# Consideration of Heterogeneity and Non-linear Deformation of the Base in the Design of Rigid Foundations

## Considération de l'hétérogénéité et de la déformation non-linéaire d'une base dans le calcul des fondations rigides

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SUMMARY

This report considers problems connected with the distribution of reactive pressures along the foot of rigid plates pressed into a non linearly strained and heterogeneous base with plane strain. As a result of the investigations it is concluded that the real base soil properties result in a significant change in the curve of reactive pressures, which in a number of cases facilitates the performance of the bases.

SOMMAIRE

Cet article traite problèmes relatifs à la distribution des pressions réactives sous des plaques rigides enfoncées dans une base non-linéairement déformée et héterogéne avec un déformation plane. Comme résultat de ces études une conclusion importante est obtenue, indiquant que les propriétés réelles du sol de la base amènent un changement considérable de la courbe des pressions réactives, ce qui facilite en nombre de cas le fonctionnement des bases.

THE PROBLEM OF STRESSES AND STRAINS of a half-space which is continuously inhomogeneous in depth was first raised by Klein (1948, 1949) without taking into consideration creep and non-linear deformation of the soil. However, as is clear from the relevant experiments (Tsytovich, 1963), for cohesive, especially for viscous clay soils, as well as for frozen soils (Vyalov, 1959) and for all kinds of soils under significant loads, the non-linear creep of the soil skeleton is of great importance.

In the mathematical description of the creep of soils at the present stage of technical development, soils are considered as one-component bodies, and the peculiarities of their deformation as multi-phase disperse systems are not taken into consideration, though such problems are already posed (Tsytovich, 1964). The performance of a soil as a one-phase system has been well described, proceeding from the plasticity theory (Prager, 1958) and associated non-linear creep (Rabotnov, 1948). This theory is used as the basis of the present investigation. Described below are the results of some analytical solutions, obtained by Yu. K. Zaretsky (1963) with the advice and general supervision of N. A. Tsytovich.

### CALCULATION OF RIGID FOUNDATIONS

The calculation of rigid foundations consists firstly in the determination of the curve of reactive pressures originating along their foot. The relationship to which the shifts of the points of the contact region should comply in the case of plane strain will be

$$u(x, t) + r(x) = c(t) + \gamma(t)\cdot x, \qquad (1)$$

where $u(x,t)$ is the vertical shift in time of the base boundary as a result of its deformation; $r(x)$, the equation of the rigid stamp surface; $c(t)$, the approach in time of the stamp to the base in the direction of the $y$ axis due to reciprocating movement; $\gamma(t)\cdot x$, the approach in time of the stamp to the base due to the turn of the stamp relative to the origin of the co-ordinates.

The vertical shifts in time of the base boundary as result of its deformation will be (Zaretsky, 1963):

$$u(x, t) = D\left\{ \int_s [(1 + L)\{p^\nu(s, t)\}]^{1/\nu}[1/|s - x|^\alpha]ds \right\}^\nu \qquad (2)$$

Here the integral operator

$$L\{y(t)\} = \int_0^t K(t, \tau)y(\tau)d\tau,$$

is introduced where $K(t,\tau)$ is the kernel of creep soils. Parameter $\alpha$ equals

$$\alpha = 1 - (1 - \eta)/\nu.$$

The stress-strain state of a base obeys the following invariant deformation regularities:

$$e_i = \sigma_i^\nu/A;$$
$$e = e_x + e_y + e_z = 0;$$
$$\nu > 1; \qquad (3)$$

where $A = A_0\, y^\eta$; $e_i$ is the intensity of shear strain; $\sigma_i$ is the intensity of shear stress; and $e$ is the volumetric change.

