The Use of the Energy Concept in Soils Mechanics

Utilisation de la notion d'Energie en Mécanique du sol

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Summary

By equating the shearing energy, and energy stored by volume changes under hydrostatic pressure, a failure criterion is provided based on the evaluation of the energy transferred to a soil by exterior loading and desiccation. It is shown that there is no fundamental difference between the energy of ' desiccation ' and the volumetric strain energy caused by normal loads, although the former is identified with the cohesive strength (even in the saturated range).

Sommaire

En formulant l'énergie de cisaillement et l'énergie emmagasinée par changements de volume sous pression hydrostatique on obtient un critère de rupture basé sur l'évaluation de l'énergie transférée à un sol par dessiccation et par charges extérieures.

L'auteur démontre qu'il n'existe pas de différence essentielle entre l'énergie « de dessiccation » et celle des changements de volume provoquées par des charges normales, quoique cette énergie de dessiccation soit identique à la force de cohésion (même dans la région de saturation).

Introduction

It is proposed to study the properties of a soil water system in terms of macroscopic or observable variables, in particular the external pressure p and volume change dV, and their conjugate quantities, the pore pressure u and water volume change dV_m. This procedure is in line with thermodynamics which derives relations between such macroscopic variables, considering them to be statistical averages of underlying microscopic or molecular processes. The behaviour of a soil-water system under stress also, of course, depends ultimately on the energy levels of the constitutent molecules notably in the ionised water films between soil particles. But, in any statistical treatment, these molecular properties are described by "thermodynamic" coefficients, for example the bulk modulus K. These coefficients are not constants in that they generally depend on the moisture content m, and in order to derive the exact functional relationship it is necessary to apply the methods of quantum statistical mechanics. This is beyond the scope of this paper, but the word "thermodynamic coefficient" or simply "coefficient" will be retained in order to emphasize its variable nature.

Before using thermodynamic methods to derive a failure criterion, it is necessary to find a relationship between the external or 'elastic' variables dV, p and the internal or 'soil water' variables namely dV_m and u.

The coefficient α

Imagine an elementary cube of soil, with a volume V, subjected to an ambient pressure p ; let the immediate pressure increase in the pore water be some fraction $u = \alpha\, p \; (0 \leqslant \alpha \leqslant 1)$ and the resulting water volume change ΔV_m, be some fraction β of ΔV. Then the energy change per unit volume in the water, due to the pressure p, will be

$$w_w = \tfrac{1}{2}\, u \frac{\Delta V_m}{V} = \tfrac{1}{2}\, \alpha \beta\, p \frac{\Delta V}{V}$$

while the observed energy change of the soil element due to a volumetric strain $\varepsilon = \dfrac{\Delta V}{V}$ will be

$$w_v = \tfrac{1}{2}\, p \frac{\Delta V}{V}$$

If the crystalline matter of the soil grains is incompressible, then $w_v = w_w$, that is

$$u \frac{\Delta V_m}{V} = \frac{\Delta V}{V} . p \qquad \dots . \quad (1)$$

So that $\alpha\beta = 1$ or

$$\left(\frac{\delta u}{\delta p}\right)_{V_m} = \left(\frac{\delta V}{\delta V_m}\right)_p = \alpha \qquad \dots . \quad (2)$$

Equation (2) has been derived independently by J.D. COLEMAN (CRONEY et al., 1958),

Now let us assume that when a quantity of water has been driven out of the element the pore pressure u falls to zero. There will then be no further expulsion of water but the energy per gram weight of the remaining pore fluid will have been reduced by the amount $\Delta\mu_1 = \nu_1\, u$ where $\nu_1 = 1$ cm^3/gm.wt. is the specific volume of the water, and the change $\Delta\mu_1$ in the thermodynamic potential μ_1 is equal to the work $\left(p \dfrac{\Delta V}{V}\right)$ necessary to bring a gram of pore water from its initial state to its state after the application of p and subsequent drainage. Thus, the pore water will now be the seat of a higher negative potential which can be measured, for example, by bringing the soil element into contact with a column of pure water and lowering the pressure on this water until there is no flow. In the equilibrium condition the potential in the pure water (which must be equal to $-\nu_1\, u$) will be the same as the potential of the water in the soil. In order to prevent mixing of the soil grains with the pure water, contact is established through a membrane permeable only to water and the device becomes a "suction plate" apparatus.

