The Effect of Inclined Loads on the Stability of a Foundation

Contribution à l'étude des fondations soumises à une charge inclinée sur la verticale

by Á. Kézdi, D. T. Sc., Technical University for Architecture, Civil and Communication Engineering, Budapest, Hungary

Summary

The author maintains that failure of a foundation is due to shearing force, and he determines, the strength of a wedge shaped body subjected to an inclined uniformly distributed load. The results obtained are applied to the case of punching strength, which is a function of the angle of friction and of the inclination of the resultant load. A reduction factor is given for determining the limiting value of the inclined load.

The author also determines the distributed load acting at ground level, which causes a state of plastic limit. The nature of distribution of the load is an unambiguous function of the angles ε and Φ.

Sommaire

La stabilité des semelles n'est pas exprimée correctement par la condition $H/V \leqslant \operatorname{tg} \varphi$, car lorsque la semelle glisse, il se produit une rupture en fondation. Afin de faire avancer la discussion théorique de la question, l'auteur étudie la résistance d'un massif prismatique tronqué chargé par une distribution uniforme et inclinée. Les résultats sont appliqués au calcul de la résistance au poinçonnement de fondations peu profondes en fonction de l'angle Φ et de l'inclinaison ε de la force résultante. L'auteur donne un facteur de réduction pour la détermination de la valeur limite de la force inclinée.

Dans la deuxième partie, en supposant des surfaces planes de glissement, l'auteur détermine la distribution des forces sur la surface limite qui provoque un état plastique limite du massif semi-indéfini.

The most important part of an investigation into the stability of foundations subjected to an inclined and occasionally eccentric load is the determination of the factor of safety against failure under shear. This is generally done by comparing the available frictional resistence with that component of the resultant load acting on the foundation-plane which is parallel to this plane. The effect of the normal force is generally investigated separately : we comput the value of maximum edge stress, calculated on the assumption of linear distribution of contact pressure.

This method is open to criticism from many aspects. The division of the resultant load into components and the separate investigations of their effects is an approximation to what occurs in practice : the soil is collectively affected by the contact-pressure, the distribution of which depends on many factors, such as the inclined eccentric resultant load, the stiffness and dimensions of the foundation slab, and the nature of the soil itself. The foundation will fail when stability vanishes. It is precisely for this reason that the coefficient of friction between the soil and a given foundation cannot be determined by a shearing failure test, because this will not give the coefficient of friction, but the limiting value of failure corresponding to a resultant load acting at a given inclination. This is the reason why the quotient of tangential and normal loads obtained from such tests was considerably lower than that resulting from a direct shear test (see e.g. Leonhardt, 1951). Yet there are two considerable differences between thrust on the surface and a direct shear test. This is partly because the soil specimen is laterally confined in a shear-box, and partly because the area of shear is stipulated and therefore the load acts over a prescribed area ; conversely, in the case of an actual foundation these conditions are not fulfilled, and there will be failure of the foundation.

The problem has been solved only for certain special cases (Sokolowski, 1954) ; in practice, the limiting value of an inclined eccentric load is calculated by semi-empirical methods for sand ($c = 0$) and for clay ($\Phi = 0$). (See, e.g. Brinch Hansen, 1957, and Skempton, 1951). The author does not provide a complete theoretical solution, his aim being only to promote precise theoretical investigations by making two contributions, namely :

1. the theoretical determination of the inclined punching strength of the material for $\gamma = 0$;

2. An investigation of the general stress-state of the semi-plastic semi-space in the case of a plane sliding surface.

1. Inclined punching strength in a semi-plastic medium

The author seeks to determine the limiting value of a uniformly distributed inclined load acting in a given direction, and affecting the truncated wedge shown in Fig. 1

