de la rigidité de l'ossature de la superstructure. On étudie des structures de différentes rigidités, incluant celles infinement souples ou infinement rigides, posées sur n'importe quel type de sol. Ce procédé, basé sur la méthode des déformations et présenté sous forme de matrice, permet de déterminer les pressions sur le sol et les tassements, ainsi que les moments fléchissants dans la fondation et dans l'ossature.

Comme exemple, l'auteur analyse un bâtiment symétrique à quatre montants. En particulier on étudie la rigidité de l'ensemble de la construction en fonction de celle du radier et de celle de l'ossature. On indique quand il est admissible dans le calcul des pressions sur le sol et des tassements de la construction de remplacer celle-ci par un radier de même rigidité. En outre on détermine l'influence de l'action réciproque entre la construction et le sol sur les moments fléchissants dans le radier, lorsqu'on fait varier la rigidité de la première et la compressibilité du second.

THE CONTACT PRESSURE, the bending moments, and the settlements of a foundation beam and a foundation slab are dependent on the relative stiffness of the structure and the soil. The rigidity of the structure is a function of the rigidity of the superstructure and the foundation, and in the usual methods of calculation it is only the rigidity of the foundation that is included (De Beer, 1948; De Beer and Krsmanovitch, 1952; Grasshoff, 1955; Kany, 1959; Ohde, 1942). The rigidity of the superstructure cannot, with the given methods of calculation, be included if it is supported on the foundation slab by positional columns or walls, as is the case with modern framed structures (Fig. 1a). The superstructure can be connected to the foundation through hinged, fixed-end flexible or fixed-end rigid columns and these systems are often statically indeterminate.

Solutions exist for special cases of the superstructure. Krsmanovitch (1955) has put forward an iteration method for hinged supports between the superstructure and foundation, and Grasshoff (1957) has examined the influence of perfectly rigid and perfectly flexible superstructures supported on a foundation through hinged and fixed-end rigid columns. Generally, the superstructure is neither entirely flexible nor rigid. Furthermore, in most cases the structure is supported on the foundation, not by hinged or fixed-end rigid columns, but by fixed-end flexible columns. In this paper a method is given in which all possible cases of rigidity of the structure, including the extreme conditions, are considered.

DEVELOPMENT OF A GENERALLY VALID METHOD FOR THE ANALYSIS OF FOUNDATIONS INCLUDING THE TOTAL FLEXURAL RIGIDITY OF A FRAMED STRUCTURE

This method is developed for foundation beams and for foundation slabs which are flexible in only one direction. It is based on the elastic theory for both structure and soil. In the derivation of the method the assumption of the isotropic half-space is used for the calculation of the stress distribution in the subsoil. However, the method also allows any other settlement analysis of the soil to be considered.

This method is developed for any framed structure with any form of loading system (Fig. 1a). The superstructure is represented with an equivalent moment of inertia $J_{\text{ü}}$, which can be calculated exactly from deflection considerations of the superstructure or approximately from the formula of Meyerhof (1953). The columns and the foundation have moments of inertia J_{St} and J_G, respectively.

The foundation is considered to be divided into parts, in this case $(n + n' - 1)$ parts, and the middle of each part

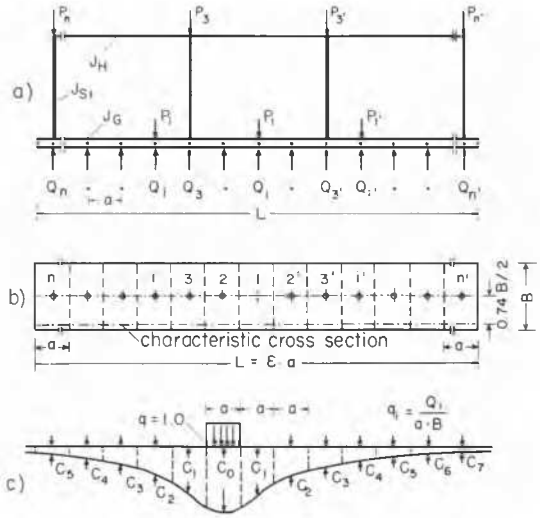

FIG. 1. Framed structure on an elastic continuous footing. (a) Loading diagram; (b) division of foundation; (c) Influence line for the settlements in the characteristic cross-section.

