The Influence of Foundation Depth on Stress Distribution

Influence de la profondeur de la fondation sur la répartition des contraintes

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Summary

Foundation settlements are usually estimated by the use of the classical Boussinesq solution for a point load acting on the surface of a semi-infinite elastic solid. This means that the soil above the level of the foundation is assumed to be taking no part in the distribution of stress due to the load on the foundation.

In calculating stresses within a mass of soil when the foundations are relatively deep, it may be advisable to use solutions derived for the case of forces applied within a semi-infinite elastic medium. The author deals with the general solutions for the vertical stresses in a semi-infinite elastic plate due to a uniform limited length line load acting in the interior of the plate and the vertical stresses in a semi-infinite solid due to the action of a uniformly distributed load on a rectangular area and a strip load acting in the interior of the solid.

The influence of the depth of a foundation on stress distribution is studied by means of photoelasticity.

Sommaire

La détermination du tassement des constructions demande la connaissance des contraintes provoquées dans le sol. Généralement on calcule les contraintes par la théorie de Boussinesq, pour un solide élastique semi-infini soumis à sa surface libre à une force ponctuelle verticale. L'application de cette théorie est valable pour les fondations en surface. Elle l'est moins pour les fondations profondes parce qu'elle néglige l'influence de la profondeur sur la répartition des contraintes. Pour celles-ci, il est indiqué de se référer au problème du corps élastique semi-infini soumis à une force verticale à une profondeur au-dessous de sa surface.

Dans ce rapport est traité la solution générale des contraintes verticales provoquées par une charge uniformément répartie sur une bande infinie ou sur un rectangle appliqué à une certaine profondeur dans un solide semi-infini.

L'influence de la profondeur de la fondation sur la répartition des contraintes est étudiée par la méthode photoélasticimétrique.

Introduction

The state of stress in a semi-infinite plate loaded in the interior by a force perpendicular to its straight boundary was solved by Melan [1]. The three-dimensional case of a force acting in the interior of a semi-infinite solid was solved by Mindlin [2].

Equations for stresses, due to a load on a flexible area can be derived by integration of equations for concentrated loads. The vertical stress is of greater significance in practical problems and the author has therefore concentrated his attention upon it.

Vertical stress due to a uniform continuous load plane stress

The vertical components of stress σ_2 due to a force acting vertically in the interior of a semi-infinite plate with horizontal boundary were given by equations derived by Melan. Vertical stress due to a flexible uniform continuous load, as shown on Fig. 1, can be obtained by integration of equation,

$$\sigma_2 = \frac{p}{\pi}\int_{x_1}^{x_2}\left[\frac{1+\mu}{2}\left(\frac{(z-h)^3}{r_1^4}+\frac{(z+h)\,[2hz+(z+h)^2]}{r_2^4}-\frac{8hz(z+h)x^2}{r_2^6}\right)+\frac{1-\mu}{4}\left(\frac{z-h}{r_1^2}+\frac{3z+h}{r_2^2}-\frac{4zx^2}{r_2^4}\right)\right]dx,$$

where

$$r_1^2 = x^2 + (z-h)^2$$
$$r_2^2 = x^2 + (z+h)^2.$$

Solving this integral, a general equation is obtained for the vertical stress due to a uniform load acting below the boundary (vide Ref. [3]). From this equation it is possible to determine stresses produced along the vertical axis of

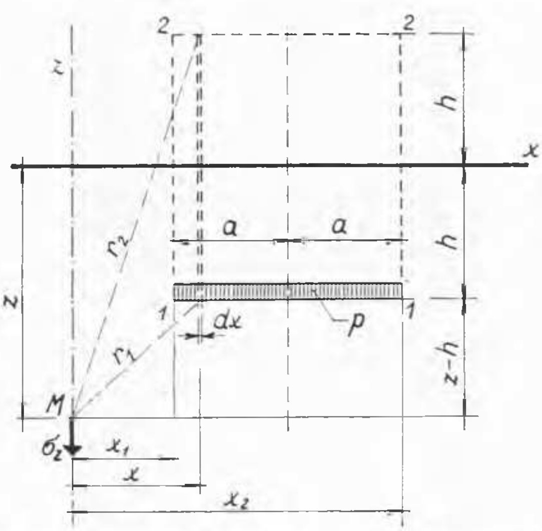

Fig. 1 Uniform load acting in the interior of a semi-infinite plate.
Charge uniformément répartie à l'intérieur d'une plaque semi-infinie.