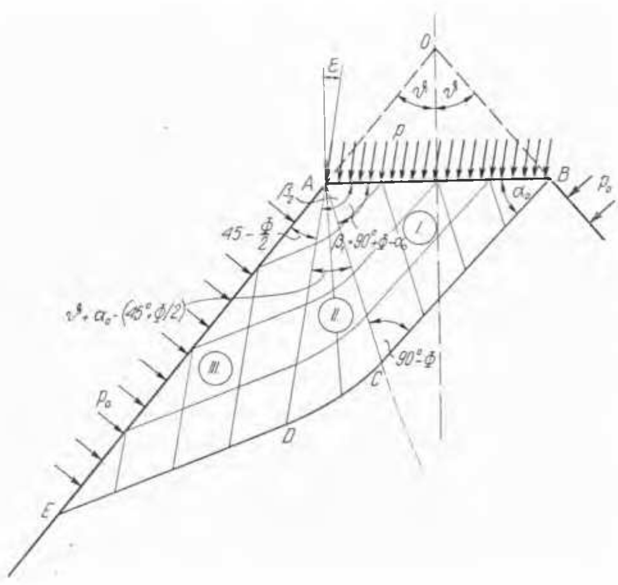

Fig. 1 Slip lines in a truncated wedge.
Lignes de glissement dans un prisme tronqué.

under a condition of failure $\tau = \sigma \operatorname{tg} \Phi + c$ in a weightless state. Stresses acting on the horizontal surface are : $p_n = p \cos \varepsilon$ and $p_t = p \sin \varepsilon$. Sliding surfaces are composed of three sections. In range I there are plane sliding surfaces; the angle of slope is a function of the direction of the force, i.e. of the angle ε. In range III the free surface AE is acted upon by uniformly distributed normal stresses; thus p_o is a principal stress, the sliding surfaces are planes, intersecting each other at an angle $90° - \Phi$ and the direction of the first principal stress is at an angle $45° - \Phi/2$. In range II, the sliding surfaces are — in compliance with Prandtl's well known solution — logarithmic spirals and rays starting out from point A, respectively. The network of sliding surfaces has no symmetry plane, and the left-hand side pattern indicating a lower resistance, can only develop. The stress-state is the same for all points in range I. It is sufficient, therefore, to investigate the point A, only.

The stress-components are (Fig. 2) :

$$\left.\begin{aligned} \sigma_z &= \tau \frac{1 + \sin \Phi \sin (2\alpha_o - \Phi)}{\sin \Phi \cos \Phi} - c \cot \Phi \\ \sigma_x &= \tau \frac{1 - \sin \Phi \sin (2\alpha_o - \Phi)}{\sin \Phi \cos \Phi} - c \cot \Phi \\ \tau_{xz} &= \tau \frac{\cos (2\alpha_o - \Phi)}{\cos \Phi} \end{aligned}\right\} \quad \ldots \quad (1)$$

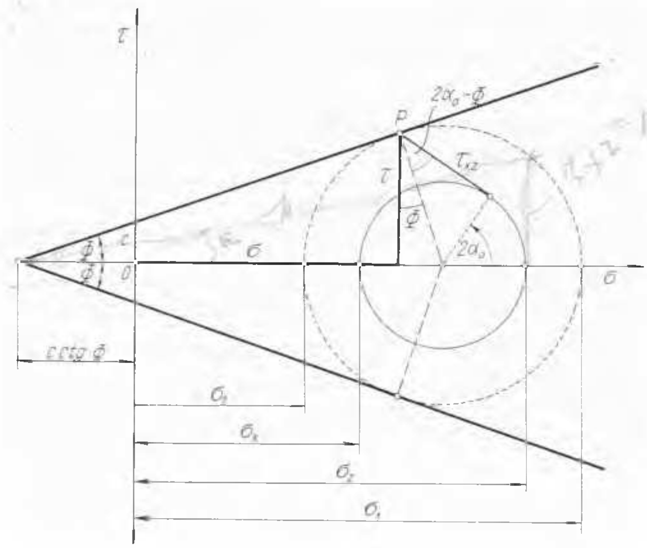

Fig. 2 Mohr's circle of stresses.
Cercle de Mohr des contraintes.

