Plastic Equilibrium in Soil

Contribution au problème de la pression du sol

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Summary

The author considers the problem of finding solutions describing the plane plastic equilibrium in soils when the unit weight γ is different from zero. The investigation leads to a partial differential equation in polar coordinates R and φ which can be reduced to an ordinary type of the second order and degree. To solve this last equation, however, is difficult and only one solution has been obtained. It describes Rankine's state of stress in an ideal non-cohesive soil with horizontal or inclined surface.

Generally it can be said that in a soil with an angle of shear strength the stresses must increase in direct proportion to the radius vector R in order to obtain a continuous stress distribution satisfying the equilibrium condition and Coulomb's failure criteria.

The author gives some approximate solutions which appear to show that Prandtl's solution is too conservative. It appears that the stresses in the radial shear zones increase more rapidly with increasing angles of shear strength Φ than Prandtl's solution indicates.

Sommaire

Cette communication traite du problème de trouver des solutions décrivant l'équilibre plastique d'un sol quand sa densité γ n'est pas nulle. L'investigation conduit à une équation aux dérivées partielles en coordonnées polaires R et φ qui peut être réduite à un type ordinaire des seconds ordre et degré. Il est pourtant difficile de résoudre cette dernière équation et jusqu'à présent on n'a trouvé qu'une solution. Elle décrit la distribution de pression de Rankine dans un sol idéal non-cohérent avec surface horizontale ou inclinée.

En général on peut dire que dans un sol ayant un angle de cisaillement, les pressions doivent augmenter linéairement avec R pour obtenir une distribution de pression continue satisfaisant les conditions d'équilibre et la condition de rupture de Coulomb.

Quelques solutions approximatives sont présentées et elles semblent indiquer que celle de Prandtl est trop pessimiste. En particulier il semble que les pressions dans les zones radiales de cisaillement augmentent plus rapidement avec l'angle de cisaillement Φ que l'indique la solution de Prandtl.

1. Introduction

In a mass which follows the law of Coulomb at failure, the plane plastic equilibrium was first investigated by Prandtl under the assumption that the unit weight γ of the mass was equal to zero. He obtained his solution by means of a stress function (F) in polar coordinates (R, φ) of the form $F = u(R) \cdot v(\varphi)$.

The solution when $\gamma \neq 0$ is known only for an ideal cohesive mass and the stress function has the form $F = u(R) \cdot v(\varphi)$. It is possible to show that also in the general case the stress function F has the shape $F = u(R) \cdot v(\varphi)$. Naturally the functions $u(R)$ and $v(\varphi)$ are not the same for the different cases related above.

2. Plastic equilibrium described with differential equations

The plane equilibrium of a mass with the unit weight γ is determined by the following two equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = \gamma \tag{1}$$

These equations are satisfied if

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + a \cdot \gamma \cdot y + A$$
$$\sigma_y = \frac{\partial^2 F}{\partial x^2} + b \cdot \gamma \cdot y + B \tag{2}$$
$$\tau = -\frac{\partial^2 F}{\partial x\, \partial y} + (1 - b) \cdot \gamma \cdot x + D$$

Where F is an arbitrary function of x and y, a, b, A, B and D are any constants.

For a mass with the angle of shear strength Φ and the cohesion c the failure condition can be written, in accordance with Coulomb's theory :

$$\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2} = K(\sigma_y + \sigma_x) + 2C \quad \ldots . \tag{3}$$

where

$$K = \sin \Phi$$
$$C = c \cdot \cos \Phi$$

When changing to polar coordinates R and φ the following expressions are obtained

$$\sigma_y + \sigma_x = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + (a + b) \cdot \gamma \cdot y + A + B = \frac{\partial^2 F}{\partial R^2} + \frac{1}{R^2} \cdot \frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{R} \cdot \frac{\partial F}{\partial R} + (a + b) \cdot \gamma \cdot R \cdot \sin \varphi + B + A \quad \ldots . \tag{4a}$$