A curve of $\log_{10}\mu_1$, againt $\Delta V_m = \nu_1\, \Delta W_m$, where ΔW_m is the weight of water expelled, is generally called the pF

moisture content curve and the work put into the soil to bring it from the saturated condition to a moisture content $m = \frac{Wm}{Ws}$ is equal to the area under this curve or, from equation (1),

$$w_v = \frac{1}{V}\int u\, dV_m = \frac{1}{V}\int u\, v_1\, dW_m = -\frac{W_s}{V}\int^m \mu_1\, d_m = \int^\varepsilon p\, \frac{dV}{V} \quad (3)$$

This equation will be assumed to be correct whether the moisture is lost by desiccation or by the application of the ambient pressure p, which is compatible with the fact that points on the pF curve have been obtained by using a consolidometer.

However, equation (1) and therefore equation (3) will no longer be valid if $\alpha = 0$ because then there can be no volume change $d\,V$ even though $dV_m \neq 0$. This will be the case for the so-called incompressible or coarse-grained materials such as sand, which are seen to be incapable of storing energy due to desiccation.

The energy under a general stress system

In general, an element in a loaded soil such as a foundation will not be subjected to a uniform pressure p but rather to three normal stresses σ_x, σ_y, σ_z and three shearing stresses τ_{xy}, τ_{yz}, τ_{zx}. However, each normal stress may be resolved into two components

$$\sigma_x = S_x + p\,; \qquad \sigma_y = S_y + p\,; \quad \sigma_z = S_z + p$$

where p is the average normal stress given by

$$p = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \quad (4)$$

So that, substituting above, it is found that

$$S_x + S_y + S_z = 0$$

In other words the deviator stresses S_x, S_y, S_z can cause no volume change, since by definition

$$\varepsilon = \frac{\Delta V}{V} = \frac{p}{K}\,; K = \text{bulk modulus} \quad (5)$$

Therefore, the value of p given by equation (4) is the correct value to use for the volumetric strain energy in the general case. It is also shown by HOSHINO (1957) that p is the normal stress on the eight octahedral planes, making 45° to the three principal planes which carry the normal stresses σ_1, σ_2, σ_3 and no shearing stresses. The shearing stress τ_G on these octahedral planes is given by

$$\tau_G = \tfrac{1}{3}\left[(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{zx}^2\right]^{\frac{1}{2}} \quad (4a)$$

Both τ_G and p are so-called invariants of the stress tensor which means that their numerical value is unchanged for any change of the coordinate system, that is, they are independent of the frame of reference, so it is natural to express the total strain energy w in terms of τ_G and p. Then, per unit volume,

$$w = w_v + w_d = \tfrac{1}{2}\left(p\frac{\Delta V}{V} + \tau\,\Delta\gamma\right) = \tfrac{1}{2}\left(p\frac{\Delta V}{V} + \tfrac{3}{2}\,\tau_G\,\Delta\gamma_G\right) \quad \ldots\ (6)$$

(TIMOSHENKO, (1934)).

Let us now imagine that our elementary soil element has an energy due to volumetric strain alone of

$$w_v = \tfrac{1}{2}\,p\,\frac{\Delta V}{V} = \tfrac{1}{2}\,\frac{p^2}{K}$$

Now, if a shearing stress τ_G is applied there will be a strain energy of distortion

$$w_d = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tau_G\,\Delta\,\gamma_G = \tfrac{3}{4}\cdot\frac{\tau_G^2}{G}$$

This will grow until $w_d = w_v$ or the radius τ_G of the failure circle is given by

$$\frac{p^2}{K} = \tfrac{3}{2}\,\frac{\tau_G^2}{G} \quad (7)$$

Equation (7) leads us to the next section.