τ means the tangential stress acting on the sliding surface, α_o is its slope-angle. Stress-components acting at point A are known : $\tau_{xz} = p \sin \varepsilon$ and $\sigma_z = p \cos \varepsilon$. Replacing these into equation (1), τ can be expressed, while for the determination of the slope-angle α_o the following equation can be deduced :

$$\frac{p}{c} = \frac{\cos \Phi \cos (2\alpha_o - \Phi)}{\sin \varepsilon - \sin \Phi \cos (2\alpha_o - \Phi + \varepsilon)} \quad \ldots \quad (2)$$

In range I, the sliding surfaces are really planes, as in the case if $p =$ a constant, we have $\alpha_o =$ a constant.

In range II, Prandtl's stress-state is predominant, for $\gamma = 0$ we have from Kötter's equation

$$\frac{d\tau}{d\beta} - 2\tau \tan \Phi = 0 \quad (3)$$

$$\tau = K e^{-2\beta \tan \Phi}$$

Consequently, the shear stress on the boundary surface between ranges I and II is :

$$\tau = K e^{-2\beta_1 \tan \Phi}$$

The normal stress on AC is : $\sigma = (\tau - c) \cot \Phi$; for the vertical stress we have (Fig. 3) :

$$\sigma_z = p \cos \varepsilon = \sigma + \tau \tan \Phi + \frac{\tau}{\cos \Phi} \sin (2\alpha_o - \Phi)$$

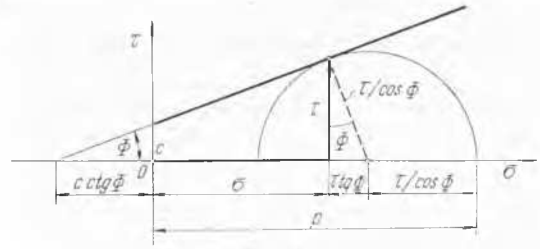

Fig. 3

Substituting values of σ and τ into the above equation, we obtain :

$$p \cos \varepsilon = K e^{-2\beta_1 \tan \Phi} \left[\frac{1 + \sin (2\alpha_o - \Phi) \sin \Phi}{\sin \Phi \cos \Phi} \right] - c \cot \Phi \quad (4)$$

In range II, p_0 is a main stress, therefore :

$$p_0 = \sigma + \tau \tan \Phi - \frac{\tau}{\cos \Phi}.$$

Shear-stress on plane AD is :

$$\tau = K e^{-2\beta_2 \tan \Phi}$$

i.e.

$$p_0 = K e^{-2\beta_2 \tan \Phi} \frac{1 - \sin \Phi}{\sin \Phi \cos \Phi} - c \cot \Phi \quad (5)$$

Dividing equation (4) by equation (5), we get : as (see Fig. 1).

$$\beta_1 = 90° + \Phi - \alpha_o$$

and

$$\beta_2 = 45° + \Phi/2 + \theta$$

the above relation obtains the following form :

$$\frac{p}{c} \cos \varepsilon = \frac{p_0}{c} e^{(2\theta + 2\alpha_o - 90° - \Phi) \tan \Phi} \frac{1 + \sin (2\alpha_o - \Phi) \sin \Phi}{1 - \sin \Phi} + \cot \Phi \left[e^{(2\theta + 2\alpha_o - 90° - \Phi) \tan \Phi} \frac{1 + \sin (2\alpha_o - \Phi) \sin \Phi}{1 - \sin \Phi} - 1 \right] \quad (6)$$

Equations (2) and (6) are already sufficient for determination of the two unknowns. In the special case $p_o = 0$ and $\theta = \pi/2$ (horizontal ground level, bearing capacity beneath a strip foundation centrally loaded by an inclined force), the angle α_o can be determined by the following equation :

$$\frac{1}{\sin \Phi \cos \varepsilon} \left[e^{(\pi/2 + 2\alpha_o - \Phi) \tan \Phi} \frac{1 + \sin (2\alpha_o - \Phi) \sin \Phi}{1 - \sin \Phi} - 1 \right] \quad (6a)$$

$$= \frac{\cos (2\alpha_o - \Phi)}{\sin \varepsilon - \sin \Phi \cos (2\alpha_o - \Phi - \varepsilon)}$$

