Statics and Dynamics of Soil

Statique et Dynamique des sols

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Summary

This paper deals with the plane state of stress and the plain state of strain in a granular material without a liquid phase. It is shown, that the stress-pattern in any state of equilibrium differs from the stress-pattern during motion. In general it is impossible to derive the shape of a rupture-line from an investigation of an equilibrium state of stress, since a rupture-line belongs to a dynamic state of the soil.

The pattern of rupture-lines has to be derived from kinematical considerations. It seems to be irrational to carry out stability-investigations with the aid of elements belonging to a state of motion.

Sommaire

Cette communication traite de la contrainte plane et de la déformation simple dans un matériau à grains, sans passer par une phase liquide. Les auteurs démontrent que la contrainte à n'importe quel stade d'équilibre diffère de la contrainte en mouvement. Il est impossible en général de déduire la forme d'une ligne de rupture de la recherche d'une contrainte dans un état d'équilibre, puisque une ligne de rupture implique un état de mouvement du sol. Les lignes de rupture découlant de considérations cinématiques, il semble irrationnel de procéder à des recherches sur la stabilité, à l'aide d'éléments dépendant du mouvement.

1. This paper deals with the plane state of stress and strain in a granular material without a liquid phase, and of which the frictional properties may be described by the equation

$$\tau = \sigma \tan. \varphi$$

in which τ denotes the shear stress
σ denotes the normal stress
and φ denotes the angle of friction

The real cohesion C is assumed to be zero. The angle of friction φ can never exceed the value φ_m, which is a value characteristic of the particular granular material being considered.

2. The state of stress at any point is determined by three variables, for instance by

the normal stress σ_x in the direction of the x-axis,
the normal stress σ_y in the direction of the y-axis,
the shear stress τ being the same in both the directions of the x-axis and the y-axis.

Assuming a specific weight γ, the equilibrium in the directions x and y furnishes the equilibrium equations

$$\frac{\delta\sigma_y}{\delta y} + \frac{\delta\tau}{\delta x} = \gamma \qquad (1)$$

$$\frac{\delta\sigma_x}{\delta x} + \frac{\delta\tau}{\delta y} = 0 \qquad (2)$$

From these two equations only it is impossible to obtain a solution of the state of stress. One equation is lacking and the problem therefore is statically indeterminate. In such cases the missing equation will be derived by formulating a law of elastic deformation. This seems to be practically impossible, in the case of a soil, because our knowledge about the elastic deformation properties of soils is still very incomplete. At present, it is therefore practically impossible to calculate the state of stress caused by known loads upon the soil.

3. We can therefore only calculate particular states of stress In order to obtain the third missing equation we supply the equation of deformation by any assumption about the state of stress. In soil mechanics the so called "limiting condition" is often used for this purpose. This assumes that at every point there are two directions (the φ-directions) in which the angle of internal friction φ attains its maximum value φ_m. This requires that all the stress circles in the stress diagram are tangential to the Coulomb envelopes. Expressed in the principal stresses ρ_1 and ρ_2, the third equation becomes

$$\sin \varphi_m = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

Substituting in equation (1) and (2) the variables ρ_1, ρ_2 and the angle α between the ρ_1 — direction and the x — axis for the variables σ_x, σ_y and τ, three equations are obtained for ρ_1, ρ_2 and α from which the three variables may be calculated. Since we are dealing with partial differential equations, a solution can only be obtained which applies to a number of boundary conditions.

Lately, de JOSSELIN DE JONG [Litt. 1] has given a graphical solution of this special state of stress. Starting from the boundaries, he constructs a pattern consisting of two systems of φ-lines (which he denotes as slip-lines *); the value of the total stress on the φ — lines is indicated at each point of intersection of two φ — lines. The angle of intersection equals $\frac{\pi}{2} - \varphi_m$. Along the φ — lines the total stress changes according to the equations of Kötter. To each value of φ_m there is a certain pattern of φ-lines.

The state of stress, described by the equations (1), (2) and (3), is the state of stress at border-equilibrium, and will be denoted by the symbol S_{be}.

4. The existence of S_{be} within a certain area is a fairly arbitrary assumption. Obviously it is impossible to prove, that the loads acting on the soil are able to create this S_{be}. Likewise it is impossible to calculate how far this S_{be} may extend.

* The denomination « slip-line » may be misleading, because these lines describe an equilibrium state of stress, in which no slipping occurs.

If one succeeds in the graphical construction of a φ — lines pattern without gaps, it is obvious that the soil may be in equilibrium under the acting loads. At any rate, one state of stress (the S_{be}) is imaginable, which is able to assure equilibrium. It is not possible, however, to prove that even this special state of stress really will occur.

At the S_{be} the angle of internal friction obtains its maximum in every direction. Along the φ — directions this has to be φ_m, but in every other direction φ has a maximum value. In connection with the internal equilibrium an increase of the φ existing along an arbitrary direction would require an increase of φ_m along the φ — directions. The latter increase is impossible, because the material is not able to supply it.

Therefore, the S_{be} may be regarded as the state of stress which activates the frictional resistance of the soil up to its maximum value.

If one does not succeed in constructing a φ — lines pattern without gaps, two possibilities exist. Either the gap may be filled up by a minor stress pattern, not being the S_{be}, or it is impossible to fill the gap. The latter case means that even a full activation of the internal friction resistance is not able to assure equilibrium. The soil will move and plastic deformation will result.

