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# A Model Law for Simultaneous Primary and Secondary Consolidation

## Une loi de modèle pour la consolidation primaire et secondaire simultanée

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### Summary

The author first considers primary consolidation, showing that the time curve may be conveniently plotted on a combined square root and logarithmic basis. A simple yet approximately accurate formula has been developed for primary consolidation.

He then investigates both primary and secondary consolidation, introducing three new constants  $t_s$ ,  $K_s$  and  $c_s$  for determining characteristics. A model law has been developed for simultaneous primary and secondary consolidation, and the author suggests that the test results should be interpreted and used in a new way, to give the correct settlement of the prototype at different times.

### Sommaire

L'auteur examine tout d'abord le cas de la consolidation primaire et montre que la courbe du tassement en fonction du temps peut être tracée en une échelle combinée de racine carrée et logarithmique. Il en déduit une formule simple mais relativement exacte pour la consolidation primaire. Il considère ensuite l'ensemble, consolidation primaire et secondaire, en introduisant trois nouvelles constantes  $t_s$ ,  $K_s$  et  $c_s$ . Il en tire une loi applicable à des essais sur modèles pour la consolidation primaire et secondaire et propose un nouveau mode d'application des résultats d'essais pour obtenir des prévisions de tassements pour différentes époques.

### 1. Introduction

In most consolidation tests it is found that some secondary consolidation will occur in addition to the primary one. When the test results are used for calculating the rates of settlement for actual foundations, it is generally assumed that primary consolidation runs its course first, and that thereafter secondary consolidation sets in. At least, the rates of settlement are usually calculated simply by applying the well-known model law for primary consolidation to the time curve found in the test.

It is evident, however, that secondary consolidation must start as soon as an effective stress is developed, and the two processes must therefore actually proceed simultaneously from the very beginning, although of course with different time rates, because they follow different model laws.

The author suggests an approximate model law for simultaneous primary and secondary consolidation, which may enable a more satisfactory calculation of settlement rates to be made. Only the case of one-dimensional consolidation is considered, in which an extensive, homogeneous clay layer of thickness  $2H$  (drained at both sides) or  $H$  (drained at one side only) is subjected to a vertical stress increase  $p$  throughout the layer. The mean specific compression  $\varepsilon$  (per unit of thickness) at any given time  $t$  is sought.

### 2. Primary consolidation alone.

In the case of no secondary consolidation, it is assumed that an effective stress increase  $d\bar{\sigma}$  will produce immediately a specific compression :

$$d\varepsilon = \frac{d\bar{\sigma}}{K_v} \quad (1)$$

which will remain constant, as long as the effective stress is not changed. In the case of varying effective stresses through the thickness of the layer (1) is still valid when  $d\varepsilon$  and  $d\bar{\sigma}$  are interpreted as mean values.

From the classical theory of consolidation it is known, that up to at least 50 per cent consolidation the following equation gives a very close approximation :

$$U = \frac{\varepsilon}{\varepsilon_c} = \frac{\bar{\sigma}}{p} = \sqrt{\frac{4}{\pi} T_v} = \sqrt{\frac{4kK_v}{\pi\gamma_w H^2} t} = \sqrt{\frac{4c_v}{\pi H^2} t} = \sqrt{\frac{t}{t_v}} \quad (2)$$

Here we have defined a characteristic time  $t_v$  by the equation :

$$t_v = \frac{\pi H^2}{4c_v} = \frac{\pi\gamma_w H^2}{4kK_v} \quad (3)$$

As we must have  $U = 1$  for  $t = \infty$ , (2) should be modified somewhat in order to cover the whole interval  $0 < t < \infty$ . The simplest function fulfilling the necessary requirements is :

$$U = \sqrt[n]{\frac{t^n}{t^n + t_v^n}} \quad (4)$$

It is further known that for  $T_v = \pi : 4 = 0.785$ , which corresponds to  $t = t_v$ , we shall have  $U = 0.89$ . This gives :

$$0.89 = \sqrt[n]{0.5} \quad n = \frac{\log 0.5}{2 \log 0.89} = 3 \quad (5-6)$$

$$U = \sqrt[6]{\frac{t^3}{t^3 + t_v^3}} = \sqrt[6]{\frac{T_v^3}{T_v^3 + 0.5}} \quad (7)$$

This function covers the whole interval  $0 < T_v < \infty$  with less than 1 per cent error in  $U$  anywhere.