$$\sigma_y - \sigma_x = \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} + (b-a)\cdot\gamma\cdot y + B - A = \frac{\partial^2 F}{\partial R^2}\cos 2\varphi - \frac{\partial^2 F}{\partial R\,\partial\varphi}\cdot\frac{2\sin 2\varphi}{R} - \frac{\partial^2 F}{\partial\varphi^2}\frac{\cos 2\varphi}{R^2} + \frac{\partial F}{\partial\varphi}\cdot\frac{2\sin 2\varphi}{R^2} - \frac{\partial F}{\partial R}\cdot\frac{\cos 2\varphi}{R} + (b-a)\cdot\gamma\cdot R\cdot\sin\varphi + B - A \tag{4b}$$

$$2\tau = -\frac{\partial^2 F}{\partial R^2}\sin 2\varphi - \frac{\partial^2 F}{\partial R\,\partial\varphi}\cdot\frac{2\cos 2\varphi}{R} + \frac{\partial^2 F}{\partial\varphi^2}\cdot\frac{\sin 2\varphi}{R^2} + \frac{\partial F}{\partial\varphi}\cdot\frac{2\cos 2\varphi}{R^2} + \frac{\partial F}{\partial R}\cdot\frac{\sin 2\varphi}{R} + 2(1-b)\cdot\gamma\cdot R\cos\varphi + 2D \tag{4c}$$

The expressions (4) are put into equation (3). If $(b-a)$ is chosen equal to $2\,(1-b)$ and this quantity is called p this equation becomes.

$$\sqrt{\left[\frac{\partial^2 F}{\partial R^2} - \frac{1}{R^2}\cdot\frac{\partial^2 F}{\partial\varphi^2} - \frac{1}{R}\cdot\frac{\partial F}{\partial R} - p\cdot\gamma\cdot R\cdot\sin\varphi + (B-A)\cos 2\varphi - 2D\sin 2\varphi\right]^2 + \left[\frac{2}{R}\cdot\frac{\partial^2 F}{\partial R\,\partial\varphi} - \frac{2}{R^2}\cdot\frac{\partial F}{\partial\varphi} - p\cdot\gamma\cdot R\cos\varphi - (B-A)\sin 2\varphi - 2D\cos 2\varphi\right]^2} = K\left[\frac{\partial^2 F}{\partial R^2} + \frac{1}{R^2}\cdot\frac{\partial^2 F}{\partial\varphi^2} + \frac{1}{R}\cdot\frac{\partial F}{\partial R} + 2\,(1-p)\cdot\gamma\cdot R\cdot\sin\varphi + A + B + \frac{2C}{K}\right] \tag{5}$$

With the substitution

$$F = G - R^2\left[\frac{A+B}{4} + \frac{B-A}{4}\cos 2\varphi - \frac{D}{2}\sin 2\varphi\right] + \frac{\gamma p}{4}R^3\sin\varphi \tag{6}$$

where G is another function of R and φ, the equation (5) becomes

$$\sqrt{\left[\frac{\partial^2 G}{\partial R^2} - \frac{1}{R^2}\cdot\frac{\partial^2 G}{\partial\varphi^2} - \frac{1}{R}\cdot\frac{\partial G}{\partial R}\right]^2 + \left[\frac{2}{R}\cdot\frac{\partial^2 G}{\partial R\,\partial\varphi} - \frac{2}{R^2}\cdot\frac{\partial G}{\partial\varphi}\right]^2} = K\left[\frac{\partial^2 G}{\partial R^2} + \frac{1}{R^2}\cdot\frac{\partial^2 G}{\partial\varphi^2} + \frac{1}{R}\cdot\frac{\partial G}{\partial R} + 2\gamma\cdot R\cdot\sin\varphi + \frac{2C}{K}\right] \tag{7}$$

and the corresponding stresses according to equation (4)

$$\sigma_y + \sigma_x = \frac{\partial^2 G}{\partial R^2} + \frac{1}{R^2}\cdot\frac{\partial^2 G}{\partial\varphi^2} + \frac{1}{R}\cdot\frac{\partial G}{\partial R} + 2\cdot\gamma\cdot R\sin\varphi \tag{8a}$$