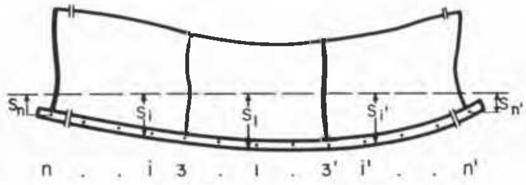

FIG. 2. Ground settlement and deformed structure.

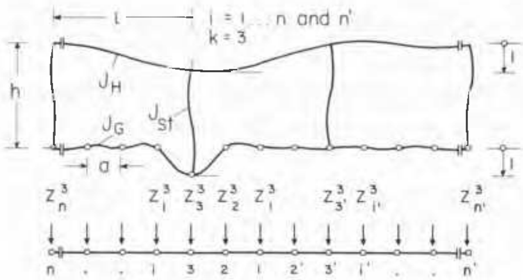

FIG. 3. Deformed structure for the unit settlement at support 3 (with positive reaction forces Z_i^k and the settlements at all other points equal to zero).

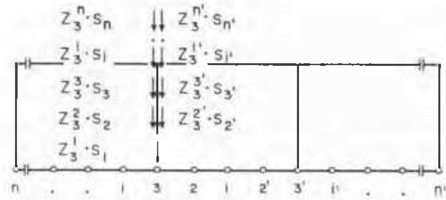

FIG. 4. Positive reaction forces $Z_i^k \cdot s_k$ at support 3 due to the real settlements s_k ($k = 1 \ldots n$ and n').

is taken to represent an imaginary support (Figs. 1a and 1b). The problem is to specify the contact pressure at each of these supports, and the method shown in this paper to calculate them uses deformations. We introduce unit settlements at the imaginary supports of the foundation in turn. From this we derive an equilibrium equation at each of the $(n + n' - 1)$ points of the foundation, and from these we find the $(n + n' - 1)$ contact pressures. This is shown below.

At first, the superstructure and foundation are considered on an unyielding subsoil. The imagined supports of the foundation are rigid in this case, and the reaction forces in them are determined from the loads of the structure. These reactions can be found for all possible forms of loading according to the laws of statics. In the example given here, we consider the loads of the structure to be concentrated as single loads acting over the imaginary supports (Fig. 1a). This means that the loads are transferred to the supports without bending in the structure, and thus the reaction forces in the supports are equal to the loads directly over them. The equilibrium equation for the imaginary support 3, for example, is

$$P_3 = Q_3$$

and for the support i

$$P_i = Q_i. \tag{1}$$

Because of the settlement of the subsoil the structure has a trough-shaped deformation (Fig. 2). These settlements cause additional forces in the structure and also additional reactions in the contact pressure forces. Because the settlements and deformations are a function of the flexural rigidity of the structure and the compressibility of the subsoil, it follows that the additional contact pressure forces also depend on them. The problem would be solved if the settlement at each of the $(n + n' - 1)$ imaginary supports could be found (Fig. 2). Because these settlements are unknown, we apply at the imaginary supports, in turn, unit settlement, i.e. $s_k = 1$ and keep the settlements at all other points on the foundation zero. The deformed structure for the unit settlement at the point $k = 3$ is shown in Fig. 3, and this causes reaction forces Z_i^3 in each support. Generally, the deformed structure due to a unit settlement at point k causes reaction forces Z_i^k at each support, where the index i indicates the position of the support. If at each support i ($i = 1 \ldots n$ and $1 \ldots n'$) unit settlements $s_k = 1$ are applied, then for each support i, k reaction forces Z_i^k ($k = 1 \ldots n$ and $1 \ldots n'$) can be found. For the real, still unknown settlements s_k, the magnitude of the reaction forces is $Z_i^k \cdot s_k$. For an example, the reaction forces $Z_i^k \cdot s_k$ for the real settlements s_k ($k = 1 \ldots n$ and $1 \ldots n'$) are shown in Fig. 4 for support 3.

The final reaction force Q_i can now be determined from $Q_i = P_i$ obtained for the structure when considered on an unyielding subsoil and on rigid supports (Eq 1) plus the sum of the reaction forces

$$\sum_{k=1}^{k=n \text{ and } n'} Z_i^k \cdot s_k$$

due to the real, still unknown settlements s_k ($k = 1 \ldots n$ and $1 \ldots n'$). Now the equilibrium equation for support 3 is given by

$$\sum_{k=1}^{k=n \text{ and } n'} Z_3^k \cdot s_k + P_3 = Q_3$$

and for the support i

$$\sum_{k=1}^{k=n \text{ and } n'} Z_i^k \cdot s_k + P_i = Q_i. \tag{2a}$$

In this way we find $(n + n' - 1)$ equations in which the $(n + n' - 1)$ unknown contact pressure forces and $(n + n' - 1)$ unknown settlements are involved.