A failure criterion for a soil water system

The equating of the maximum octahedral stress to a constant has a precedent in the Von Mises failure criterion for steel, where experiment has shown that the strength is independent of the hydrostatic pressure p. This is not so for soil, so the constant is replaced by the volumetric strain energy as suggested in equation (7) which expresses the fact that failure will occur when the energy of distortion is equal to the energy already stored in the soil by desiccation and hydrostatic pressures. But there can be no strength increase due to hydrostatic pressure if there is no volume change, such as may occur in a saturated clay which cannot drain; hence the importance of the intergranular stress concept, since a fall in the pore pressure inevitably follows the expulsion of pore water.

Equation (7) is approximate to the extent that it is assumed that no volume changes take place due to the action of the shearing stresses. If these should cause an expansion there would be a strength decrease, otherwise an increase of strength would be caused by compression. In either case the volume change effect may be included by replacing the left hand side of (7) by p_1^2 where

$$\frac{p_1^2}{K} = \left(p\frac{\Delta V}{V} + p_0\frac{dV}{V}\right) \quad \ldots\ (7a)$$

Here p_0 is the hydrostatic pressure at failure and $\frac{dV}{V}$ the volumetric strain caused by the shearing stresses, which is negative for expansion.

Then from (7) the shearing stress τ_G at failure is

$$\tau_G = \sqrt{\frac{2\,G}{3\,K}}\,.p. \quad (7b)$$

This may be compared with the Mohr-Coulomb failure criterion when the radius τ_G of the Mohr circle at failure is given by

$$\begin{aligned}\tau_G &= \tau_f \operatorname{Cos}\Phi = c\operatorname{Cos}\Phi + \sigma\operatorname{Sin}\Phi \\ &= (c\operatorname{Cot}\Phi + \sigma)\operatorname{Sin}\Phi = p\operatorname{Sin}\Phi \quad \ldots\ (8)\end{aligned}$$

since, as indicated previously the hydrostatic pressure p is supposed to include effects of desiccation, *i.e.* the term $c\operatorname{Cot}\Phi$, as well as any normal stress application σ. In effect, p is the pressure measured from the apex of the failure cone with a central angle of Φ, where, as explained, a rational value of Φ can only be given by a drained test.

Comparison of equations (7b) and (8) then leads to a definition of this failure angle.

$$\text{Sin}\,\Phi = \sqrt{\frac{2\,G}{3\,K}} = \sqrt{\frac{1-2\nu}{1+\nu}} \qquad \ldots . \quad (9)$$

It is now apparent that Φ is a function of the Poisson's ratio ν only. ν may be measured in the laboratory on a cylindrical specimen under compression σ_z if the pressure σ_r, necessary to prevent any lateral displacement, is known. Then $\nu = K_0/1 + K_0$ if K_0 = coefficient of earth pressure at rest $= \frac{\sigma_r}{\sigma_z}$.

σ_z, σ_r are principal stresses, σ_1, σ_3 since there are no shearing stresses on the surfaces of the cylindrical sample and the normal and shearing stresses on planes at 45° to these principal planes are respectively $p = \frac{1}{2}(\sigma_1 + \sigma_3)$ and $\tau_G = \frac{1}{2}(\sigma_1 - \sigma_3)$.

Fig. 1 is a plot of equation (9) together with experimental results obtained at the National Building Research Institute Laboratories and by others (BISHOP, 1958). The agreement is good over almost the entire range of natural soils, and, as predicted, a value $\nu = \frac{1}{2}$ corresponds to an "angle of friction" Φ of zero, so that no strength increase should follow the application of a normal stress σ. A Poisson's ratio of $\frac{1}{2}$ means, no volume change since

$$K = \frac{E}{3\,(1-2\nu)}$$

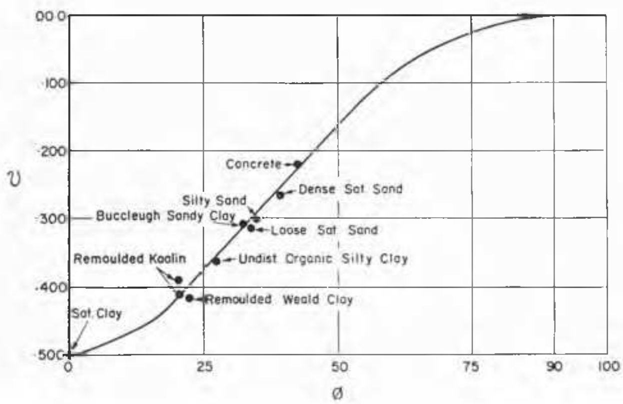

Fig. 1 Relationship between angle of friction and Poisson's ratio.
Relation entre l'angle de frottement et le coefficient de Poisson.

