Stresses and Displacements in a Limited Layer of Uniform Thickness, Resting on a Rigid Base, and Subjected to an Uniformly Distributed Flexible Load of Rectangular Shape

Contraintes et déformations dans une couche d'épaisseur uniforme et limitée, reposant sur une base rigide soumise à une charge rectangulaire, uniforme et souple

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Summary

The author has developed formulae for calculating stresses and displacements in an elastic layer of uniform thickness, resting on a rigid foundation. It is assumed that the contact between the elastic layer and the rigid base is smooth. The case of loading an elastic layer with a flexible load of rectangular shape is treated in detail.

In the second part the numerical evaluations are given for various values of the fundamental parameters $\frac{h}{c}$ and $\frac{c}{d}$ (h being the tickness of the compressible layer, c and d being the half sides of the loaded area) and for a Poisson's ratio $\nu = 0{\cdot}5$.

Due to the limited thickness of the elastic layer, the normal stresses along the compressible layer are concentrated near the loaded area and the displacements w_z are greater than the values obtained for an isotropic elastic half-space. These differences become significant when the ratio $\frac{h}{c}$ is less than 2·5.

Sommaire

Dans la première partie du travail l'auteur déduit des formules avec lesquelles on peut déterminer les contraintes et déformations dans une couche élastique limitée, d'épaisseur uniforme, qui est placée sur une base rigide. Entre les deux surfaces de la couche élastique et de la base rigide on suppose le contact glissant. Le cas dans lequel la surface supérieure de la couche élastique est chargée d'une charge mobile de forme rectangulaire est étudié en détail.

Dans la deuxième partie l'auteur a calculé numériquement quelques exemples pour les différentes valeurs des paramètres fondamentaux $\frac{h}{c}$ et $\frac{c}{d}$ (h signifie l'épaisseur de la couche compressible, c et d sont les demi-côtés de la surface chargée) avec le coefficient de Poisson égal à 0,5.

En raison de l'épaisseur limitée de la couche élastique les contraintes normales σ_z se présentent concentrées dans le domaine de la surface chargée le long de la couche compressible et dans ce domaine les déplacements w_z sont plus grands que ceux que nous donne le calcul pour le solide semi-infini élastique et isotrope. Les différences sont plus sensibles pour le quotient $\frac{h}{c} < 2{,}5$.

Introduction

The problem of the magnitude and distribution of stress and deflexions in an elastic layer of uniform thickness has been closely studied in recent years. The results of theoretical investigations have been applied in practice to the design and construction of slab foundations, and to the estimation of stress and settlement in multi-layer systems of soil.

In the construction of foundations we are often dealing with soils, which have been deposited in layers. During the last twenty or thirty years the well-known formulae of Boussinesq which apply to a homogeneous, isotropic, semi-infinite, elastic solid were normally used. Woinowsky-Krieger (1933), Marguerre (1933), Biot (1935), Burminster (1945), and Fox (1948) contributed solutions for layered soil systems. It was Burminster, however, who in 1956 solved the complete problem of stresses and displacements in "a two — layer rigid base soil system" and illustrated the practical use and importance of this problem in soil mechanics and foundation engineering.

The author has developed a method for calculating the stresses and displacements in an elastic limited layer of uniform thickness, resting on a rigid base. It is assumed that the layer is loaded by a flexible load of rectangular shape. The contact between the elastic layer and the rigid basis is assumed to be smooth. In numerical examples the results are given for various values of fundamental parameters $\frac{h}{c}$ and $\frac{c}{d}$ (h being the thickness of the compressible layer, c and d being the half sides of the loaded area) and for Poisson's ratio $\nu = 0{\cdot}5$. It is thereby assumed that the ratio between the horizontal dimensions of the layer and of the loaded area is large enough to permit the application of resulting values to a compressible layer of infinite horizontal dimensions.