$$\sigma_y - \sigma_x = \frac{\partial^2 G}{\partial R^2}\cos 2\varphi - \frac{\partial^2 G}{\partial R\,\partial\varphi}\cdot\frac{2\sin 2\varphi}{R} - \frac{\partial^2 G}{\partial\varphi^2}\cdot\frac{\cos 2\varphi}{R^2} + \frac{\partial G}{\partial\varphi}\cdot\frac{2\sin 2\varphi}{R^2} - \frac{\partial G}{\partial R}\cdot\frac{\cos 2\varphi}{R} \tag{8b}$$

$$2\tau = -\frac{\partial^2 G}{\partial R^2}\sin 2\varphi - \frac{\partial^2 G}{\partial R\,\partial\varphi}\cdot\frac{2\cos 2\varphi}{R} + \frac{\partial^2 G}{\partial\varphi^2}\cdot\frac{\sin 2\varphi}{R^2} + \frac{\partial G}{\partial\varphi}\cdot\frac{2\cos 2\varphi}{R^2} + \frac{\partial G}{\partial R}\cdot\frac{\sin 2\varphi}{R} \quad \ldots. \tag{8c}$$

It is interesting to note that p, A, B and D do not appear in the expressions for the stresses.

2.1. *Solution for an ideal cohesive mass* [$C = c$ and $K = 0$] :

From equation (7) one has

$$\sqrt{\left[\frac{\partial^2 G}{\partial R^2} - \frac{1}{R^2}\cdot\frac{\partial^2 G}{\partial\varphi^2} - \frac{1}{R}\cdot\frac{\partial G}{\partial R}\right]^2 + \left[\frac{2}{R}\cdot\frac{\partial^2 G}{\partial R\,\partial\varphi} - \frac{2}{R^2}\cdot\frac{\partial G}{\partial\varphi}\right]^2} = 2c \tag{9}$$

The solution has the shape

$$G = R^2 \cdot v \tag{10}$$

where v is a function of φ alone and follows from the equation

$$\sqrt{(v'')^2 + 4(v')^2} = 2c \quad \ldots. \tag{11}$$

The solutions of (11) are

$$v = \frac{c}{2} \cdot \sin [2(\pm \varphi + \beta] - \frac{\alpha}{2} \tag{12a}$$

$$v = \pm c\varphi + \varepsilon \tag{12b}$$

α, β and ε are arbitrary constants.

The corresponding stress functions describe the Rankine state of stress (12a) and the radial shear states of stress (12b) in an ideal cohesiv mass. The stresses are in polar coordinates

$$\sigma_x = -\alpha - c \sin 2\beta + \gamma \cdot R \cdot \sin \varphi \tag{13a}$$

$$\sigma_y = -\alpha + c \sin 2\beta + \gamma \cdot R \cdot \sin \varphi \tag{13b}$$

$$\tau = -c \cdot \cos 2\beta \tag{13c}$$

$$\sigma_x = \pm 2c\varphi \pm c \sin 2\varphi + \gamma \cdot R \cdot \sin \varphi + 2\varepsilon \quad \ldots. \tag{14a}$$

$$\sigma_y = \pm 2c\varphi \mp c \sin 2\varphi + \gamma \cdot R \cdot \sin \varphi + 2\varepsilon \quad \ldots. \tag{14b}$$

$$\tau = \mp c \cdot \cos 2\varphi \tag{14c}$$

2.2 *Solutions for an ideal non-cohesive mass* $[C = 0 \,;\; K \neq 0]$:

$$\sqrt{\left[\frac{\partial^2 G}{\partial R^2} - \frac{1}{R^2} \cdot \frac{\partial^2 G}{\partial \varphi^2} - \frac{1}{R} \cdot \frac{\partial G}{\partial R}\right]^2 + \left[\frac{2}{R} \cdot \frac{\partial^2 G}{\partial R\, \partial \varphi} - \frac{2}{R^2} \cdot \frac{\partial G}{\partial \varphi}\right]^2} = \left[\frac{\partial^2 G}{\partial R^2} + \frac{1}{R^2} \cdot \frac{\partial^2 G}{\partial \varphi^2} + \frac{1}{R} \cdot \frac{\partial G}{\partial R} + 2\gamma \cdot R \cdot \sin \varphi\right] \quad \ldots. \tag{15}$$