By substituting in Eq 1 the expression for the shift of a non linearly strained base, Eq 2, we obtain an equation including the desired function of distribution of contact pressures, $p(x,t)$. This equation is then replaced with a system of integral equations:

$$D^{1/\nu} \int_s g(s, t)[1/|s - x|^\alpha]ds = [c(t) + \gamma(t)x - r(x)]^{1/\nu} \qquad (4)$$

$$(1 + L)\{p^\nu(s, t)\} = g^\nu(s, t) \qquad (4a)$$

Eq 4 is the Volterra equation of the second order and its solution is given in the form:

$$p(s, t) = [(1 - N)\{p^{\nu}(s, t)\}]^{1/\nu}, \quad (5)$$

where $N\{y(t)\}$ denotes the integral operator

$$N\{y(t)\} = \int_0^t R(t, \tau)y(\tau)d\tau.$$

Here $R(t,\tau)$ is the resolvent of the Kernel $K(t,\tau)$. To finally determine the curve of contact pressures $p(x,t)$ the solution of Eq 4 must be obtained. Constants $c(t)$ and $\gamma(t)$ are found from the static conditions:

$$\int_s p(x, t)dx = P(t); \quad \int_s p(x, t)xdx = M(t), \quad (6)$$

where $P(t)$ is the external load acting on the base, and $M(t)$ the external moment of forces.

With $\alpha < 1$ Eq 4 is the integral equation of Fredholm of the first order with a weak singularity. A solution of this equation which is free from singular integrals taken in the meaning of Cauchy can be obtained with the method suggested by Krein (1955). The present work employs another method reducing Eq 4 with finite limits to an equation with infinite limits and a Kernel depending upon the difference of the arguments, which permits effective use for its solution of the bilateral Laplace transform. This method suggested by Zaretsky is simple enough and gives a closed solution.

## A METHOD FOR SOLVING AN INTEGRAL EQUATION OF TYPE 4

Let us consider the solution of the integral equation

$$\int_{-1}^{1} \phi(x)[1/|y - x|^{\alpha}]dx = f(y). \quad (7)$$

First of all replace variables $x = th(\xi/2)$ and $y = th(\zeta/2)$. After elementary transformations Eq 7 will be written as follows:

$$\int_{-\infty}^{\infty} H(|\zeta - \xi|)\Phi(\xi)d\xi = F(\zeta). \quad (8)$$

Here

$$\Phi(\xi) = \phi(\xi)/ch^{2-\alpha}(\xi/2);$$

$$F(\zeta) = 2f(\zeta)/ch^{\alpha}(\zeta/2);$$

$$H(\zeta - \xi) = [sh\tfrac{1}{2}(\zeta - \xi)]^{-\alpha}.$$

The kernel of integral Eq 8 depends upon the difference of the arguments and at $\alpha < 1$ belongs to space $U_2$. Suppose that function $F(\zeta)$ also belongs to class $U_2$ (to be integrated with a square), and try to find a solution to Eq 8 in the following form:

$$\Phi(\xi) = \int_{-\infty}^{\infty} G(|\zeta - \xi|)F(\xi)d\xi, \quad (9)$$

where $G(|\zeta - \xi|)$ is the resolvent kernel of Eq 8. To obtain a solution to Eq 9 use the bilateral Laplace transform:

$$Z^*(p) = p\int_{-\infty}^{\infty} \exp(-p\tau)\cdot Z(\tau)d\tau$$

to integral Eq 8. With the help of this transform and the theorem of multiplication of representations reduce integral Eq 8 to an algebraic form:

$$\Phi^*(p) = pF^*(p)/H^*(p)$$

or

$$\Phi^*(p) = (1/p)[p^2/H^*(p)]F^*(p). \quad (10)$$

A solution in the form of Eq 10 is possible if representations $p^2/H^*(p)$ and $F^*(p)$ have common zones of convergence. It is then easy to see that Eq 9, written in the representations, coincides with Eq 10 if it is assumed that

$$G^*(p) = p^2/H^*(p). \quad (11)$$

The representation of the Kernel $H(\xi)$ is found with the help of the following expression:

$$H^*(p) = (2^{\alpha}/\cos\pi\alpha/2) \times p\cos\pi p[\Gamma(\alpha/2 + p)\Gamma(\alpha/2 - p)/\Gamma(\alpha)] \quad (12)$$

$$R_e(\alpha/2 - p) > 0;$$

$$R_e(1 - \alpha) > 0.$$

In accordance with Eq 11, and taking into consideration Eq 12, the representation of the resolvent Kernel will be given in the form:

$$G^*(p) = \frac{p\Gamma(a + p)\Gamma(a - p)}{2^{\alpha+2}\pi\sin(\pi\alpha/2)\Gamma(1 - \alpha)}[2\cos\pi p - (1 + \cos\pi\alpha)/\cos\pi p], \quad (13)$$

where $a = 1 - \alpha/2$. Functions $\cos\pi p$ and $\sec\pi p$ should be represented in the form of power rows in $p$:

$$\cos\pi p = \sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}}{(2k)!}p^{2k}; \quad \sec\pi p = \sum_{k=0}^{\infty}\frac{\pi^{2k}|E_{2k}|}{(2k)!}p^{2k}; \quad -1/2 < R_e p < 1/2. \quad (14)$$

Hereinafter $E_{2k}$ are Euler's numbers; $\Gamma(x)$ is the gamma function.

The existence of representation is determined by the conditions $R_e(1 - \alpha) > 0$ and $\max(-½, -R_e a) < R_e p < \min(½, R_e a)$.

On the other hand, we have seen that the representation of the Kernel of integral Eq 8 exists at $R_e(1 - \alpha) > 0$ and $R_e(\alpha/2 - p) > 0$. Fulfilment of the above limitations imposed on the Kernel $H(|\zeta - \xi|)$ shows that parameter $\alpha$ should be within the limits $-1 < \alpha < 1$.

By using these correlations, and on the basis of the theorem of multiplication of representations and differentiation of originals, the desired solution $\phi(\xi)$ will be represented in the final form:

$$\phi(\xi) = -[(1 - \alpha)\,\mathrm{ch}^{2-\alpha}(\xi/2)]/4\pi\sin(\pi\alpha/2) \times \left[\sum_{k=0}^{\infty}\pi^{2k}\frac{(-2)^{k+1} + (1 + \cos\pi\alpha)|E_{2k}|}{(2k)!}\cdot\frac{d^{2k}}{d\xi^{2k}}\left(\int_{-\infty}^{\infty} f(\zeta)\frac{d\zeta}{\mathrm{ch}^{\alpha}(\zeta/2)[\mathrm{ch}\,\frac{1}{2}(\zeta - \xi)]^{2-\alpha}}\right)\right]. \quad (15)$$

After function $\phi(\xi)$ is found, the change to old variable $x$ is accomplished with the help of reverse replacement of variable $\xi = \ln[(1 + x)/(1 - x)]$.

Let us now consider the solution of Eq 7 for some special case of the term on its right-hand side

$$f_n(y) = c_n y^n. \tag{16}$$

By substituting $f_n(\zeta) = c_n\zeta^n$ in Eq 15 and by making the corresponding calculations, we shall obtain the desired solution. But in this case it is expedient to follow another procedure using Eq 10.

Representation $\Phi_n{}^*(p)$ will be written as follows:

$$\Phi_n{}^*(p) = [2c_n \cos(\pi\alpha/2)/\pi]\cdot F(-n, \alpha/2 + p; \alpha; 2)\cdot[\pi p/\cos \pi p]. \tag{17}$$

The hypergeometric function included in Eq 17 can be represented as the finite series:

$$F(-n, \alpha/2 + p; \alpha; 2) = A_n + \sum_{m-1}^{n} B_{n,m}p^m, \tag{18}$$

where factors $A_n$ and $B_{n,m}$ equal, respectively:

$$A_n = 1 + \sum_{s=1}^{n} \beta_{n,s}a_{s,s}; \quad B_{n.m} = \sum_{s=m}^{n} \beta_{n,s}a_{s-m,s},$$

The factors $a_{i,k}$ equal

$$a_0 = 1;\ a_{i,k} = \sum_{j=i}^{k} (\alpha/2 + j - i)(\alpha/2 + j - i + 1) \ldots (\alpha/2 + j - 1)$$

and factors $\beta_{n,k}$ equal

$$\beta_{n.k} = (-2)^k n(n-1)\ldots(n-k+1)/\alpha(\alpha+1)\ldots(\alpha+k-1)\cdot k!$$

Now, with the help of Eq 18 it is easy to change over to the originals in Eq 17.