If α_0 is known, the value of the punching strength can be determined from equation (2). The value of α_o can vary between the limits 0 and $45° + \Phi/2$; as for $\alpha_o < 0$ the sliding surface would leave the body; for $\alpha_o = 45° + \Phi/2$, however, the direction of p becomes vertical. Fig. 4 shows

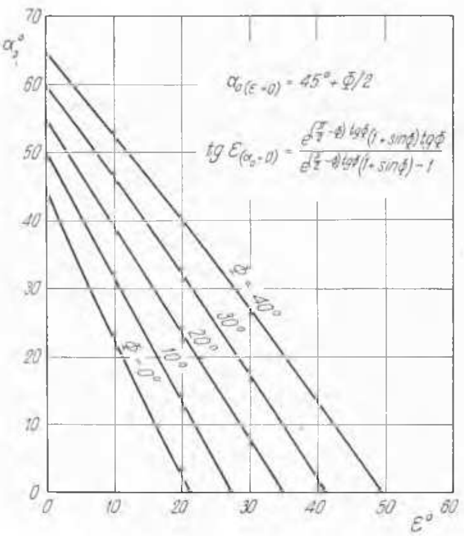

Fig. 4 Relationship between the direction of load and the angle of slip line in the first section.
Relation entre la direction des pressions et l'angle de la ligne de glissement dans la première zone.

the relation between α_o and ε, by using the parameter Φ. It is easy to prove that α_o will be equal to zero (the main sliding surface is horizontal within the range I) if

$$\tan \varepsilon = \tan \varepsilon_l = \tan \Phi \frac{e^{(\pi/2 - \Phi)\tan \Phi}\,(1 + \sin \Phi)}{e^{(\pi/2 - \Phi)\tan \Phi}\,(1 + \sin \Phi) - 1} \qquad \text{.... (7)}$$

This is the case corresponding to failure under shear. The values of ε_l as a function of Φ are shown in Fig. 5. Thus if the inclination of the resultant force were greater, equilibrium would not be possible. The values of the punching

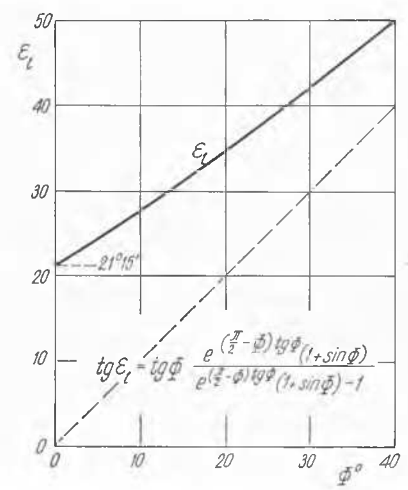

Fig. 5 Limiting inclination of pressures.
Valeur limite de l'inclinaison ε en fonction de Φ.

strength plotted as a function of Φ and ε to be found in Fig. 6. For $\varepsilon = \varepsilon_l$ the values of p/c are the followings :