For a symmetrical structure with symmetrical loading it is only necessary to have n equilibrium equations. In this case, the equation for support 3 is

$$\sum_{k=1}^{k=n} Z_3^k \cdot s_k + P_3 = Q_3$$

and for the support i

$$\sum_{k=1}^{k=n} Z_i^k \cdot s_k + P_i = Q_i. \tag{2b}$$

In the following, for the Eqs 2b for the symmetrical structure with symmetrical loading, we express the settlements s_k ($k = 1 \ldots n$) as a function of the contact pressures q_i ($i = 1 \ldots n$). The relation between the settlements and the contact pressures for a homogeneous subsoil or for stratified layers of subsoil are shown according to Figs. 1a to 1c, below:

$$\begin{aligned} s_1 &= [c_0 \cdot q_1 + 2c_1 \cdot q_2 + \ldots + 2c_{n-1} \cdot q_n] \cdot a/E_s \\ s_2 &= [c_1 \cdot q_1 + (c_0 + c_2) \cdot q_2 + \ldots + (c_{n-2} + c_n) \cdot q_n] \cdot a/E_s \\ &\cdot \\ &\cdot \\ &\cdot \\ s_n &= [c_{n-1} \cdot q_1 + (c_{n-2} + c_n) \cdot q_2 + \ldots + (c_0 + c_{2n-2}) \cdot q_n] \cdot a/E_s. \end{aligned} \tag{3a}$$

c_i ($i = o \ldots 2n - 2$) are the values of the influence line for the settlements for $q = 1$, $a = 1$ and for the compressibility modulus of the soil $E_s = 1$. As the foundation cannot be deformed in the direction B (Fig. 1b) since we consider only deformations in one direction, the settlement analysis is executed at the so-called "characteristic cross-section" (Fig. 1b) (Grasshoff, 1953).

The relation between the settlements and the contact pressures shown in Eq 3a can now be expressed in the matrix form as follows:

$$\bar{s} = \frac{a}{E_s} \cdot \bar{A} \cdot \bar{q}, \text{ where } \bar{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \text{ and } \bar{s} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}. \tag{3b}$$

The elements a_{ik} of the matrix $\bar{A}$ result from the values c_i of the influence line of the settlements according to Eq 3a.

If we introduce the unknown contact pressures q_i ($i = 1 \ldots n$) as a single column vector $\bar{q}$, the settlements s_i in Eq 3a can be written as scalar products between the row vector $\bar{a}^i$ and the column vector $\bar{q}$, as follows

$$s_i = a/E_s \cdot \bar{a}^i \cdot \bar{q} (i = 1 \ldots n), \tag{3c}$$

where

$$\bar{a}^i \cdot \bar{q} = a_{i1} \cdot q_1 + a_{i2} \cdot q_2 + \ldots + a_{in} \cdot q_n.$$

The reaction forces Z_i^k for the unit deformations according to Fig. 3 are determined in the most suitable way with the formula usually used in connection with the deformation method in statics. Details are given in the literature (Ostenfeld, 1926). These reaction forces Z_i^k for the unit settlements $s_k = 1$ are determined now for the length $a = 1$ and for the flexural rigidity of the foundation (to which the rigidities of the structure and the columns are referred) $EJ_G = 1$. If these reaction forces, which are calculated with $EJ_G = 1$ and $a = 1$, are Z_i^{*k}, the real magnitude of Z_i^k is given by

$$Z_i^k = EJ_G/a^3 \cdot Z_i^{*k}. \tag{4}$$

If we substitute the reaction forces Z_i^k according to Eq 4 and the settlements s_k according to Eq 3c in Eq 2b, where the contact pressure forces $Q_i = q_i \cdot a \cdot B$, the equilibrium equation for the support i becomes:

$$\frac{EJ_G}{a^3 \cdot B \cdot E_s} \sum_{k=1}^{k=n} Z_i^{*k} \cdot \bar{a}^k \cdot \bar{q} + \frac{P_i}{a \cdot B} = q_i. \tag{5}$$