5. Since in a dry, granular material there is no influence which forces the plastic deformation to be a steady movement, this deformation generally will be a non--steady or accelerated one.

The equilibrium-equations (1) and (2), therefore become :

$$\frac{\delta\sigma_y}{\delta y} + \frac{\delta\tau}{\delta x} = \gamma - \frac{\gamma}{g} a_y \qquad (1\text{ m})$$

$$\frac{\delta\sigma_x}{\delta x} + \frac{\delta\tau}{\delta y} = -\frac{\gamma}{g} \cdot a_x \qquad (2\text{ m})$$

a_x and a_y denoting the acceleration in x-and y-direction respectively.

During the plastic transformation the third equation

$$\sin \varphi_m = \frac{p_1 - p_2}{p_1 + p_2} \qquad (3)$$

keeps its validity

The state of stress during plastic deformation, denoted here by the symbol S_m, cannot yet be derived from these three equations, because there are five variables : p_1, p_2, α, a_x and a_y. The two missing equations can be obtained from considering the plastic deformation (e.g. the principle of continuity, the principle of the Saint Venant, etc.).

We will not deal here in particular with the shape of these equations, nor with the problem of how to construct the φ-lines patterns, stress fields, flow lines or strain fields.

In general the S_m can be representated by a φ -lines pattern, which may be a function of time. The state of motion (strain) here denoted by the symbol M, can be represented by a flow line pattern or a strain field.

In general S_{be} is not identical with S_m, because they depend on different equations. Therefore, the φ-lines patterns of S_{be} and S_m will be different ones.

The Kötter equations in S_m are more intricate than in S_{be}. There is no reason, why the φ-lines of S_m should be identical with the flow-lines of M.

6. The slip-line or rupture-line is the border-line between the area in motion (plastic deformation) and the area at rest. Continuity considerations lead to the statement that a rupture-line has to be an extreme flow-line of the moving area.

In general a rupture-line will not coincide with a φ-line. The angle of internal friction along a rupture-line in general will not be equal to φ_m, but will be smaller. The total stress along a rupture-line will not change according to Kötters equations.

The stability of soil is often investigated by assuming some rupture-lines of arbitrary shape. However, this is not permissible because only one system of rupture-lines is the correct one, which may be derived from the state of motion M. It is a flow-line system. Moreover, the angle of internal friction along a rupture-line in general will not be equal to φ_m, but will be smaller. The real value of φ has to be derived from the φ-lines pattern belonging to S_m.

8. In his investigations [2 and 3], de Josselin de Jong identifies slip lines belonging to M, with the φ-lines belonging to S_{be}. This is based on his assumption [3, p. 52] that disintegration of the soil can take place only in the slip-line directions. This assumption of shearing only along the slip-lines seems to be incorrect, since during plastic deformation shearing occurs in every direction. Moreover, he identifies S_{be} with S_m, and this is only true if a_x and a_y are small.

9. In his stability-investigations Brinch Hansen starts from rupture-lines, the shape of which has been investigated in model-tests. He points out, that in general rupture-lines have to be kinematically possible [5, p. 61]. This in itself is certainly true, but one may ask, what these kinematics (slip-lines and rupture-lines included) matter in an investigation of the S_{be}. Kinematics only come in when the soil moves, but at that moment the S_{be} in general does not exist any longer. Further, Brinch Hansen assumes that along the rupture-line the angle of internal friction equals φ_m and that Kötters equations are valid. In the opinion of the authors these assumptions cannot be right in general. It is noteworthy, that the shape of the rupture-lines, as found by Brinch Hansen, does not always agree with the shape required by Kötters equation.

10. In most stability investigations there is the same queer contradiction. Stability is a state of equilibrium, in which there is no movement or plastic deformation. In this equilibrium problem, however, a rupture-line is assumed, but a rupture-line belongs to a state of motion and does not belong to a state of equilibrium. It seems to be irrational to carry out an investigation of the soil at rest with the aid of elements belonging to a state of motion. Since it is in practice impossible to calculate the real state of stress caused by the acting loads, a study of stability has to be a study of the S_{be} only, without bringing into the picture elements of plastic deformation, i.e. rupture-lines or kinematic considerations.

11. A soil in equilibrium can be described by only one "image", here denoted by the symbol I. This one image is a stress field, denoted by the symbol IS_e. If the equilibrium of the soil is a border equilibrium, we write IS_{be}.

A moving soil cannot be described by one image; we need two images. The state of stress may be described by an IS_m, the state of strain (plastic deformation) by IM_m. The latter may be a strain field, a strain rate field or a flow-lines pattern. If, for any reason, a steady movement occurs, we write IS_{sm} and IM_{sm}. A soil in the state of equilibrium apparently has an $IM = 0$.

A soil in a state of steady flow (accelerations are zero or small) has $IS_{sm} = IS_{be}$. The IM_{sm} depends merely on kinematical considerations (including the kinematic boundary conditions); the IS_{sm} depends merely on the pure stress equations (1), (2), and (3) and their boundary conditions on stress. IS and IM however are connected by stress-strain relations, given by the properties of the material.

In general, however, a non-steady flow will occur. IS_m will differ from IS_{be}. The whole pattern of IS and IM and their interrelations as well, becomes far more intricate. Both are a function of time.

Litterature

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