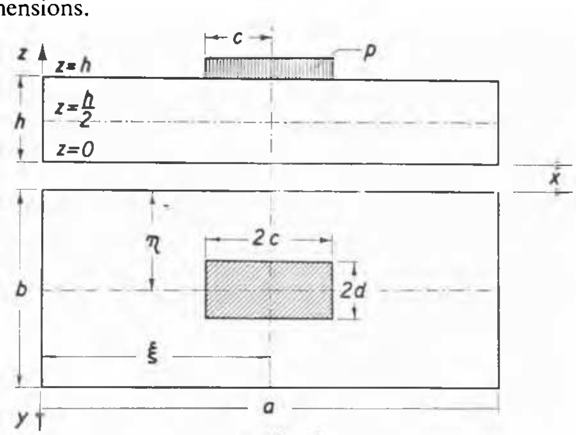

Fig. 1

Basic equations

An elastic layer is assumed with length a, width b and height h, resting on a rigid base and loaded on the surface ($z = h$) with a flexible load of rectangular shape (Fig. 1). Neglecting the weight of the layer itself the differential equations of equilibrium are :

$$(1 - 2\nu)\Delta u + \frac{\partial \Theta}{\partial x} = 0$$

$$(1 - 2\nu)\Delta v + \frac{\partial \Theta}{\partial y} = 0 \qquad (1)$$

$$(1 - 2\nu)\Delta w + \frac{\partial \Theta}{\partial z} = 0.$$

ν = Poisson's ratio,
Θ = Volume expansion.

For displacements u, v and w, acting in the coordinate directions x, y, z, we take, as it has been done by Woinowsky-Krieger in his solutions for determination of stresses and displacements in thick plates, relations in the form of double trigonometrical series :

$$u = \sum_m \sum_n U \sin \alpha x \cos \beta y,$$

$$v = \sum_m \sum_n V \cos \alpha x \sin \beta y, \qquad (2)$$

$$w = \sum_m \sum_n W \cos \alpha x \cos \beta y.$$

$\alpha = \frac{m\pi}{a}$, $\beta = \frac{n\pi}{b}$, U, V, W are Fourier's coefficients, which are variables of coordinate z.

The solutions, obtained by substituting expressions (2) in equations (1) are biharmonic functions :

$$U = \frac{1}{2G}[(C_1\alpha + C_5\beta) \sinh \gamma z + (C_2\alpha + C_6\beta) \cosh \gamma z + C_3\alpha z \cosh \gamma z + C_4\alpha z \sinh \gamma z], \qquad (3)$$

$$V = \frac{1}{2G}[(C_1\beta + C_5\alpha) \sinh \gamma z + (C_2\beta + C_6\alpha) \cosh \gamma z + C_3\beta z \cosh \gamma z + C_4\beta z \sinh \gamma z], \qquad (4)$$

$$W = \frac{1}{2G}\{[(3 - 4\nu)C_4 - C_2\gamma] \sinh \gamma z + [(3 - 4\nu)C_3 - C_1\gamma] \cosh \gamma z - C_4\gamma z \cosh \gamma z - C_3\gamma z \sinh \gamma z\}, \quad \ldots . \quad (5)$$

where $\gamma^2 = \alpha^2 + \beta^2$.

$C_1 - C_6$ are integration constants.

Boundary conditions

The boundary conditions for this case may be expressed as follows :

1. When $z = 0$; $w = 0$, $\tau_{zx} = 0$, $\tau_{zy} = 0$ .... (6)
2. When $z = h$; $\sigma_z = -p = -\sum\sum_{mn} a_{mn} \cos \alpha x \cos \beta y$, $\tau_{zx} = 0$, $\tau_{zy} = 0$ .... (7)

The following expressions for constants are obtained :

$$C_1 = 0,$$

$$C_2 = 2a_{mn} \frac{\gamma h \cosh \gamma h - (1 - 2\nu) \sinh \gamma h}{\gamma^2(\sinh 2\gamma h + 2\gamma h)},$$

$$C_3 = 0, \qquad (8)$$

$$C_4 = -2a_{mn} \frac{\sinh \gamma h}{\gamma(\sinh 2\gamma h + 2\gamma h)},$$

$$C_5 = 0,$$
$$C_6 = 0.$$

In the next step the solutions for displacements u, v, w and stresses σ_x, σ_y, σ_z, τ_{xz}, τ_{yz}, τ_{xz} were developed.

Numerical examples

The numerical evaluations of stresses σ_z (— is compressive stress) and displacements w_z in the z coordinate direction were carried out for Poisson's ratio $\nu = 0{\cdot}5$ and for a number of values of the basic parameters. For loads uniformly distributed over rectangular areas applied on the surface of an elastic layer and for various ratios of the width of the area to the thickness of elastic layer and area side ratio, c/d.