Substituting a by L/ϵ, where L is the length of the slab and ϵ the number of elements with the length a, we have the term

$$\frac{EJ_G}{a^3 \cdot B \cdot E_s} = \frac{EJ_G \cdot \epsilon^3}{L^3 \cdot B \cdot E_s} = K_r \cdot \epsilon^3 \tag{6}$$

where K_r is the relative stiffness of the foundation to that of the soil. With this relative stiffness K_r, Eq 5 becomes

$$K_r \cdot \epsilon^3 \sum_{k=1}^{k=n} Z_i^{*k} \cdot \bar{a}^k \cdot \bar{q} - q_i = -\frac{P_i \cdot \epsilon}{L \cdot B} \tag{7}$$

Varying i from $1 \ldots n$ we now have n equations for the n contact pressures q_i. In Eq 7 Z_i^{*k} depends only on the rigidity of the structure, that is, the geometrical dimensions, the ratio of the rigidity of the superstructure to that of the foundation, and the way in which the superstructure is connected to the foundation. Since different compressibilities of the subsoil can be taken into account by varying K_r, we can find the contact pressures as a function of both the rigidity of the structure and the compressibility of the subsoil.

Eq 7 is now arranged in terms of the unknown contact pressures q_i and, to this purpose, the reaction forces Z_i^{*k} are expressed by the matrix

$$\bar{Z} = (Z_{i,k}^*) \qquad \begin{aligned} i &= 1 \ldots n, \\ k &= 1 \ldots n. \end{aligned}$$

Then the equations for the contact pressures q_i are given by

$$(\bar{Z} \cdot \bar{A} - \bar{D}) \cdot \bar{q} = \bar{p}. \tag{8}$$

The matrix from equation system 8 is given by the product of the two matrices $\bar{Z}$ and $\bar{A}$ minus the diagonal matrix $\bar{D}$. In this case, the elements of the diagonal matrix are:

$$d_{i,i} = 1/K_r \cdot \epsilon^3 \qquad (i = 1 \ldots n)$$

and

$$d_{i,k} = 0 \qquad (i \neq k).$$

The right-hand side of the equation, $\bar{p}$, includes the reaction forces P_i in the imagined perfectly rigid supports of the foundation slab due to the loads of the structure, so that

$$\bar{p} = -P_i/(K_r \cdot \epsilon^2 \cdot L \cdot B) \qquad (i = 1 \ldots n).$$

With the contact pressures q_i, which are calculated from Eq 8, we find according to Eq 3c the settlements at each of the imaginary supports of the foundation. Superimposing the separate deformation figures according to Fig. 3, using the real settlements instead of unit settlements, we find the bending moments of the structure by using the method given by Ostenfeld.