$\Phi°$	0	10	20	30	40
p/c	2,76	3,23	3,65	3,95	4,52

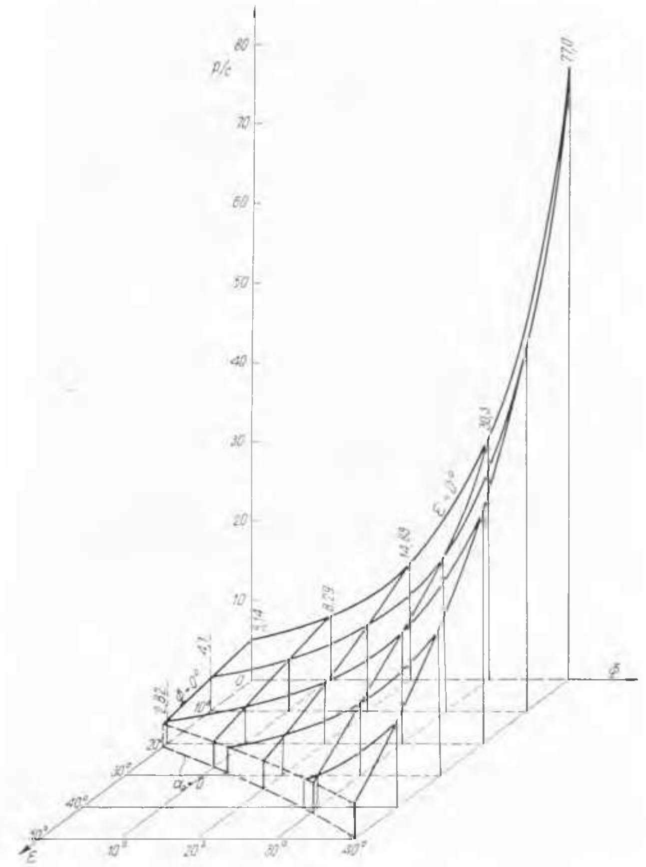

Fig. 6 Punching strength as a function of Φ and ε.
Résistance au poinçonnement en fonction de Φ et ε.

If the load acts vertically ($\varepsilon = 0$), the ultimate bearing capacity is obtained from Prandtl's formula; in this case $\alpha_o = 45° + \Phi/2$.

With regard to the applicability of the analyses for $\Phi = 0$, it is worth examining this case separately (JAKY, 1945). Carrying out in equations (5) and (6) the transition for the slope angle of the sliding surface, we obtain the relation-ship :

$$\cot \varepsilon = \frac{1 - \frac{\pi}{2} + 2\theta + \sin 2\alpha_o + 2\alpha_o}{\cos 2\alpha_o} \qquad (8)$$

while for the inclined strength the relationship

$$\frac{p}{c} = \frac{\cos 2\alpha_o}{\sin \varepsilon} \qquad (9)$$

is obtained. If $\theta = \pi/2$, i.e. if bearing capacity for inclined pressure is concerned, we have

$$\cot \varepsilon = \frac{1 + \frac{\pi}{2} + \sin 2\alpha_o + 2\alpha_o}{\cos 2\alpha_o} \qquad (8a)$$

The value of the limit-angle is $\varepsilon_l = 21°15'$, then $p = 2{,}76c$; the vertical component thereof is : $p_n = \left(1 + \frac{\pi}{2}\right)c$; as the limit value of the vertical load according to Prandtl is $p_n = 2\left(1 + \frac{\pi}{2}\right)c$, in the case $\varepsilon = \varepsilon_l$ the vertical component of inclined pressure is only the half of the limit-value of vertical pressure. The limit value of inclined pressure is shown in Fig. 7 as a function of tan ε; it can be seen that for $\Phi = 0$ the relationship may reasonably be assumed as linear, i.e.

$$p \cong 2c(2{,}6 - 3{,}0 \tan \varepsilon). \tag{10}$$

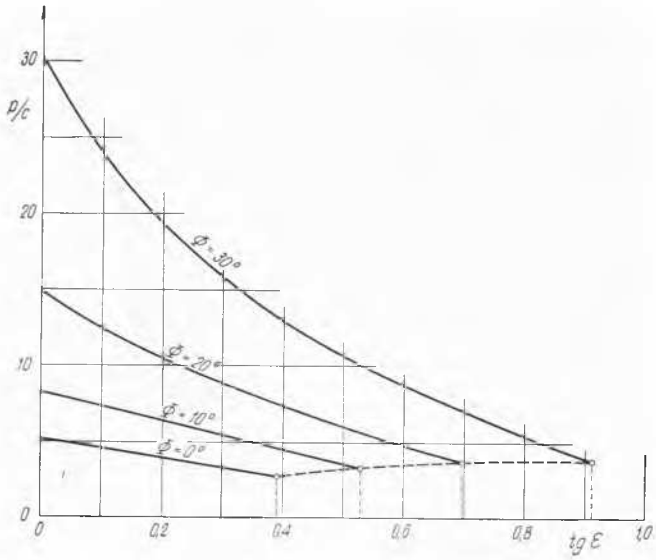

Fig. 7 Punching strength as function of tan ε.
Résistance au poinçonnement en fonction de tg ε.