Consolidation or settlement curves are usually plotted using either a  $\sqrt{t}$  — scale or a  $\log t$  — scale. Actually it will be most practical to employ both, namely a  $\sqrt{t}$  — scale below  $t_v$  and a  $\log t$  — scale above  $t_v$ . In order to get a smooth time curve we must for  $t = t_v$  have the same scale increase. This gives the following condition, in which  $L$  represents the length of a decade in the  $\log t$  — scale :

$$\frac{d\sqrt{t}}{dt} = \frac{d(L \log t)}{dt} \quad \frac{0.5}{\sqrt{t_v}} = \frac{L \log e}{t_v} \quad \dots (8-9)$$

$$\sqrt{t_v} = L \cdot 2 \log e = 0.87 L \quad \dots (10)$$

This shows that the total length of the  $\sqrt{t}$  — scale should be equal to 87 per cent of the length of a decade in the  $\log t$  — scale. Fig. 1 shows the theoretical consolidation curve plotted in such a diagram.

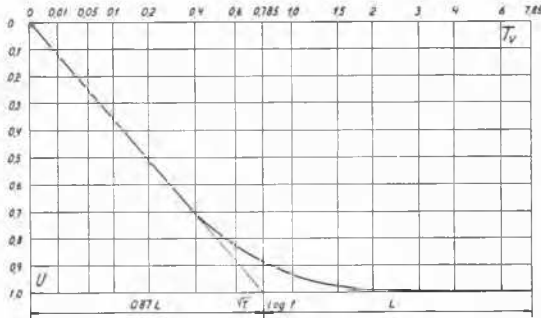


Fig. 1 Theoretical consolidation curve in  $\sqrt{t}$  —  $\log t$  — diagram.  
Courbe de consolidation théorique; graphique en  $\sqrt{t}$  et  $\log t$ .

The main practical advantage of the proposed combined time scale is, that both the first and the last parts of the time curve will be approximately straight lines. The theoretical point of 0 per cent consolidation is found by drawing the first straight line to intersection with the vertical axis corresponding to  $t = 0$ .

### 3. Primary and secondary consolidation together

The term secondary consolidation refers to the fact that the compression will usually increase with time, even when the effective stress is kept constant. Moreover, it seems that for  $t = 0$  we have actually also  $\varepsilon = 0$ . Under these circumstances any distinction between primary and secondary compression will be artificial and arbitrary.

It is known that for reasonably great values of  $t$  the secondary compression will increase in proportion to  $\log t$ . Hence, if at the time  $t_0$  an effective stress increase  $d\sigma_0$  is applied, it will at any later time  $t$  have produced a specific compression, for which the simplest expression will be :

$$dz = \frac{d\bar{\sigma}_0}{K_s} \log \frac{t_s + t - t_0}{t_s} \quad (11)$$

The modulus  $K_s$  and the time  $t_s$  must be regarded as characteristic constants for clay in the actual state of consolidation.

The total specific compression at the time  $t$  will then be :

$$\varepsilon = \frac{1}{K_s} \int_{t_0=0}^{t_s=t} \log \frac{t_s + t - t_0}{t_s} \cdot \frac{d\bar{\sigma}_0}{dt_0} dt_0 \quad (12)$$

We consider now the *beginning* of the consolidation process. In analogy with (2) we assume provisionally that  $\bar{\sigma}$  is approximately proportional to  $\sqrt{t}$  :

$$\bar{\sigma} = \beta \sqrt{t} \quad \bar{\sigma}_0 = \beta \sqrt{t_0} \quad (13-14)$$

