The Theory of Probability and Statistics in Relation to the Rheology of Soils

Quelques problèmes de la mécanique statistique et de la rhéologie des sols

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Summary

The authors discuss the application of the theories of probability and statistics to soil mechanics. New results of rheological investigations undertaken by the authors are described, and experimental equations of elastic deformation and viscous flow of soils are given. A theoretical method of determining the rate of mining subsidence is put forward, taking creep into account and a comparison with field measurements.

Sommaire

Le rapport expose le problème d'application de la théorie des probabilités et de la statistique à la mécanique des sols. Les résultats des recherches de l'après-action élastique et du fluage visqueux sont présentés, ainsi que la méthode d'évaluation du fluage dans le calcul des affaissements de la surface du sol au dessus des exploitations minières. Les résultats des calculs sont comparés avec les données des observations in situ et d'essais des modèles faits dans un appareil centrifuge.

The probability approach to problems of soil mechanics was first suggested in the U.S.S.R. by G. POKROVSKY as far as in 1933 [1,2]. His fundamental principles were :

1. The stress distribution in soil is controlled by the law of probability (usually normal).

2. The settlement is a process of transition of a foundation to a more stable state. Boltzman's theorem as to proportionality of entropy change to logarithm of probability is applicable to a structure-foundation system.

3. The strength of soil is determined by the maximum local stress coinciding with the weakest points of the material and statistically distributed over the whole volume of soil.

These principles will now be supplemented by the statistical treatment of sounding or penetration data or by some other simple methods of soil profile determination.

Let us consider for example the consolidation problem. It is evident that Terzaghi's model completely coincides with the rheological Blizard-Cauer "ladder" model made up of the Kelvin-Foigt's models (Fig. 1).

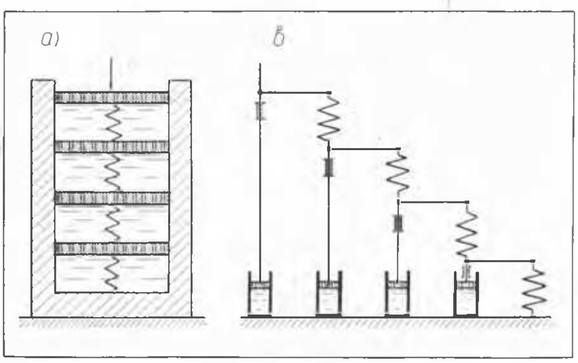

Fig. 1 Terzaghi's consolidation model (a) and Blizard-Cauer's ladder structure (b).
Le modèle de consolidation de Terzaghi (a) et la structure en « escalier » de Blizard-Cauer (b).

When the number of elements of such a model is increasing infinitely the parabolic type equation is obtained [4] :

$$\frac{\partial \sigma}{\partial t} = k^2 \frac{\partial^2 \sigma}{\partial z^2}; \; k^2 = \frac{E}{\lambda} \tag{1}$$

coinciding with Terzaghi's consolidation equation. Thus C_v is nothing but the ratio of compression modulus E to the coefficient of viscosity λ.

Consolidation may be regarded as a process of successive broadening of deformation due to drainage. The application of Kolmogoroff's equation gives the following result :

$$\frac{\partial \sigma(\tau,\zeta\,;t,z)}{\partial t} = -\frac{\partial}{\partial z}[A(t,z)\cdot\sigma[\tau,\zeta\,;t,z)] + \frac{\partial^2}{\partial z^2}[B(t,z)\cdot\sigma(\tau,\zeta\,;t,z)] \tag{2}$$

where σ — effective stress ; τ and ζ — preceding time and coordinate ; t and z — subsequent time and coordinate ; A and B — are characteristic of the medium. When $A = 0$ and $B =$ a constant we obtain Terzaghi's equation.

According to modern theories [6,7,8] we have two limiting conditions of clay structure : of "card-house", and of "packets" type. Any deformation under load changes the structure to a limited extent from the first to the second type.

The authors claim to have proved this by micrographic research on kaolin clay sections obtained after triaxial compression tests. Fig. 2 shows how clay particles change their orientation towards the shear surfaces [9].

The orientation begins in some separate points where local lack of homogeneity leads to stress concentration or it begins close to the points where there are already particles, orientated by chance in the direction of shear. Then orientation spreads outwards from the points or origin owing to the redistribution of stresses and overloading of adjacent parts, as the strength of sections with orientated particles decreases.

The presence of sand or silt grains hampers or even interrupts the process of orientation, thereby creating zones of

Fig. 2 Microphotograph of a zone of shear in a primary kaolin sample before rupture. A number of orientated clay particles bands is clearly seen. Increase 80×.