It is interesting to compare this result with the formula of Skempton — somewhat transformed, published by Brinch-Hansen (1957). According to this, the bearing capacity of a strip-foundation ($L \to \infty$) on the surface ($D = 0$) is

$$p = 5c(1 - 1{,}3H/V);$$

Taking into consideration that $H/V = \tan \varepsilon$, we can write :

$$p = 2c(2{,}5 - 3{,}25 \tan \varepsilon); \quad .... \tag{11}$$

i.e. the two results are nearly identical. According to the numerical values plotted in Fig. 7, it can be observed that for $\Phi = 10°$ and $20°$ a similar regularity appears, i.e. when calculating the limit bearing capacity of a strip foundation, the effect of the inclination of the resultant force can be taken into account by a simple reduction. So, the competent strength valid for an inclined force is :

$$\frac{p}{c} = \alpha(1 - 1{,}15 \tan \varepsilon) \tag{12}$$

where α denotes the value of p/c valid for a vertical load depending on the angle of friction only. This relation can also be used even up to 20°.

2. Stress state of the infinite semi space in the case of plane sliding surfaces

The sliding surfaces of the semi-space being in a plastic limit state are plane if $c = 0$ and the ground level is unloaded or uniformly loaded, respectively. If $c \neq 0$ and $\gamma \neq 0$, then for an inclined ground level curved sliding surfaces will be obtained (see e.g. Jelinek, 1947). Even in that case, if plane sliding surfaces are assumed, this could be possible only for a not uniformly distributed load and for a certain given inclination of the load. This investigation shall be carried out in the following.

We start from equation (1) expressing co-ordinated stresses with the help of tangential stresses acting on the sliding surface. We suppose that α = const and the sliding surface as being plane. Then from (1) :

$$\left.\begin{aligned} \sigma_z &= A\tau - c \cot \Phi \\ \sigma_x &= B\tau - c \cot \Phi \\ \tau_{xz} &= C\tau \end{aligned}\right\} \tag{13}$$

Substituting the above into Cauchy's differential equations we obtain :

$$A\frac{\partial \tau}{\partial z} + C\frac{\partial \tau}{\partial x} = \gamma; \quad C\frac{\partial \tau}{\partial z} + B\frac{\partial \tau}{\partial x} = 0.$$

From these equations we have :

$$\left.\begin{aligned} \frac{\partial \tau}{\partial z} &= \gamma \frac{B}{AB - C^2} = \gamma \tan \Phi[1 - \sin \Phi \sin (2\alpha - \Phi)] \\ \frac{\partial \tau}{\partial x} &= \gamma \frac{C}{C^2 - AB} = -\gamma \tan \Phi \sin \Phi \cos (2\alpha - \Phi) \end{aligned}\right. \tag{14}$$

Thus, the value of τ can be obtained by integration :

$$\tau = z\gamma \tan \Phi\,[1 - \sin \Phi \sin (2\alpha - \Phi)] - x\gamma \tan \Phi \sin \Phi \cos (2\alpha - \Phi) + K. \quad .... \tag{15}$$

We wish to apply this result for the semi-space with a horizontal ground level. Boundary conditions are (Fig. 8) : in the case of $z = 0$ and $x = 0$:

$$\begin{aligned} \tau &= \tau_0 = K \\ \sigma_z &= p_o \cos \varepsilon \\ \tau_{xz} &= p_o \sin \varepsilon; \end{aligned}$$