The solution has the form

$$G = \gamma \cdot z \cdot R^3 \tag{16}$$

where z is a function of φ alone and follows from the equation

$$\sqrt{(3z - z'')^2 + (4z')^2} = K(9z + z'' + 2 \sin \varphi) \quad \ldots. \tag{17}$$

It is a very delicate problem to solve this equation and only one of the solutions can be presented. It has the shape

$$z = \alpha \cdot \sin \varphi + \beta \cos \varphi + \partial \sin 3\varphi + \varepsilon \cos 3\varphi \tag{18}$$

where α, β, ∂ and ε have the following values

$$\alpha = \frac{K(K+\lambda)}{4(1-K^2)} \tag{19a}$$

$$\beta = \pm \frac{K\sqrt{1-\lambda^2}}{4(1-K^2)} \tag{19b}$$

$$\partial = \frac{K[K(1-2\lambda^2) - \lambda]}{12(1-K^2)} \tag{19c}$$

$$\varepsilon = \pm \frac{K\sqrt{1-\lambda^2}(1+2\lambda K)}{12(1-K^2)} \quad \ldots. \tag{19d}$$

λ is an arbitrary constant which links the coefficients together.

This solution gives the following expressions for σ_x, σ_y and τ in Cartesian coordinates

$$\sigma_x = \frac{(1+\lambda K)^2}{1-K^2} \cdot \gamma \cdot y \pm \frac{K\sqrt{1-\lambda^2}(1+\lambda K)}{1-K^2} \cdot \gamma \cdot x \tag{20a}$$

$$\sigma_y = \frac{1-\lambda^2 K^2}{1-K^2} \cdot \gamma \cdot y \pm \frac{K\sqrt{1-\lambda^2}(1-\lambda K)}{1-K^2} \cdot \gamma \cdot x \tag{20b}$$

$$\tau = -\frac{K^2(1-\lambda^2)}{1-K^2} \cdot \gamma \cdot x \mp \frac{K\sqrt{1-\lambda^2}(1+\lambda K)}{1-K^2} \cdot \gamma \cdot y \quad \ldots. \tag{20c}$$

It can be observed that for the lines

$$y = \mp \frac{K\sqrt{1-\lambda^2}}{1+\lambda K} \cdot x$$

σ_x, σ_y and τ are all equal to zero. This means that these lines are unloaded surfaces, whose inclinations depend on the λ value. If the surface is to be horizontal, λ must have the value ± 1 and with these values introduced in equation (20) the well-known expressions for active and passive pressure are obtained.

$$\sigma_x = \frac{1 \pm K}{1 \mp K} \cdot \gamma \cdot y = \gamma \cdot \tan^2\left(\frac{\pi}{4} \pm \frac{\Phi}{2}\right) \cdot y$$
$$\sigma_y = \gamma \cdot y \tag{21}$$
$$\tau = 0$$

If the inclination α of the free surface $\left(\tan \alpha = \frac{K\sqrt{1-\lambda^2}}{1+\lambda K}\right)$ is studied as a function of λ, this inclination has a maximum for $\lambda = -K$ and is equal to $\Phi \cdot \left(\tan \alpha = \frac{K}{\sqrt{1-K^2}} = \tan \Phi\right)$

This means that the angle of repose for an ideal non-cohesive mass is equal to the angle of shear strength.

At present no other solution of (15) has been found and it looks as if the easiest way to obtain other solutions is to integrate the equation numerically.

If the derivatives of (16) are put into equation (8) one obtains the following expressions for the stresses

$$2\sigma_y = \gamma\cdot R[z(9 + 3\cos 2\varphi) - 4z'\sin 2\varphi + z''(1 - \cos 2\varphi) + 2\sin\varphi]$$
$$2\sigma_x = \gamma\cdot R[z(9 - 3\cos 2\varphi) + 4z'\sin 2\varphi + z''(1 + \cos 2\varphi) + 2\sin\varphi] \quad \ldots\ldots \quad (22)$$
$$2\tau = \gamma\cdot R[-3z\sin 2\varphi - 4z'\cos 2\varphi + z''\sin 2\varphi]$$

It can thus be seen that when a non-cohesive mass is in plastic equilibrium the stresses increase in direct proportion with the radius vector R.