### CONTACT PRESSURES ALONG THE FOOT OF A RIGID RECTANGULAR PLATE

Let us consider firstly the particular problem of the distribution of contact pressures along the foot of a rigid rectangular plate of a width $2a$ under the effect of a central force $P(t)$. In this case the region of the contact is constant and is given at section $-a \leqslant x \leqslant a$ while in the right-hand side of Eq 4 it should be supposed that $\gamma(t) = 0$; $r(x) = 0$. On the basis of the results obtained above with $f(y) = c_0 = \text{const.}$ and also taking into account Eq 4a we shall have:

$$p(x, t) = (1/\pi)\cdot\cos(\pi\alpha/2)[1/(a^2 - x^2)^{(1-\alpha)/2}] \times [(1-N)\{c_0^{\nu}(t)\}]^{1/\nu}. \tag{19}$$

As value $c_0(t)$ generally is not known, we shall determine it on the basis of static condition, Eq 6. From this condition we find:

$$[(1-N)\{c_0^{\nu}(t)\}]^{1/\nu} = \{\Gamma(\alpha/2)\Gamma[(1-\alpha)/2]\}[1/2a^{\alpha}\sqrt{\pi}]P(t). \tag{20}$$

Further, introducing Eq 20 again into Eq 19 we finally obtain

$$p(x, t) = \frac{\Gamma[(1-\alpha)/2]\Gamma(1+\alpha/2)\cos(\pi\alpha/2)}{\sqrt{\pi}} \cdot \frac{P(t)/a}{\pi[1-(x/a)^2]^{(1-\alpha)/2}} \tag{21}$$

where $\Gamma(x)$ is still a gamma function, and $\alpha = 1 - (1-\eta)/\nu$. Supposing $\eta = 0$, the solution will coincide with one made by Arutyanyan (1959).

The effect of the non-linearity of the deformation law $\nu = 1/n$ and of the heterogeneity of the mechanical properties of the base $\eta$ on the distribution of contact pressures is illustrated in Fig. 1. The comparison of these curves shows

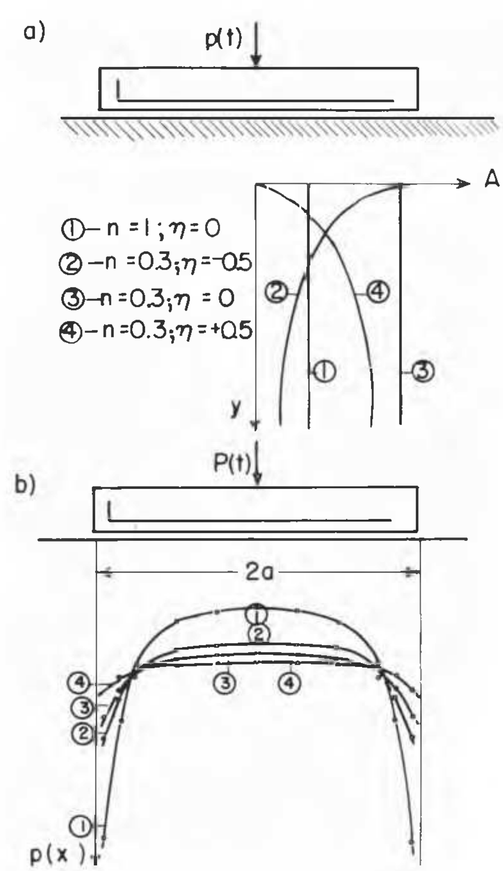


FIG. 1. Curves of contact pressures under rigid plate affected by central force.

that although the peculiarity at the plate edges $x = \pm a$ is preserved, the distribution of pressures becomes more uniform with an increase in $\alpha = 1 - n(1-\eta)$ (here $n = 1/\nu$) from zero to unity.

Thus, taking into account real properties of the base soils results in a reduction in the bending moment, which acts on the base, i.e. enables the lightening of the base structure.