Differentiating (14) and introducing it in (12) we get :

$$\begin{aligned} \varepsilon &= \frac{\beta}{K_s} \int_{t_0=0}^{t_s=t} \log \frac{t_s + t - t_0}{t_s} \cdot \frac{dt_0}{2\sqrt{t_0}} = \\ &= \frac{2\beta}{K_s} \left[ \sqrt{t_0} \log \frac{\sqrt{t_s + t} - \sqrt{t_0}}{e\sqrt{t_s}} + \right. \\ &\quad \left. + \sqrt{t_s + t} \log \frac{\sqrt{t_s + t} + \sqrt{t_0}}{\sqrt{t_s + t} - \sqrt{t_0}} \right]_{t_0=0}^{t_s=t} = \\ &= \frac{2\beta\sqrt{t}}{K_s} \left[ \sqrt{\frac{t+t_s}{t}} \log \frac{\sqrt{t+t_s} + \sqrt{t}}{\sqrt{t_s}} - \log e \right] \quad (15) \end{aligned}$$

As will be seen later, the characteristic time  $t_s$  is generally only a fraction of a second, and it can therefore usually be neglected, whenever it should be added to an observed time  $t$  (but not when it stands alone). Consequently we find, reintroducing  $\bar{\sigma}$  by means of (13) :

$$\varepsilon \sim \frac{\beta\sqrt{t}}{K_s} \log \frac{4t}{e^2 t_s} = \frac{\bar{\sigma}}{K_s} \log \frac{4t}{e^2 t_s} \quad (16)$$

The rate of compression must be a function of the average excess pore water pressure, and hence of the average effective stress. From (1) — (3) we can deduce this function :

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon_c}{2\sqrt{t}t_v} = \frac{p^2}{2K_v t_v \bar{\sigma}} = \frac{2kp^2}{\pi\gamma_w H^2 \bar{\sigma}} = \frac{p^2 c_s}{2K_s H^2 \bar{\sigma}} \quad (17)$$

We have here defined a new characteristic constant :

$$c_s = \frac{4kK_s}{\pi\gamma_w} \quad (18)$$

Differentiating (16) and inserting in (17) we get the following differential equation for  $\bar{\sigma}$  :

$$\frac{d\bar{\sigma}}{dt} \cdot \frac{1}{K_s} \log \frac{4t}{e^2 t_s} + \frac{\bar{\sigma}}{K_s} \cdot \frac{\log e}{t} = \frac{1}{\bar{\sigma}} \cdot \frac{p^2 c_s}{2K_s H^2} \quad (19)$$

with the solution :

$$\bar{\sigma} = p \sqrt{\frac{c_s t}{H^2}} \cdot \frac{\sqrt{\log(4t : e^3 t_s)}}{\log(4t : e^2 t_s)} \quad (20)$$

Inserting this in (16) we get the following equation, which should be approximately valid for the *beginning* of the consolidation process :

$$\varepsilon_0 = \frac{p}{K_s} \sqrt{\frac{c_s t}{H^2}} \log \frac{4t}{e^3 t_s} \sim \frac{p}{K_s} \sqrt{\frac{c_s t}{H^2}} \log \frac{t + 5t_s}{5t_s} \quad (21)$$

Here we have put  $e^3 = 20$ , and have made a small modification in order to avoid getting negative values of the quantity under the square root sign.

We shall finally consider the very *advanced* process. At this stage the specific compression must be approximately

equal to that, which would have been obtained, if the final effective stress  $p$  had been applied instantly at the time  $t_0 = 0$ :

$$\epsilon_\infty = \frac{p}{K_s} \log \frac{t + t_s}{t_s} \quad (22)$$

Just as in the case of primary consolidation alone, the author has developed expressions for the specific compression at the beginning and at the end of the process. We can therefore in a similar way combine them to an approximate model law for the whole process:

$$\epsilon = \frac{p}{K_s} \sqrt{\frac{6}{c_s^3}} \left[ t \log \frac{t + 5t_s}{5t_s} \right]^3 + 1 : \left[ \log \frac{t + t_s}{t_s} \right]^6 \quad (23)$$

#### 4. Evaluation of consolidation tests

The developed model law contains three unknown quantities:  $K_s$ ,  $c_s$  and  $t_s$ . They can be found by comparison with the observed time curve from a consolidation test (sample thickness  $2H_0$ ).