Microphotographie de la zone de cisaillement dans un échantillon du kaolin primaire avant rupture. On voit un certain nombre de bandes des particules orientées. Grossissement 80×.

compactness with orientation in other directions. Experiments with mixtures of montmorillonite clays and different quantities of fine sand (from 40 to 80 per cent) have shown that the greater the amount of clay, the greater will be the deformation and the rate of deformation under the same load during the first 5-10 days.

The orientation of particles loads to an increase of repulsion forces and to negative pore pressure set up in the zone of orientation. As a result, water from the neighbouring parts of the soil will be sucked up into this zone and weakening will continue. The longer the load acts, the more intensive will orientation become and the higher will be the water content in the shear zone. The long-term strength of the soil will be correspondingly reduced.

When passing to a new state the particle also passes through an intermediate state, pushing aside the adjacent particles to gain the places for turning and displacement. Hence, some additional energy must have been accumulated a potential barrier. The activation stress σ_a necessary to overcome this barrier is evidently greater than the average value of acting stress σ.

Let us assume that the activation energy $U(\sigma_a)$ is a function of σ_a and that every particle offers the same deformation increment (leap) ρ and that each particle offers such an increment, as adopted in the Cottrell's theory of unsteady creep. Then the probability $P(\sigma_a)$ of a leap for a particle with the stress activation σ_a during the time interval dt will be equal to :

$$P(\sigma_a)\,dt = B \cdot \exp\left[- M\, U(\sigma_a)\right] dt \tag{3}$$

where B and M are system characteristics which generally depend on σ and t. The rate of deformation with the lapse of time t from the beginning of creep is :

$$V_t = \rho \int_0^{\infty} N(\sigma_a, t)\; P(\sigma_a)\, d\sigma_a \tag{4}$$

where $N(\sigma_a, t)$ is the number of particles with stress activation between σ_a and $\sigma_a + d\sigma_a$ with the lapse of time t. The longer this time the greater must be σ_a as the increase of quantity of particles causes increase of stress on the stable undisturbed parts of the structure, i.e. σ_a is an increasing function of t.

Let us assume as a first approximation, that for the unsteady stage of creep σ_a = a constant, $U(\sigma_a) = A \cdot \sigma_a$ and

$$N(\sigma_a \cdot t) = N_0 \cdot \exp\left[- P(\sigma_a) \cdot t\right]$$

we receive the solution of (4) under given σ in the form :

$$V_t = \frac{C N_0}{t} \tag{5}$$

where N_0 is the value of $N(\sigma_a, t)$ at the beginning of the process ($t = 0$) and C is a constant. With the increase of t the rate of creep decreases thus turning the soil into the stage of steady creep with constant rate.

The authors developed the following empirical equation for the rate of creep :

$$V_t = C\,\sigma^{x} \cdot t^{-\alpha} \quad \ldots . \tag{6}$$

describing approximately the rate of elastic deformation and viscous flow which take place simultaneously.

The value of α was found to be 0,75 in average for illite, 1,09 — for montmorillonite and 1,4 — for polymineral clay in a stiff consistency, i.e. rather close to 1. The difference between (5) and (6) is explained by the approximate method of deriving (5). Considering the time deformation ε_t as a sum of elastic strain ε_e, damping creep (elastic deformation) ε_a and viscous flow ε_f, the authors have determined ε_e and ε_a by the unloading curve (Fig. 3). It was found that :

$$\varepsilon_f = \varepsilon_t - (\varepsilon_e + \varepsilon_a) \quad \ldots . \tag{7}$$

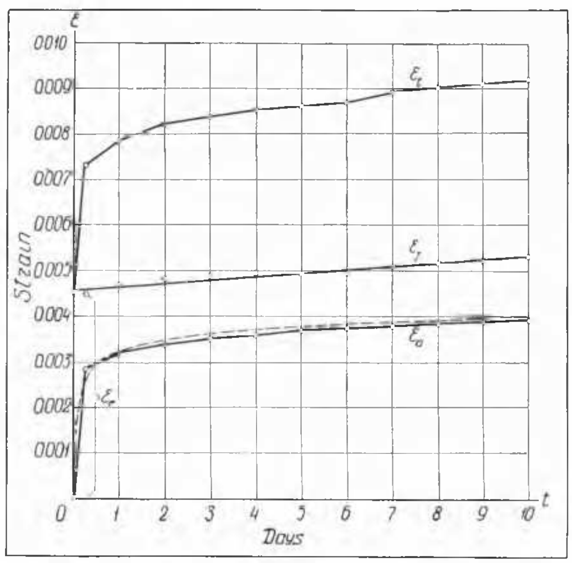

Fig. 3 Creep test diagram of a marl clay sample of stiff consistency (σ = 9·65 k/cm²). The theoretical deformation curve is shown by the dotted line.