the load p, of width $2a$ acting at a depth h below the boundary.

$$\sigma_2 = \frac{p}{\pi}\left[\operatorname{arc\,tg}\frac{a}{z-h}+\operatorname{arc\,tg}\frac{a}{z+h}+\frac{(1+\mu)(z-h)a}{2[a^2+(z-h)^2]}+\frac{[2(z+h)+(z-h)(1-\mu)]a}{2[a^2+(z+h)^2]}+\frac{(1+\mu)2hz(z+h)a}{[a^2+(z+h)^2]^2}\right] \quad \ldots (1)$$

Vertical stress due to a uniform rectangular area load

The vertical components of stress due to a force acting vertically in the interior of a semi-infinite solid with horizontal boundary were given by an equation derived by Mindlin. Vertical stress due to a flexible rectangular area load with dimensions $2a \times 2b$, as shown in Fig. 2, produced at a point M, is given by the double integral,

$$\sigma_2 = -\frac{q}{8\pi(1-\mu)}\int_{x_1}^{x_2} dx \int_{y_1}^{y_2}\left[-\frac{(1-2\mu)(z-h)}{R_1^3} + \frac{(1-2\mu)(z-h)}{R_2^3} - \frac{3(z-h)^3}{R_1^5} - \frac{3(3-4\mu)z(z+h)^2 - 3h(z+h)(5z-h)}{R_2^5} - \frac{30hz(z+h)^3}{R_2^7}\right]dy,$$

where

$$R_1 = \sqrt{x^2 + y^2 + (z-h)^2}$$
$$R_2 = \sqrt{x^2 + y^2 + (z+h)^2}.$$

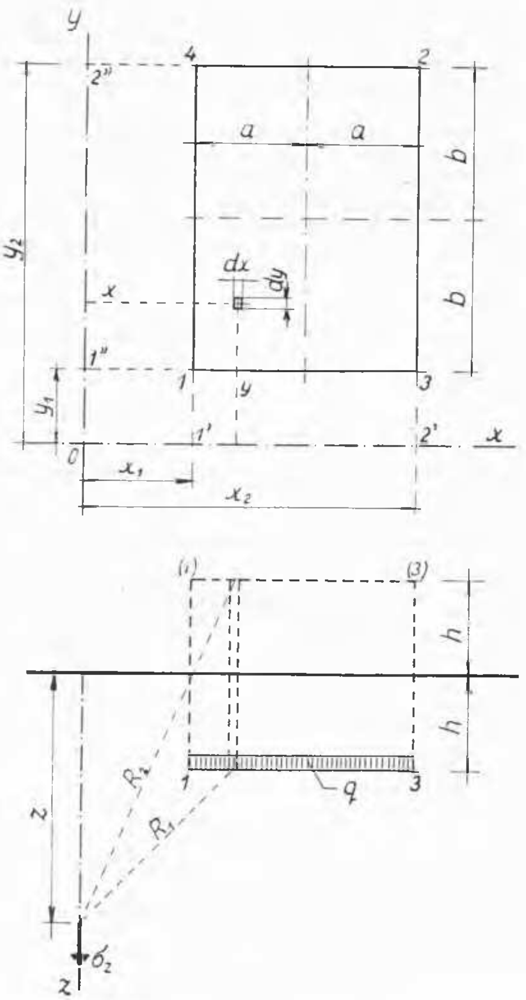

Fig. 2 Uniformly loaded rectangular area acting in the interior of a semi-infinite solid.

Charge rectangulaire uniformément répartie à l'intérieur du corps semi-infini.