For this purpose the time curve is best plotted in the combined  $\sqrt{t} - \log t$  - diagram. It will then usually be found that the first and the last parts of the curve can be approximated by two straight lines. By changing the units of the time scale it can further be obtained that these two lines will intersect at (or at least in the vicinity of) the boundary line between the  $\sqrt{t}$  - and the  $\log t$  - scales. The time curve will then furnish us with three observed quantities  $t_c$ ,  $\epsilon_c$  and  $\epsilon_s$ , as defined in fig. 2.

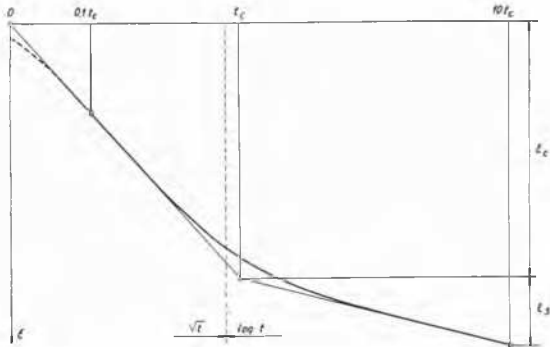


Fig. 2 Characteristic quantities of observed time curve.  
Caractéristiques de la courbe de consolidation en fonction du temps.

For the sake of brevity we shall use the following symbols:

$$A = \log \frac{t_c + 50t_s}{50t_s} \quad B = \log \frac{t_c}{t_s} \quad (24-25)$$

As will be seen from (21), the beginning of the curve is theoretically not quite a straight line. The straight line drawn through the points representing the first test results will, however, correspond approximately to the theoretical tangent for  $t = 0.1 t_c$ . Differentiating (21) with respect to  $\sqrt{t}$ , and putting  $t = 0.1 t_c$ , we can find the inclination of the first straight line:

$$\frac{d\epsilon_1}{d\sqrt{t}} = \frac{p(A + \log e)}{K_s H_0} \sqrt{\frac{c_s}{A}} \quad (26)$$

Combining this with (21) for  $t = 0.1 t_c$  we get the following equation for the first straight line:

$$\epsilon_1 = \frac{p}{K_s H_0} \sqrt{\frac{c_s}{A}} \left[ (A + \log e) \sqrt{\frac{t}{t_c}} - \frac{\log e}{\sqrt{10}} \right] \quad (27)$$

The equation for the second straight line is easily derived from (22):

$$\epsilon_2 = \frac{p}{K_s} \log \frac{t}{t_s} \quad (28)$$

For  $t = t_c$  we shall have  $\epsilon_1 = \epsilon_2$ . This gives:

$$c_s = \frac{A}{t_c} (BH_0)^2 : \left[ A + \log e - \frac{\log e}{\sqrt{10}} \right]^2 \sim \frac{A}{t_c} \left[ \frac{BH_0}{A + 0.297} \right]^2 \quad (29)$$

Multiplying (26) by  $\sqrt{t_c}$  we find:

$$\epsilon_c = \frac{p(A + \log e)}{K_s H_0} \sqrt{\frac{c_s t_c}{A}} \quad (30)$$

whereas (28) yields:

$$\epsilon_s = \frac{p}{K_s} \quad K_s = \frac{p}{\epsilon_s} \quad (31-32)$$

Eliminating  $K_s$  and  $c_s$  from (29-31) we find:

$$\frac{\epsilon_c}{\epsilon_s} = \frac{A + 0.434}{A + 0.297} B \quad (33)$$

Inserting (24-25), this equation can be solved for  $t_s$  by trial. For  $\epsilon_c/\epsilon_s > 2$ , which will usually apply, the following approximate formula gives very nearly the correct result:

$$B = \log \frac{t_c}{t_s} \sim \frac{\epsilon_c}{\epsilon_s} - \left[ \frac{1.1 \epsilon_s}{\epsilon_c} \right]^2 - 0.13 \quad (34)$$

The procedure is now to find first  $t_s$  from (34); if necessary check or revise the result by means of (33).  $A$  is then calculated from (24),  $c_s$  from (29) and  $K_s$  from (32). Finally, the coefficient of permeability may be derived from (18):

$$k = \frac{\pi \gamma w c_s}{4 K_s} \quad (35)$$

When  $t_s$ ,  $c_s$  and  $K_s$  have been found as described above, the compressions of a similar layer in nature with a drainage path  $H$  can be found by means of the model law (23).