Let us consider secondly the problem of a rigid rectangular plate of a width $2a$ pressed into a heterogeneous base (characterized by parameter $A = A_0 y^{\eta}$) with a non-linearity value $\nu = 1$. Force $P(t)$ is applied with eccentricity $e$.

The distribution of reactive pressures along the plate foot is found from the solution of integral Eq 4. In this case $\alpha = \eta$, the region of contact $-a \leqslant x \leqslant a$, and in the right-hand side of Eq 4 $r(x) = 0$ should be assumed. On the basis of the obtained results it is easy to find:

$$p(x, t) = (1/\pi)\cdot\cos(\pi\eta/2)[1/(a^2 - x^2)^{(1-\eta)/2}] \times[(1-N)\{c_0(t) + (x/\eta)c_1(t)\}]. \tag{22}$$

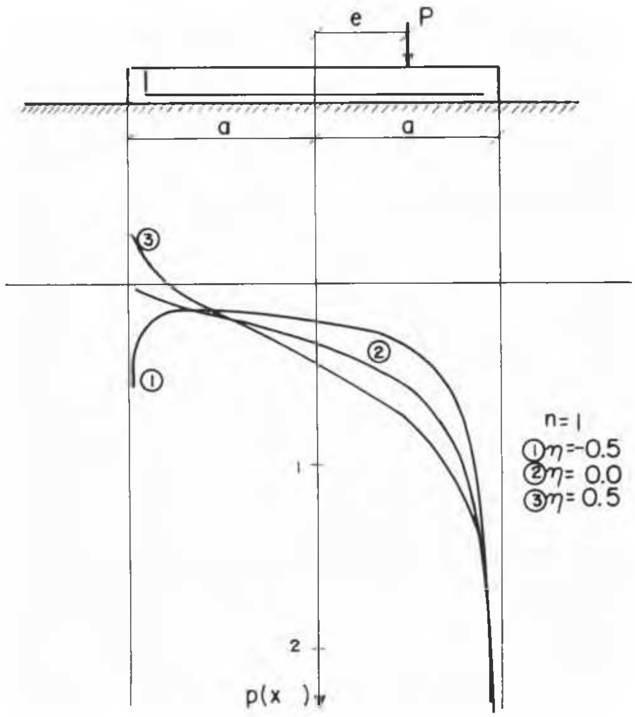


FIG. 2. Curves of contact pressures under rigid plate affected by force applied eccentrically.

The constants $c_0(t)$ and $c_1(t)$ will be determined from the static conditions of Eq 6.

By using the values of integrals

$$2\int_0^a \frac{dx}{(a^2 - x^2)^{(1-\eta)/2}} = a^{\eta} B\left(\frac{1}{2}, \frac{1+\eta}{2}\right);$$

$$2\int_0^a \frac{x^2 dx}{(a^2 - x^2)^{(1-\eta)/2}} = a^{2+\eta} B\left(\frac{3}{2}, \frac{1+\eta}{2}\right)$$

where $B(u,v)$ is the beta function, and by assuming that $M(t) = P(t) \cdot e$, we shall finally obtain:

$$p(x, t) = \frac{\Gamma(1+\eta/2)\Gamma\left(\frac{1-\eta}{2}\right)\cos\frac{\pi}{2}\eta}{\sqrt{\pi}} \cdot \frac{P(t)/a}{\pi[1-(x/a)^2]^{(1-\eta)/2}}\left[1 + (1+\eta/2)\frac{2e}{a}\left(\frac{x}{a}\right)\right]. \quad (23)$$

Fig. 2 illustrates curves of reactive pressures under effect of point force $P(t)$ applied at a distance of a quarter of the width from the base centre with three values of the heterogeneity factor $\eta = -0.5$, $\eta = 0$, and $\eta = 0.5$.

The analysis of Eq 23 obtained shows that tensile stresses do not appear under the plate if the eccentricity of the force applied does not exceed $e = a/(2 + \eta)$. In this case, when parameter $\eta \to 1$, eccentricity approaches ⅙ of the structure foot width. With $\eta \to -1$ extreme eccentricity tends to a half-width of the bed, $a$, and if $\eta = 0$, to a quarter of the bed width, $a/2$.

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