These numerical examples were carried out with the intention of indicating how the stresses and displacements vary with the thickness of the elastic layer compared with the width of the loaded area and with the area side ratio, on the one hand, and on the other hand, how much these values vary from Boussinesq's vertical stress distribution in a homogeneous deposit.

The stresses σ_z are calculated for two horizontal planes $\left(z = 0 \text{ and } z = \frac{h}{2}\right)$.

$\frac{c}{d}$	$\frac{h}{d}$	$\frac{a}{h}$	Coordinate		Stresses σ_z in the plane	
			x	y	$z = 0$	$z = \frac{h}{2}$
1	2	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·5616 p	— 0·7835 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·3835 p	— 0·4571 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·1194 p	— 0·0491 p
1	5	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·1217 p	— 0·2498 p
			$\frac{9a}{20}$	$\frac{a}{2}$	— 0·1105 p	— 0·1938 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·0694 p	— 0·0782 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0150 p	— 0·0578 p
1	10	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·0325 p	— 0·0744 p
			$\frac{19a}{40}$	$\frac{a}{2}$	— 0·0317 p	— 0·0695 p
			$\frac{19a}{40}$	$\frac{19a}{40}$	— 0·0282 p	— 0·0616 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·0176 p	— 0·0182 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0040 p	— 0·0031 p

$\frac{c}{d}$	$\frac{h}{d}$	$\frac{a}{h}$	Coordinate x	Coordinate y	Stresses σ_z in the plane $z = 0$	Stresses σ_z in the plane $z = \frac{h}{2}$
2	2	8	$\frac{a}{2}$	$\frac{a}{2}$	— 0·7410 p	— 0·8808 p
			$\frac{7a}{16}$	$\frac{a}{2}$	— 0·6254 p	— 0·7529 p
			$\frac{a}{2}$	$\frac{7a}{16}$	— 0·5110 p	— 0·5359 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·3842 p	— 0·4412 p
			$\frac{3a}{8}$	$\frac{7a}{16}$	— 0·2636 p	— 0·2678 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·1509 p	— 0·0701 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0056 p	— 0·0002 p
2	4	8	$\frac{a}{2}$	$\frac{a}{2}$	— 0·3219 p	— 0·5097 p
			$\frac{a}{2}$	$\frac{7a}{16}$	— 0·2804 p	— 0·3881 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·2197 p	— 0·2923 p
			$\frac{3a}{8}$	$\frac{7a}{16}$	— 0·1921 p	— 0·2256 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·1873 p	— 0·1840 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0665 p	— 0·0330 p
			$\frac{a}{2}$	$\frac{a}{4}$	— 0·0472 p	— 0·0263 p
2	10	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·0622 p	— 0·1333 p
			$\frac{a}{2}$	$\frac{19a}{40}$	— 0·0606 p	— 0·1242 p
			$\frac{9a}{20}$	$\frac{a}{2}$	— 0·0564 p	— 0·1050 p
			$\frac{a}{2}$	$\frac{9a}{20}$	— 0·0561 p	— 0·1016 p
			$\frac{9a}{20}$	$\frac{19a}{40}$	— 0·0552 p	— 0·0988 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·0354 p	— 0·0402 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·0333 p	— 0·0279 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0077 p	— 0·0029 p
			$\frac{a}{2}$	$\frac{a}{4}$	— 0·0073 p	— 0·0026 p