Also, the angle does not seem to vary significantly with moisture content and normal pressure which implies that the Poisson's ratio remains sensibly constant and, therefore, that there must be a reasonably constant relationship between the shear modulus at failure G_f and the bulk modulus K, that is from equation(9)

$$G_f = \tfrac{3}{2} \cdot \frac{(1-2\,\nu)}{(1+\nu)} K. \qquad (9a)$$

In order to verify this relationship saturated samples of a silty clay (from Buccleugh near Johannesburg) were first mixed in a vacuum and then extruded into cylinders 1 in. in diameter and 2 in. long. These were placed on suction plates or in desiccators to be brought to a known pF before being tested in a triaxial apparatus. Apart from drained unconfined tests, samples were first consolidated under cell pressures of 40 and 80 lb./sq.in. before being failed in drained shear. During compression, the volume change was measured and K determined at various moisture contents. The characteristic point for this soil is given in Fig. 1 corresponding to an average Poisson's ratio of just over 0·30, so that G_f could be evaluated from equation (9a).

The stress strain curves are shown in Fig. 2 for three different

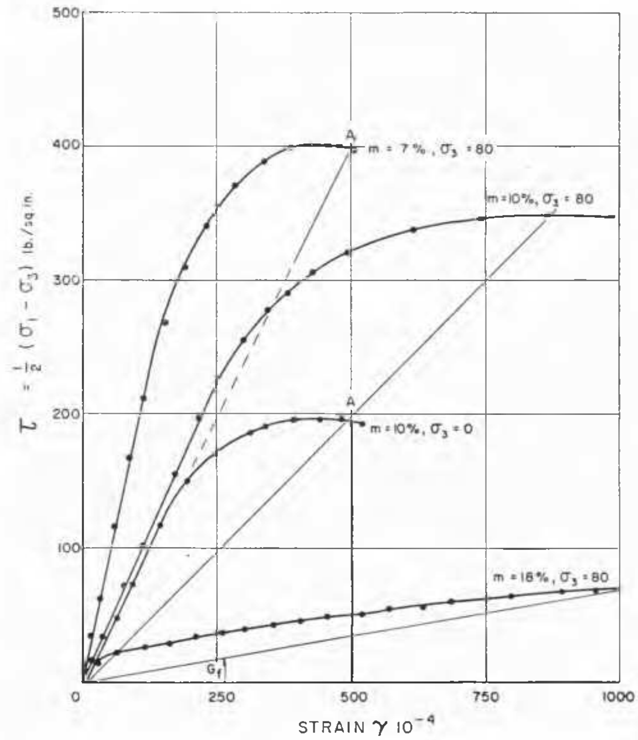

Fig. 2 Stress-strain curves for a Buccleugh silty clay.
Courbe tension-déformation pour une argile silteuse de Buccleugh.

moisture contents. In each case the linearised curve with a slope of G_f is included. Now, according to the failure hypothesis, failure should occur when the energy $\int \tau\, d\gamma$ given by the area under the stress strain curve is equal to $\frac{3}{2}$ times the triangular area $\frac{1}{2}\,\frac{\tau_G{}^2}{G_f}$. And the point of failure coincides very closely with the points of intersection A at the maximum proving ring stress. Now because the shear modulus G_f is a coefficient which varies with the moisture content, greater strains are needed before the stress-strain curve intersects the straight line if the moisture content and confining pressure is high. This fact helps to explain changes in the stress strain curve with moisture content and confining pressure.