Solving this integral and using the notation shown in Fig. 3, we get the general equation in the form

$$\sigma_2 = -\frac{q}{8\pi(1-\mu)}\Bigg[-2(1-\mu)\left(\operatorname{arc\,tg}\frac{x_2y_2}{vs_{2v}} - \operatorname{arc\,tg}\frac{x_2y_1}{vs_{3v}} - \operatorname{arc\,tg}\frac{x_1y_2}{vs_{4v}} + \operatorname{arc\,tg}\frac{x_1y_1}{vs_{1v}}\right) - 2(1-\mu)$$
$$\left(\operatorname{arc\,tg}\frac{x_2y_2}{us_{2u}} - \operatorname{arc\,tg}\frac{x_2y_1}{us_{3u}} - \operatorname{arc\,tg}\frac{x_1y_2}{us_{4u}} + \operatorname{arc\,tg}\frac{x_1y_1}{us_{1u}}\right)$$
$$-\frac{x_2vs_{2v}}{y_2s_{x2v}^2} + \frac{x_2vs_{3v}}{y_1s_{x2v}^2} + \frac{x_1vs_{4v}}{y_2s_{x1v}^2} - \frac{x_1vs_{1v}}{y_1s_{x1v}^2} + \frac{x_2v^3}{y_2s_{y2v}^2s_{2v}}$$
$$-\frac{x_2v^3}{y_1s_{y1v}^2s_{3v}} - \frac{x_1v^3}{y_2s_{y2v}^2s_{4v}} + \frac{x_1v^3}{y_1s_{y1v}^2s_{1v}} - \frac{(3-4\mu)zu - h(5z-h)}{u}$$
$$\cdot\left(\frac{x_2s_{2u}}{y_2s_{x2u}^2} - \frac{x_2s_{3u}}{y_1s_{x2u}^2} - \frac{x_1s_{4u}}{y_2s_{x1u}^2} + \frac{x_1s_{1u}}{y_1s_{x1u}^2}\right) + [(3-4\mu)zu^2 - hu(5z-h)]$$
$$\cdot\left(\frac{x_2}{y_2s_{y2u}^2s_{2u}} - \frac{x_2}{y_1s_{y1u}^2s_{3u}} - \frac{x_1}{y_2s_{y2u}^2s_{4u}} + \frac{x_1}{y_1s_{y1u}s_{1u}}\right) - 4hzu$$
$$\left(\frac{x_2s_{2u}^3}{y_2^3s_{x2u}^4} - \frac{x_2s_{3u}^3}{y_1^3s_{x2u}^4} - \frac{x_1s_{4u}^3}{y_2^3s_{x1u}^4} + \frac{x_1s_{1u}^3}{y_1^3s_{x1u}^4}\right) - \frac{6hz}{u}\left(\frac{x_2(y_2^2-u^2)s_{2u}}{y_2^3s_{x2u}^2}\right.$$
$$\left.-\frac{x_2(y_1^2-u^2)s_{3u}}{y_1^3s_{x2u}^2} - \frac{x_1(y_2^2-u^2)s_{4u}}{y_2^3s_{x1u}^2} + \frac{x_1(y_1^2-u^2)s_{1u}}{y_1^3s_{x1u}^2}\right) + 2hzu^3$$
$$\cdot\left(\frac{x_2(2y_2^2+u^2)}{y_2^3s_{y2u}^4s_{2u}} - \frac{x_2^3}{y_2s_{y2u}^4s_{2u}} - \frac{x_2(2y_1^2+u^2)}{y_1^3s_{y1u}^4s_{3u}} + \frac{x_2^3}{y_1s_{y1u}^4s_{3u}^3}\right.$$
$$\left.-\frac{x_1(2y_2^2+u^2)}{y_2^3s_{y2u}^4s_{4u}} + \frac{x_1^3}{y_2^3s_{y2u}^4s_{4u}^3} + \frac{x_1(2y_1^2+u^2)}{y_1^3s_{y1u}^4s_{1u}} - \frac{x_1^3}{y_1s_{y1u}^4s_{1u}^3}\right). \qquad (2)$$