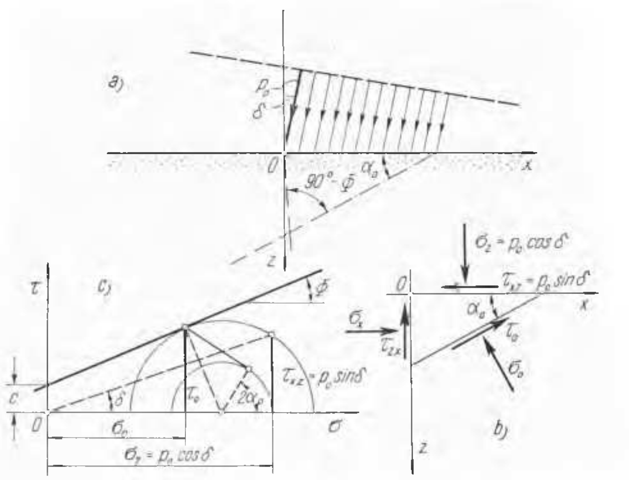

Fig. 8 Boundary conditions in the semi-space.
Conditions aux limites dans le semi-espace.

Replacing these into equations (1), we obtain for the slope of the sliding surface the same formula as in the case of a weightless medium loaded by an inclined force :

$$\frac{p_o}{c} = \frac{\cos \Phi \cos (2\alpha_o - \Phi)}{\sin \varepsilon - \sin \Phi \cos (2\alpha_o - \Phi + \varepsilon)} \tag{2}$$

If $c = 0$, then :

$$\cos (2\alpha_o - \Phi + \varepsilon) = \frac{\sin \varepsilon}{\sin \Phi} \tag{2a}$$

The constant of integration can be obtained from the known stress of the zero-point :

$$\tau_0 = \frac{p_o \sin \varepsilon \cos \Phi}{\cos (2\alpha_o - \Phi)} = K. \quad (16)$$

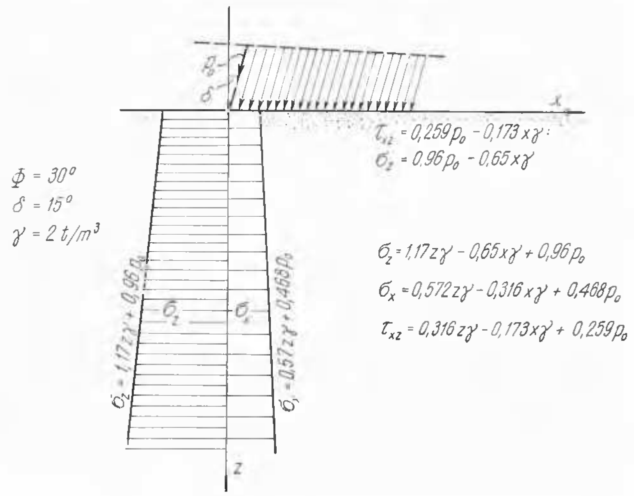

Fig. 9 Numerical example.
Exemple numérique.

and on the basis of equation (2) the tangential stress acting on the sliding surface can be determined at each point of the semi-space. On the horizontal ground level ($z = 0$) :

$$\tau = p_0 \frac{\sin \Phi \cos \Phi}{\cos (2\alpha_o - \Phi)} - x\gamma \tan \Phi \sin \Phi \cos (2\alpha_0 - \Phi) \quad \ldots . \quad (17)$$

Thus, τ and herewith the coordinated stresses and similarly the resulting stress p are linear functions of x. Fig. 9 shows a numerical example : it gives the stress distribution in the semi space in the case of a given p_o and ε, respectively Φ; $c = 0$. Consequently, at a given slope, the distribution of stresses acting on the ground level is determined, the degree of the linear decrease on the ground level is given. Assuming plane sliding surfaces, in the semi-space the plastic state — corresponding to Rankine's state — can only come into effect at these values.

References

[1] Brinch Hansen, J. (1957). General Report, *4th Int. Conf. Soil Mech. Found. Eng.*, Div. 3a. London.

[2] Skempton, A. W. (1951). The Bearing Capacity of Clays. *Proc. Build. Research Congress.* London.

[3] Leonhardt (1951). Die Rheinbrücke bei Köln-Rodenkirchen. *Die Bautechnik.*

[4] Sokolowski, V. V. (1954). Statics of earth media. Moscow.

[5] Jelinek, R. (1947). Der Coulomb'sche Halbraum, *Bauwissenschaft*, I. Ig. H. 4.