The shearing strain γ used in Fig. 2 was derived from the vertical unit strain $\varepsilon_z = \frac{d}{h}$ by using the elastic relations

$$\varepsilon_z = \frac{1}{E}(\sigma_1 - 2\,\nu\,\sigma_3)\,;\ \varepsilon_r = \varepsilon_\theta = \frac{1}{E}\left[(1-\nu)\,\sigma_3 - \nu\,\sigma_1\right]$$

Therefore

$$\gamma = \frac{\tau}{G} = \frac{1}{2\,G}(\sigma_1 - \sigma_3) = \frac{(1+\nu)}{E}(\sigma_1 - \sigma_3) = \varepsilon_z - \varepsilon_r. \qquad \ldots . \quad (10)$$

ε_r may be determined if the volumetric strain

$$\varepsilon = \frac{d\,V}{V} = \varepsilon_z + \varepsilon_y + \varepsilon_0 = \varepsilon_z + 2\,\varepsilon_r$$

is known. Otherwise γ may be evaluated from a knowledge of Poisson's ratio because equation (10) may be written

$$\gamma = \varepsilon_z\left(1 - \frac{\varepsilon_r}{\varepsilon_z}\right) = (1 + \nu)\frac{d}{h}\left\{\frac{\sigma_1 - \sigma_3}{(\sigma_1 - \sigma_3) + (1 - 2\,\nu)\,\sigma_3}\right\} \quad \ldots\ldots \quad (10a)$$

d being the vertical dial gauge reading and h the sample height. Computations of γ by both methods (10) and (10a) were found to agree closely. Again from equation (10) the horizontal displacement $u_r = r_0\,\varepsilon_r$ is given by

$$u_r = r_0(\varepsilon_z - \gamma) \simeq r_0\,(\varepsilon_z - (1 + \nu)\,\varepsilon_z) \simeq -\,r_0\,\nu\,\varepsilon_z$$

Therefore, the increase in sample area due to shearing strain is given by

$$\pi\,(r_0 - u_r)^2 = \pi\,r_0^{\,2}\,(1 + \nu\,\varepsilon_z)^2 \approx \pi\,r_0^{\,2}\,(1 + 2\,\nu\,\varepsilon_z) \quad \ldots\ldots \quad (10b)$$

The cohesion *c*

Generally, when a soil is sheared in the laboratory there has already been a significant energy stored by virtue of desiccation; therefore the failure envelope does not pass through the origin, except in very coarse grained materials which are unable to store energy when drying out because the coefficient $\alpha = 0$.

The intercept of the failure envelope with the $\sigma = o$. axis is termed the cohesion, and increases with a fall in the moisture content as predicted by equation (3) since the energy given by the area under the pF curve will be larger.

For the samples from Buccleugh the pF moisture content curve was known and therefore the energy of desiccation at a given moisture content could be evaluated. The shrinkage of the samples was also measured so that $\frac{\Delta V}{V}$ was known after equilibrium had been attained at a given moisture content and a given pF.

At a moisture content of 10 per cent the stored energy per unit volume $-\frac{W_s}{V}\int^{m}\mu_1\,dm$ or $\int^{\varepsilon} p\,\frac{dV}{V}$ was found to be 20 lb./sq.in., while the volume change was equal to 0·100. This corresponds to an average value $\bar{p}$ determined by the relation $\bar{p}\,\frac{\Delta V}{V} = \int^{\varepsilon} p\,\frac{dV}{V}$ or $\bar{p} = c\,\mathrm{Cot}\,\Phi = 200$ lb./sq.in. i.e. $c = 120$ lb./sq.in. This compares very favourably with a measured cohesion of 110 lb./sq.in.

Conclusion

A change in volume is accompanied by a change in moisture content and a change in density, and is related to a change in pressure by the coefficient K; also, the shearing resistance is proportional to the product of volume change and pressure. Therefore it is apparent that points of failure must be characterised by values of moisture content, pressure and strength which are all uniquely interrelated. These points will define a failure surface in a three-dimensional space with axes of moisture content, strength and intergranular pressure as pointed out by Roscoe et al (1958).

This note on the application of energy or thermodynamic techniques to soil failure, is only a section of a wider project on soil thermodynamics still to be published.

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