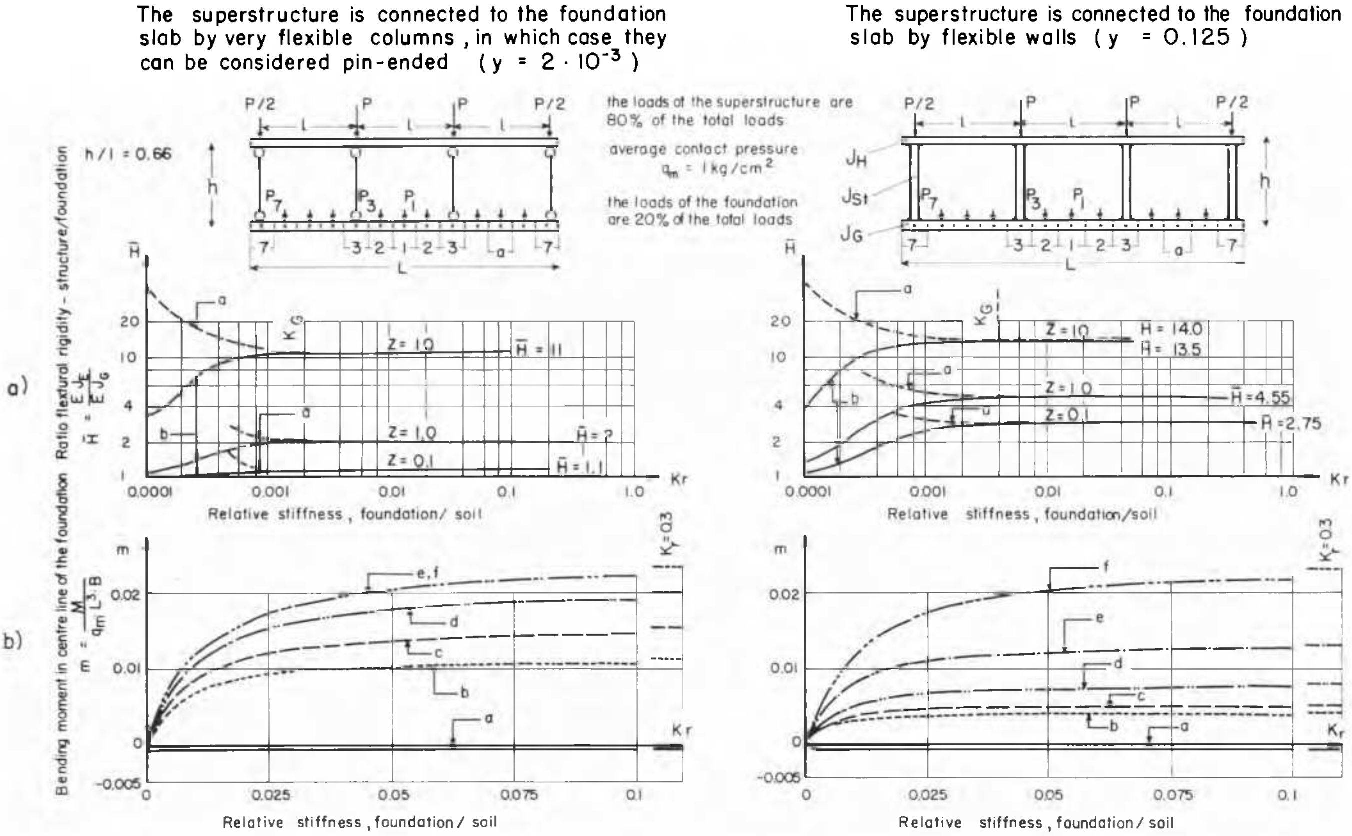

FIG. 5. Influence of foundation and superstructure rigidity.

α Ratio of flexural rigidity of the structure EJ_E to flexural rigidity of foundation EJ_G as a function of K_r. Curve a, results obtained from relative settlements Δs between the centre line of foundation and the foundation point under the external supports of superstructure. Curve b, results obtained from relative settlements Δs between the foundation points under the internal and the external supports of superstructure. $K_r = EJ_G/E_sL^3B$ = relative stiffness of foundation/soil. $z = J_{II}/J_G$; $y = J_{St}/J_G$.

β Bending moment in the centre line of foundation slab for different rigidities of the superstructure as a function of K_r. Curve a, $z = 10^2 \sim \infty$, ideal rigid superstructures; curve b, $z = 1.0$, superstructure between rigid and flexible; curve c, $z = 0.5$, superstructure between rigid and flexible; curve d, $z = 0.1$, superstructure between rigid and flexible; curve e, $z = 10^{-5} \sim 0$, ideal flexible superstructure; curve f, $z = 0$; $y = 0$, the case if only the rigidity of the foundation is considered. For a foundation slab with a flexural rigidity EJ_G for a slab thickness of 1 m and with a length $L = 18$ m and width $B = 90$ m ($L/B = 0.2$) gives the relative stiffness $K_r = 0.001$ the compression modulus $E_s = 3000$ kg/sq.cm. (compact gravel), $K_r = 0.01$ compression modulus $E_s = 300$ kg/sq.cm. (loose sand or preconsolidated clay), $K_r = 0.1$ compression modulus $E_s = 30$ kg/sq.cm. (soft clay).

APPLICATION OF THE METHOD

The method was programmed for the electronic computer I.B.M. 650. The application of this method shows to what extent the rigidity of the foundation and superstructure can be expressed by the total flexural rigidity of the structure. Furthermore, the influence of both the foundation and superstructure rigidities on the bending moments in the foundation can be investigated. For the investigation given here, a symmetrical superstructure on four columns with symmetrical loading was considered (Fig. 5). The length/width ratio (L/B) of the foundation was taken to be 0.2. These conditions can be met in practice when investigations are made for the conditions of a structure at its cross section.