Courbes de fluage pour un échantillon de marne compacte (σ = 9,65 k/cm²). La courbe théorique de fluage élastique est indiquée par le pointillé.

is a straight line passing through the origin of coordinates :

$$\varepsilon_f = b\int_0^t [\sigma(t) - \sigma_0]\,dt \qquad (8)$$

and that ε_a is :

$$\varepsilon_a = \frac{kn}{E_a}\int_0^t (t-\theta)^{n-1}\cdot \exp\,[k(t-\theta)n]\,\sigma(\theta)d\theta \quad \ldots. \quad (9)$$

where θ — time changing from 0 to t.

These factors were found to be valid for unconfined and triaxial unconsolidated compression tests. The tested soils were tertiary marl ($L_L = 57$, $P_L = 30$) and black jurassic clay ($L_L = 65$, $P_L = 29$) of stiff consistency.

Equation (9) is more exact than (6) but the latter is more convenient for coumputations. We give below an example of mining subsidence process calculation, taking into consideration rheological properties of soil [11]. The authors consider the deformation of a soil strata as a bending of a beam [12, 13] with the equation (fig. 4) :

$$Ey_1\frac{\partial^2 Uy}{\partial x^2} = -\sigma_x \qquad (10)$$

where y_1 is a distance from neutral axis.

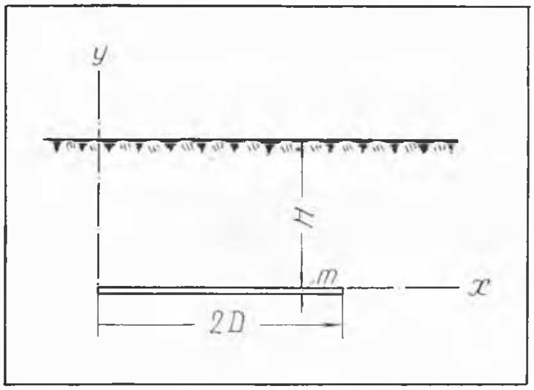

Fig. 4 Diagram corresponding to equation (15). $2D$ - breadth ; H - depth ; m - height of mining.
Diagramme correspondant à l'équation (15). $2D$ - largeur ; H - profondeur ; m - hauteur de la taille.

Assuming the incompressibility of soil and taking $x = 1$:

$$\sigma_x = \frac{1}{C}\frac{\partial^2 u_x}{\partial x\cdot\partial t}t^\alpha = -\frac{1}{C}\frac{\partial^2 u_y}{\partial y\cdot\partial t}t^\alpha \qquad (11)$$

Supposing that $y_1 = \xi^2 y$ and designating $CE = \beta$ we obtain from (10) and (11) a differential equation :

$$\beta\xi^2 y\frac{\partial^2 u_y}{\partial x^2} = \frac{\partial^2 u_y}{\partial y\cdot\partial t}\cdot t^\alpha \qquad (12)$$

The initial conditions are :

$$t = 0;\quad U_y(x,y,0) = 0$$

$$t = \infty;\quad U_y(x,y,\infty) = U_{yo}(x,y)$$

where $U_{yo}(x,y)$ is the equation of soil surface when subsidence is complete. Let us assume the solution in the form :

$$U_y(x,y,t) = U_y(x,y)\cdot\varphi(t) \quad \ldots. \quad (13)$$

In this way we receive the following equation of surface subsidence deformation under simple boundary conditions :

$$U_y W_0 \exp\left(-\frac{\beta}{b}t^{-b}\right)\left\{\Phi\left(\frac{2D-x}{\xi H}\right)+\Phi\left(\frac{x}{\xi H}\right)\right\} \quad \ldots. \quad (14)$$

Here W_0 — the maximum subsidence ; Φ ( ) — probability integral, $b = 1 - \alpha$. The rate of subsidence can be received by differentiation of (14) :

$$V = \frac{1}{2}W_0\beta t^{-\alpha}\exp\left(-\frac{\beta}{b}t^{-b}\right)\left\{\Phi\left(\frac{l+st-x}{\xi H}\right)+\Phi\left(\frac{x}{\xi H}\right)\right\} + \frac{w_0 S}{\xi H\sqrt{2\pi}}\exp\left[-\frac{\beta}{b}t^{-b}-\frac{1}{2}\left(\frac{l+st-x}{\xi H}\right)^2\right] \quad \ldots. \quad (15)$$

where $S = (2D - l)/t$ is the speed of coal face mining ; l — the breadth of mining, corresponding to the time at which subsidence begins. The calculation is confirmed by the field data (Fig. 5), but allowance for the rheological properties can be somewhat complicated.