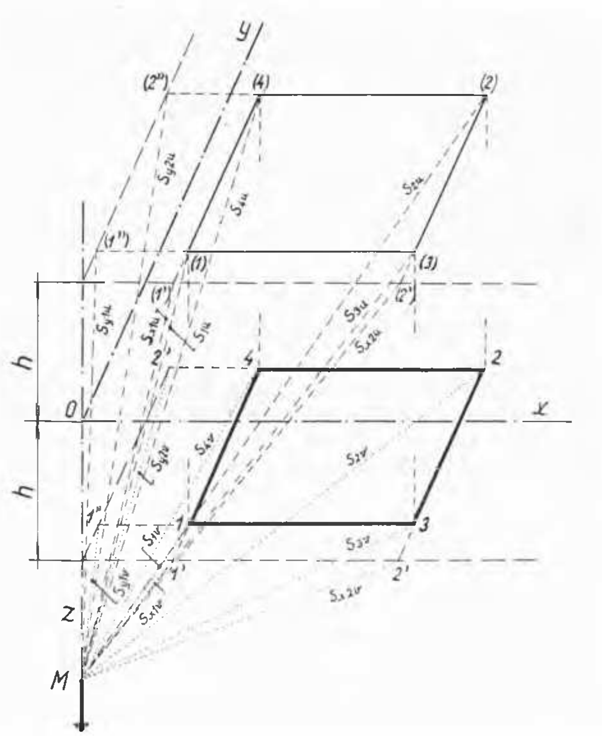

Fig. 3 Designation used fer determination of vertical stress due to a uniformly loaded rectangular area acting in the interior of a semi-infinite solid.

Notation pour la détermination des contraintes verticales produites par une charge rectangulaire uniformément répartie à l'intérieur du corps semi-infini.

The verification of the solution is shown in Ref. [3].

From the general equation special cases can be obtained. Vertical stress at points lying along the vertical through the centre of the rectangular area is given by equation

$$\sigma_2 = \frac{q}{\pi(1-\mu)}\Bigg[(1-\mu)\left(\operatorname{arc\,tg}\frac{ab}{(z-h)R_1} + \operatorname{arc\,tg}\frac{ab}{(z+h)R_2}\right) + \frac{(z-h)aR_1}{2br_1^2} - \frac{a(z-h)^3}{2br_3^2R_1} + \frac{[(3-4\mu)z(z+h) - h(5z-h)]aR_2}{2(z+h)br_2^2} - \frac{[(3-4\mu)z(z+h)^2 - h(z+h)(5z-h)]a}{2br_4^2R_2} + \frac{2hz(z+h)aR_2^3}{b^3r_2^4} + \frac{3hzaR_2r_5^2}{(z+h)b^3r_2^2} - \frac{hz(z+h)^3a}{br_4^4R_2}\left(\frac{2b^2-(z+h)^2}{b^2} - \frac{a^2}{R_2}\right)\Bigg]. \tag{3}$$

where

$R_1^2 = a^2 + b^2 + (z-h)^2$ $\qquad$ $R_2^2 = a^2 + b^2 + (z+h)^2$

$r_1^2 = a^2 + (z-h)^2$ $\qquad$ $r_2^2 = a^2 + (z+h)^2$

$r_3^2 = b^2 + (z-h)^2$ $\qquad$ $r_4^2 = b^2 + (z+h)^2$

$r_5^2 = b^2 - (z+h)^2.$

Vertical stress at points lying along the vertical through the centre of a uniformly loaded strip can be derived after application of the necessary limits as shown below.

$$\sigma_2 = \frac{2}{\pi}\left[\operatorname{arc\,tg}\frac{a}{z-h} + \operatorname{arc\,tg}\frac{a}{z+h} + \frac{a(z-h)}{2[a^2+(z-h)^2](1-\mu)} + \frac{a[(3-4\mu)z+h]}{2[a^2+(z+h)^2](1-\mu)} + \frac{2hz(z+h)a}{[a^2+(z+h)^2]^2(1-\mu)}\right]. \quad \ldots \tag{4}$$

This equation may be compared with equation (1) obtained for the corresponding two-dimensional case. It may be noted that equation (4) is valid for the case of plane strain, while equation (1) is derived for the case of plane stress. By taking into account the relationship between the elastic constants for plane stress and plane strain, equation (4) can be transformed in equation (1).

The distribution of vertical stress along the axis of a uniformly loaded rectangular area acting below the surface of the semi-infinite solid given by equation (3) is shown in Fig. 4 for a numerical example. From this figure, it can be seen that the nature of stress is tensile above the level of the load and compressive below it.