2.21. *Approximative solutions*—If $\sin\varphi$ is neglected in equation (17) the following solutions are obtained

$$z = A\cdot e^{\pm\sqrt{\frac{9K^2-1}{1-K^2}}\,\varphi} \qquad (23)$$
$$z = C_1\sin 3\varphi + C_2\cos 3\varphi$$

Another approximative solution can be obtained if γ is put equal to zero in equation (15), i.e. for a weightless mass. The solution to equation (15) has in this case the shape

$$G = x\cdot R^n \qquad (24)$$

where x is a function of φ alone and n is an arbitrary number. x follows from the equation

$$\sqrt{[n(n-2)\cdot x - x'']^2 - [2(n-1)\cdot x']^2} = K[n^2\cdot x + x''] \qquad (25)$$

The solutions are of the form

$$x = A\cdot e^{\alpha\varphi} \qquad (26)$$

where α has the following values

$$\alpha_{1,2} = \pm\sqrt{\frac{K^2n^2 - (n-2)^2}{1 - K^2}}$$
$$\alpha_{3,4} = \pm n\cdot i \qquad (27)$$

There is thus an infinite number of possible solutions when γ is considered equal to zero. If the value of n is chosen equal to 2, Prandtl's solution is obtained. To obtain a solution which gives stresses increasing directly with R, the value of $n = 3$ is introduced and the following solutions remain :

$$x = A\cdot e^{\pm\sqrt{\frac{9K^2-1}{1-K^2}}\,\varphi}$$
$$x = C_1\sin 3\varphi + C_2\cos 3\varphi \qquad (28)$$

The solutions for z and x are thus identical.

If Prandtl's expression for radial shear

$$x = A\cdot e^{\frac{2K}{\sqrt{1-K^2}}\varphi}$$

is compared with the new ones

$$z = x = A\cdot e^{\sqrt{\frac{9K^2-1}{1-K^2}}\,\varphi}$$

it can be seen that with increasing K the last expression gives stresses increasing more rapidly than that of Prandtl. This is, however, true only when K is greater than $1/\sqrt{5}$, i.e. when $\Phi \geqslant 26°6$.

It is interesting to compare this result with those obtained by the Danish Geotechnical Institute from model tests on foundations in sand. Up to an angle of shear strength of about 30° the theoretical bearing capacities show good agreement with the tests but with increasing angle of shear strength increasing differences have been obtained.

The differences between theory and test in a non-cohesive soil may thus be explained from the fact that an approximative theory has hitherto been used which in certain cases is too conservative.

2.3. *Solutions when C and K are different from zero :*

The solution for this case follows from the solution of (15) by a simple transformation.

If the new stresses are called σ_{x1}, σ_{y1} and τ_1 they are

$$\sigma_{x1} = \sigma_x - \frac{C}{K}$$
$$\sigma_{y1} = \sigma_y - \frac{C}{K} \qquad (29)$$
$$\tau_1 = \tau$$

where σ_x, σ_y and τ are the stresses obtained from the solution of (15). This follows immediately from Mohr's diagram. (See Fig. 1).

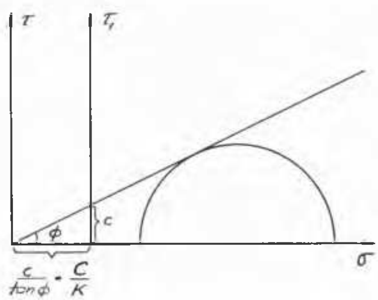

Fig. 1 Mohr's diagram illustrating coordinate transformation.
Diagramme de Mohr montrant la transformation de coordonnées.

3. Conclusions

The plane plastic equilibrium in soils, when the unit weight $\gamma \neq 0$ can be solved by inserting a stress function F in polar coordinates of a simple shape.

It is, however, difficult to solve the resulting differential equation and it seems to be necessary to integrate the equation numerically in order to obtain the most interesting solution which describes the states of stress in the radial shear zones.

From some approximate solutions it seems as if Prandtl's, solution in certain cases is too conservative. This is confirmed by model tests on foundations in sand carried out by the Danish Geotechnical Institute.

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