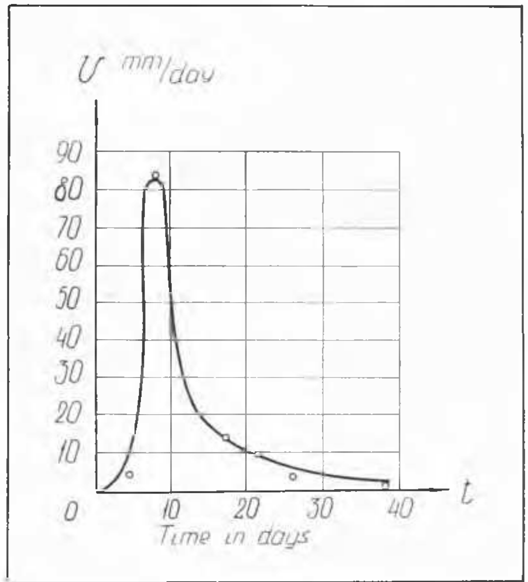

Fig. 5 Computed and observed subsistence rates.
Vitesse d'affaissement de la surface calculée et observée.

The authors therefore suggest the application of scale model research. Equations of equilibrium and of rheological state are of the following type :

$$\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\tau_{xy}}{\partial x\cdot\partial y}+\frac{\partial^2\tau_{xz}}{\partial x\cdot\partial z} = \gamma f(\sigma,\tau)\frac{\partial}{\partial t}\varphi(t) \quad \ldots. \quad (16)$$

The similarity of the scale model to the prototype is possible only if f and φ are homogeneous functions. If the model is made of the same material as the prototype, we receive the condition of similarity in the form :

$$m_e^2\cdot m_\sigma^{\delta-1}\cdot m_t^{\psi-1} = 1 \quad \ldots. \quad (17)$$

where m_e, $m\sigma$, m_t are scales, correspondingly, of length, stress and time ; δ and ψ — homogeneous characteristics of functions f and φ. The results of model experiments by the authors with distance measurement of deformations carried out by the authors are in close agreement with the calculations (Fig. 6).

A diagram of the centrifugal apparatus used by the authors for these model tests is shown in Fig. 7.

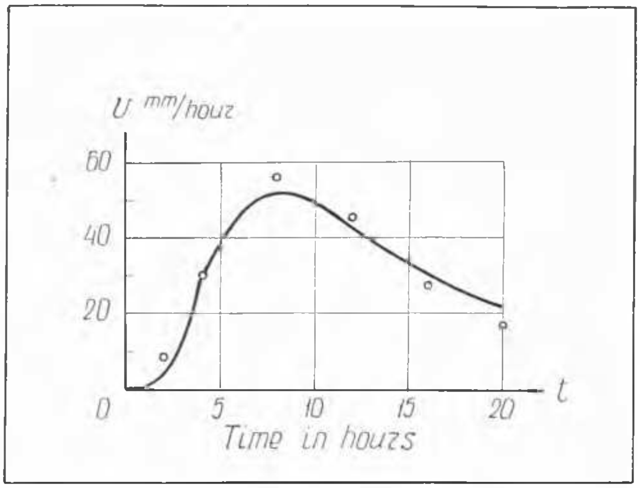

Fig. 6 The scale model test of mining in a rock at a depth of 20 m in comparaison with computation results.
Comparaison d'un essai sur modèle d'exploitation en carrière (profondeur 20 m) avec les résultats du calcul.

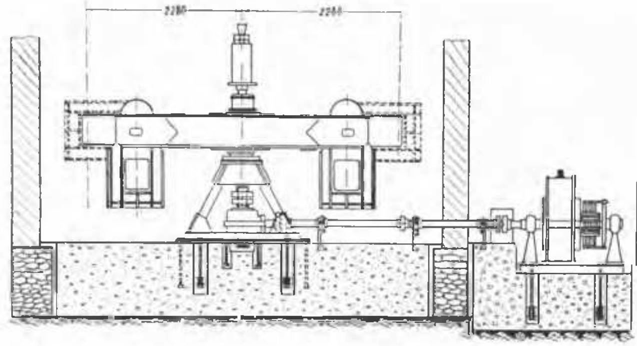

Fig. 7 Centrifugal apparatus for large scale model investigations.
Installation centrifuge pour les essais sur modèles de grande dimension.

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