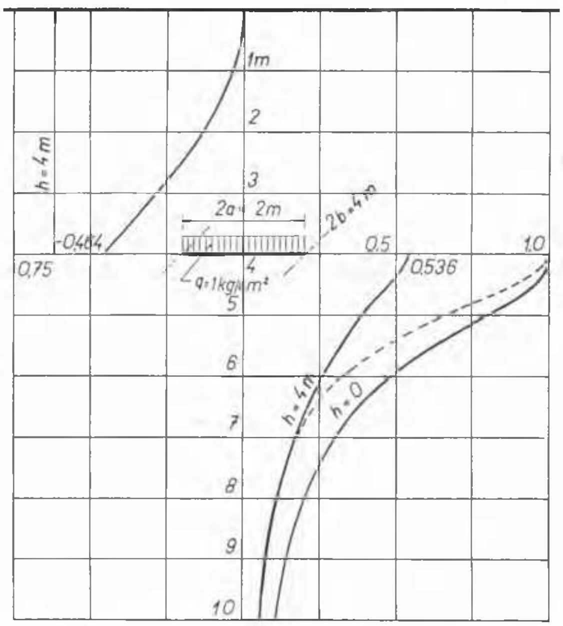

Fig. 4 Vertical stress along the axis of uniformly loaded rectangular area with dimensions 2 × 4 m acting at a depth of 4 m below the surface.
Répartition des contraintes verticales le long de l'axe de la charge rectangulaire uniformément répartie, de dimensions 2 × 4 m, appliquée à la profondeur de 4 m sous la surface.

Fig. 5 Isochromatic lines : (a) $h = 0$, (b) $h = 2b$, (c) $h = 4b$.
Les lignes isochromatiques : (a) $h = 0$, (b) $h = 2b$, (c) $h = 4b$.

Experimental study of problem

The above equations for vertical stresses due to a continuous load acting below the surface are valid under certain conditions, one of which being that continuity exists in the parts of the solid above the level of the load transmission. When considering foundations, the above condition is not strictly complied with. The extreme case of discontinuity occurs when the load is considered as acting at the bottom of the cut. It is difficult to solve this case analytically because the boundary conditions cannot be precisely defined. This case is therefore investigated by photoelastic experiments.

A rectangular plate from "elastoplex" 30 × 30 cm and 1 cm thick, provided with a rectangular notch 1 cm in width is loaded by a uniformly distributed load acting at the bottom of the notch. The distribution of vertical stress along the vertical through the centre of the notch is measured for varying depths of the notch, $h = 0$, $2b$, $4b$, where h is depth of the notch and $2b$ is the width. Typical isochromats are shown in Fig. 5.

The results of one set of experiments are given in Fig. 6. From this figure it is evident that as the depth of the notch increases the concentration of the vertical stress diminishes.

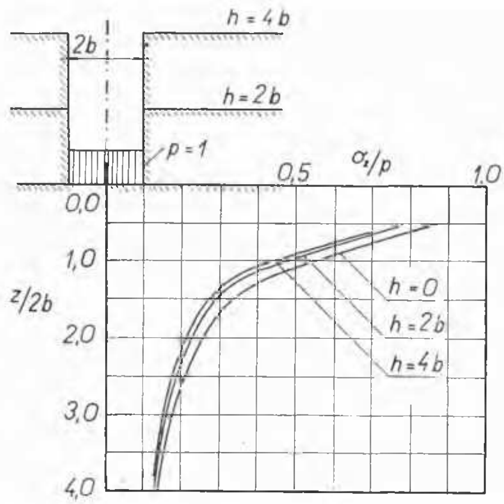

Fig. 6 Distribution of vertical stress along the axis of the uniform load as a function of depth determined from photoelastic tests.

Répartition des contraintes verticales le long de l'axe de la charge en fonction de la profondeur, déterminées par la méthode photoélasticimétrique.

Conclusions

From the results of theoretical and experimental investigations and from a study of practical cases, the following conclusions may be drawn :.

The depth of the footing below the ground level influences the concentration of vertical stresses in an elastic mass. This influence, which is a reduction, can be apparent only when the material above the level of load transmission can withstand tensile stresses. In the case of cohesionless soil the reduction of concentration is not as great as it is in cohesive soils.

The distribution of vertical stress below the foundation also depends upon the interaction between the sides of the footing and the surrounding soil. The resulting distribution will be similar to that given by the dotted line in Fig. 4.

References

[1] Melan, E. (1932). Der Spannungszustand der durch eine Einzelkraf im Inneren beanspruchten Halbscheibe, *Zamm* 12 (6), pp. 343-346.

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[3] Škopek, J. (1960). Uniformly Distributed Area Load in the Interior of a Semi-Infinite Solid, Acta Technica (Czechoslowak Academy of Sciences), 5.