#### 5. Example

In Fig. 3 is shown a time curve for a certain Danish clay, deposited in a glacial lake ( $w = 45$  per cent,  $w_P = 30$  per cent,

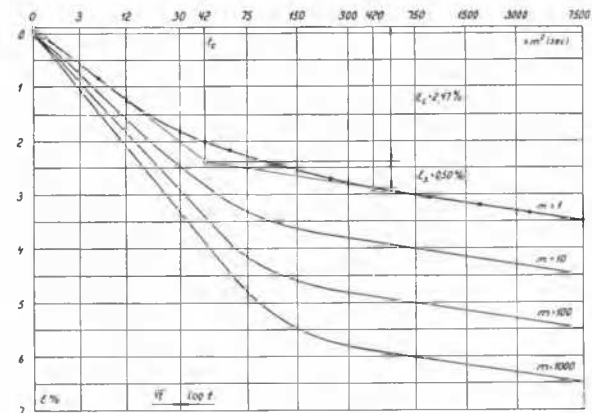


Fig. 3 Time curves for different layer thicknesses.

Courbes de consolidation en fonction du temps pour différentes épaisseurs de la couche.

$w_L = 65$  per cent). The sample had a cross section of  $10 \text{ cm}^2$  and a thickness  $2H_0 = 2 \text{ cm}$ . In the considered stage it was loaded from 30 to  $60 \text{ t/m}^2$  ( $p = 60 - 30 = 30 \text{ t/m}^2$ ).

From the shown time curve we find :

$$t_c = 42 \text{ sec.} \quad \varepsilon_c = 2.47 \text{ per cent} \quad \varepsilon_s = 0.50 \text{ per cent}$$

$$B = \log \frac{t_c}{t_s} = \frac{2.47}{0.50} - \left[ \frac{1.1 \cdot 0.50}{2.47} \right]^2 - 0.13 = 4.76$$

$$\frac{t_c}{t_s} = 57600 \quad t_s = \frac{42}{57600} = 7.3 \cdot 10^{-4} \text{ sec.}$$

$$A = \log \frac{42 + 0.0365}{0.0365} = 3.06$$

$$\frac{c_s}{c} = \frac{3.06}{42} \left[ \frac{4.76 \cdot 0.01}{3.06 + 0.297} \right]^2 = 1.46 \cdot 10^{-5} \text{ m}^2/\text{s.}$$

$$K_s = 30 \cdot 100 : 0.50 = 6000 \text{ t/m}^2$$

$$k = \frac{\pi \cdot 1 \cdot 1.46 \cdot 10^{-5}}{4 \cdot 6000} = 1.9 \cdot 10^{-9} \text{ m/s.}$$

According to current conceptions the time curve from the consolidation test should also represent the specific compression of a layer  $m$  times as thick, provided that the time scale is multiplied by  $m^2$ .

In Fig. 3 are, for  $m = 10, 100$  and  $1000$  (corresponding to  $2H = 0.2, 2$  and  $20 \text{ m}$ ) respectively shown the compressions calculated by means of the new model law. It will be seen that the difference from the classical concept (represented by the curve for  $m = 1$ ) may be quite considerable.

## 6. Conclusions

An approximate model law for simultaneous primary and secondary consolidation has been developed, and it has been shown that — at least theoretically — the conventional method for determining settlement rates can lead to considerable errors on the unsafe side.

It has here been tacitly assumed that the compressions observed in a consolidation test will represent faithfully the actual properties of the undisturbed soil. Otherwise it is evident that errors may be introduced. Sample disturbance and friction in the consolidation apparatus may result in such errors. Also, it is not known whether secondary consolidation will still proceed after many years according to the logarithmic law as presumed here.

With regard to experimental verification laboratory tests are not very suitable, because the height ratio must be great in order to show the difference clearly. Full scale settlement observations would be better, but very few have been carried on for a sufficiently long time to show the effect definitely.