$\frac{c}{d}$	$\frac{h}{d}$	$\frac{a}{h}$	Coordinate x	Coordinate y	Stresses σ_z in the plane $z = 0$	Stresses σ_z in the plane $z = \frac{h}{2}$
5	5	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·3459 p	— 0·4847 p
			$\frac{a}{2}$	$\frac{19a}{40}$	— 0·3199 p	— 0·3930 p
			$\frac{7a}{16}$	$\frac{a}{2}$	— 0·2975 p	— 0·3826 p
			$\frac{a}{2}$	$\frac{7a}{16}$	— 0·2175 p	— 0·2247 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·1675 p	— 0·2032 p
			$\frac{3a}{8}$	$\frac{19a}{40}$	— 0·1541 p	— 0·1826 p
			$\frac{3a}{8}$	$\frac{7a}{16}$	— 0·1139 p	— 0·0966 p
			$\frac{5a}{16}$	$\frac{a}{2}$	— 0·0457 p	— 0·0282 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·0443 p	— 0·0014 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0094 p	— 0·0082 p
			$\frac{a}{2}$	$\frac{a}{4}$	— 0·0031 p	— 0·0010 p
5	10	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·1346 p	— 0·2294 p
			$\frac{a}{2}$	$\frac{19a}{40}$	— 0·1316 p	— 0·2094 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·0894 p	— 0·1239 p
			$\frac{3a}{8}$	$\frac{19a}{40}$	— 0·0875 p	— 0·1205 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·0757 p	— 0·0531 p
			$\frac{a}{2}$	$\frac{a}{4}$	— 0·0171 p	— 0·0112 p
5	25	4	$\frac{a}{2}$	$\frac{a}{2}$	— 0·0246 p	— 0·0526 p
			$\frac{9a}{20}$	$\frac{a}{2}$	— 0·0214 p	— 0·0406 p
			$\frac{a}{2}$	$\frac{9a}{20}$	— 0·0213 p	— 0·0380 p
			$\frac{3a}{8}$	$\frac{a}{2}$	— 0·0137 p	— 0·0161 p
			$\frac{a}{2}$	$\frac{3a}{8}$	— 0·0127 p	— 0·0105 p
			$\frac{a}{4}$	$\frac{a}{2}$	— 0·0042 p	— 0·0040 p
			$\frac{a}{2}$	$\frac{a}{4}$	— 0·0039 p	— 0·0032 p

In Fig. 2 the distribution of vertical stresses under the centre of uniformly loaded areas throughout the elastic layer for various side ratios $\frac{c}{d}$ is illustrated. On the abscissa

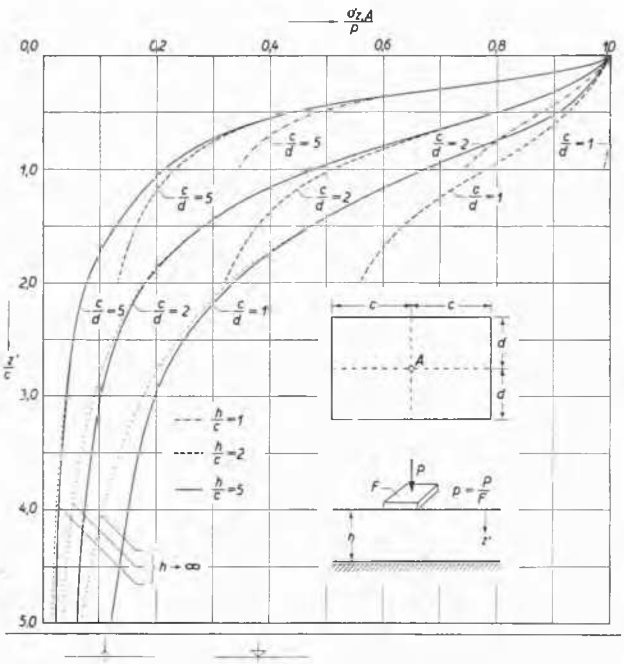

Fig. 2

the values $\frac{\sigma_z}{p}$ are related as a function of the ratio z'/c, where z' is the depth under the surface of the elastic layer and c the bigger half width of the loaded area. Dotted curves, representing Boussinesq's vertical stress distribution, are used as the basis for comparison (STEINBRENNER, 1934).

Similarly as for vertical stresses σ_z, the surface settlements w of the elastic layer were calculated. In Fig. 3, 4, 5 and 6 for various side ratios the functions $\frac{d}{h} - f_t$ are illustrated for four characteristic points. The surface settlement w_z of each of these points A, B, C or D is obtained by the equation

$$w_t = \frac{pd}{E} f_t \tag{9}$$

where E = elastic modulus.