The rigidity of the superstructure is introduced into the method according to the ratio of the moments of inertia $z = J_{II}/J_G$ of the superstructure (J_{II}) and the foundation slab (J_G), and by the nature of the connections of the columns or walls with the foundation. This latter condition is included by the ratio of the moments of inertia $y = J_{St}/J_G$ of the columns or walls (J_{St}) and the foundation (J_G). The chosen values for the parameters z and y are specified in Fig. 5. The parameter $y = 2.10^{-3}$ was ascertained from a newly erected structure, the superstructure of which was supported on the foundation by *columns*. The parameter $y = 0.125$ was derived for a structure with *wall* supports 50 cm thick on a 1-m-thick foundation slab. Normally in practice y is not greater than this value.

Flexural Rigidity of the Structure

The flexural rigidity of the structure is defined here as the rigidity of an imaginary foundation slab, which permits the same relative settlement Δs as the total structure. To calculate the rigidity of the structure, the difference in settlement of the foundation slab, stiffened by the superstructure ($z \neq 0$, $y \neq 0$), is compared with the difference in settlement for a foundation slab, not stiffened by a superstructure ($z = 0$, $y = 0$). To this purpose, the difference in settlement between α the middle of the foundation slab and the point under the external column, and β the point under the internal column and the point under the external column were calculated for both the real structure and the imaginary foundation slab. In this way, the effect of the additional rigidity of the foundation slab due to the superstructure was examined in the middle and at the positions under the internal supports of the superstructure. Details of this analysis are outside the scope of this paper, but the general results obtained are indicated in diagram (a) (Fig. 5). In these diagrams $\bar{H}$ represents the ratio of the flexural rigidity of the complete structure to the flexural rigidity of the foundation, i.e. $H = EJ_E/EJ_G$.

It follows from the results obtained:

1. The flexural rigidities of the structure in the middle and under the internal supports are equal and independent of K for values of $K > K_G$.

2. The flexural rigidity of the structure for values of $K_r > K_G$ can be calculated without any consideration for the properties of the subsoil, because for these cases the rigidity is independent of K_r. Hence, the flexural rigidity can be obtained by considering the complete structure rigidly supported under the external columns of the structure with free deformation between the supports (like a beam on two supports). This analysis can be made exactly according to the laws of statics or approximately by using the formula given by Meyerhof (1953).

3. For the relative stiffness $K_r > K_G$ the work involved in the calculation of the contact pressure can be reduced, as was done in the determination of the settlements, by avoiding the highly statically indeterminate system and using instead an imaginary foundation slab having a rigidity the same as that of the entire structure. The contact pressures and the resulting bending moments determined by this simplified method deviate from the values obtained from the exact method by up to 5 per cent.

4. For the relative stiffness $K_r < K_G$ the foundation slab is so flexible that there is a tendency for the slab to vault upwards between the columns. In this case, the rigidity of the structure can no longer be derived from the differences in settlement at the centre of the foundation slab. Because of the tendency of the slab to vault upwards the difference in settlement is reduced, and so the rigidity of the structure appears to be greater than that in the previous case for $K_r > K_G$, as is shown by the dotted lines in Fig. 5a. This is obviously wrong. Actually, the rigidity of the structure reduces with the relative stiffness K_r. This can be seen from the results obtained for the difference in settlement between the internal and external supports of the structure. The conditions which allow the rigidity of the structure to be expressed by an equivalent rigidity of a foundation slab are no longer applicable for the relative stiffness $K_r < K_G$. In this case the foundation slab and the superstructure must be regarded as one whole system.

The Relation of the Bending Moments in the Foundation Slab to the Rigidity of Structure and the Stiffness of the Subsoil

The moments in the centre of the slab are shown as a function of the relative stiffness K_r (Fig. 5b) for different rigidity conditions of the structure. These diagrams show that the bending moment increases with K_r, from zero for a perfectly flexible foundation, to a maximum for a rigid foundation, and they decrease as the rigidity of the superstructure increases. The moments in the centre of the slab are shown by curve f for the case when the superstructure is perfectly flexible. If in the design of a foundation slab, carrying a superstructure with some rigidity, the calculations are made taking into account only the rigidity of the slab, the dimensions will be greater than those actually necessary.

The influence of soil settlement and the creeping of concrete over the period of time has not been taken into consideration in this paper, but it is reasonable to suppose that the method outlined above can provide a theoretical solution to such problems.

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