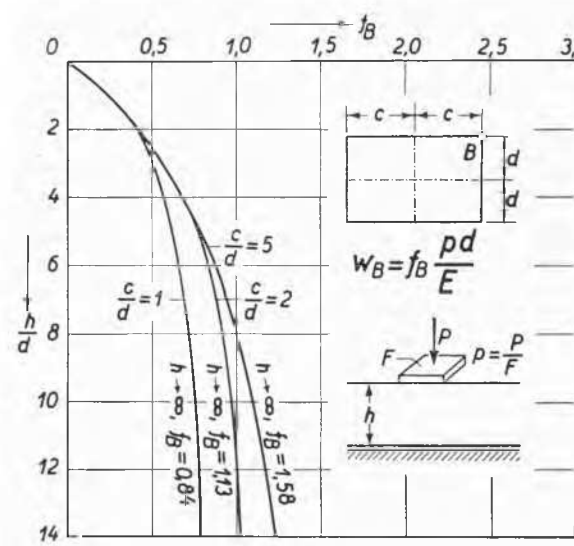

Fig. 4

In Figs. 3, 4, 5 and 6 also the values of coefficient f_t are given provided that the thickness of the elastic layer becomes infinite. These values are for Poisson's ratio $\nu = 0{\cdot}5$ summarised after SCHLEICHER (1927).

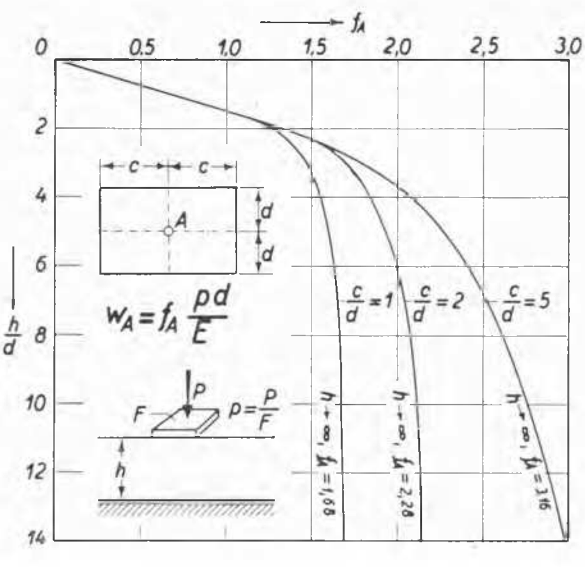

Fig. 3

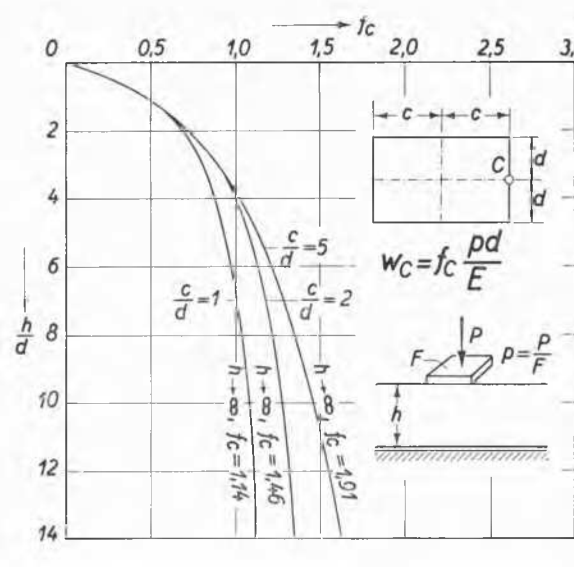

Fig. 5

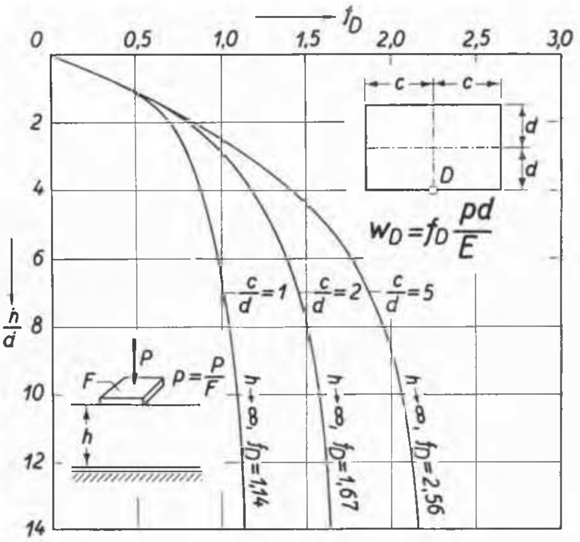

Fig. 6

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