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A Semi-Graphical Solution of Artesian Well Problems

Une solution semi-graphique du problème de rabattement de la nappe par puits filtrants

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Summary

Transient conditions in artesian wells received little attention until Theis published his non-equilibrium formula in 1935, by which the drawdown of the piezometric surfaces of an artesian well can be estimated as a function of time.

Since then, many related problems have been studied or clarified. In 1952, the author and others published a numerical procedure — a step by step method — based on transforming the general partial differential equation which governs the flow of water towards wells under the transient condition to its finite-differences form. This method checked fairly well with the Theis formula, except close to a borehole. The discrepancy in results between the two methods is probably due to the fact that in the Theis solution a hypothetical well was assumed, whereas in the numerical procedure the actual well diameter has been considered. The finite-difference method is believed to yield a more correct result, as it eliminates some of the assumptions set forth in the Theis solution.

The author describes a proposed semi-graphical procedure for determining the piezometric level of an artesian well at any time measured from the start of pumping. The method is based on both the Theis solution and the above mentioned numerical procedure. It is considered to be an improvement on the finite difference method which, although very helpful in complicated problems, is nevertheless generally laborious. The suggested method is a trial to eliminate some of the idealized assumptions made in mathematical solutions, the laboriousness of numerical solutions and a step forward in establishing a simple method of solving practical and complicated problems such as that presented by a group of wells under transient (non-equilibrium) conditions.

Introduction

A semi-graphical method, based on the finite-differences for solving single artesian well problems under a transient condition, is presented. In order to follow up the details of the procedure it has been found necessary to make a brief survey of: (1) the Mathematical equation of flow (2) the analytical method of finite-difference (3) a graphical solution based on finite-differences (which has not as yet been published). The method suggested by the author is claimed to be an improvement on the three methods of solution, and is hoped to be an advance in providing a simple method of solving other practical and complicated problems, such as a group of wells under the transient condition.

The Mathematical Equation of Flow of Water Towards Artesian Wells Under the Non-Equilibrium State [1,2]

The differential equation in terms of polar coordinates that governs the flow is as follows:

$$\left. \begin{aligned} 1/r[\partial/\partial r(r\partial h/\partial r)] &= (S/T)\partial h/\partial t, \\ \text{or} \\ (\partial^2 h/\partial r^2) + (1/r)\partial h/\partial r &= (S/T)\partial h/\partial t, \end{aligned} \right\} \quad (1)$$

Sommaire

L'évolution de la pression dans le cas des puits filtrants a été peu étudiée depuis que Theis a publié son équation par laquelle l'abaissement des surfaces piézométriques peut être calculé en fonction du temps. Depuis, certains problèmes ont été étudiés ou résolus.

En 1952 l'auteur et d'autres ont publié un procédé de calcul numérique par approximations successives, basé sur la transformation de l'équation différentielle qui régit le flux en équation aux différences finies. Cette méthode conduit à des résultats comparables à ceux de la formule de Theis, sauf au voisinage des parois. Les différences trouvées sont dues au fait que dans la formule de Theis on fait une hypothèse sur la forme du profil de rabattement, tandis que dans la méthode précédente, le diamètre du puits a été considéré. Cette méthode conduit donc à un résultat plus correct et élimine les hypothèses simplificatrices de Theis.

La présente communication a pour objet une méthode semi-graphique de détermination de la surface piézométrique d'un puits filtrant, en fonction du temps après le début du pompage. Cette méthode utilise les deux solutions précédentes. Elle constitue une amélioration de la méthode d'approximations successives qui est en général laborieuse, quoique très utile dans les cas compliqués. La méthode présentée permet d'éliminer les hypothèses des solutions mathématiques, de simplifier les calculs numériques, de résoudre simplement les problèmes compliqués, tels que celui d'un groupe de puits filtrants avant que l'équilibre ne soit atteint.

where S and T are the coefficients of storage and transmissibility respectively, h is the head at any radius r from the well centre and t is the time.

To obtain a particular solution for equation (1), a mathematical sink of constant strength was assumed [1] and the following non-equilibrium equation was deduced:

$$\left. \begin{aligned} s &= (q/4\pi T) \int_u^\infty (e^{-u}/u) du, \\ \text{or} \\ s &= (9/4\pi T) [-\text{Ei}(-r^2 S/4\pi T)], \end{aligned} \right\} \quad \dots \quad (2)$$

where the drawdown $s = h_e - h$, q is the rate of discharge, $u = r^2 S/4\pi T$ and h_e is the initial head before pumping starts,

The drawdown s_w at the well surface was suggested [2] to be determined by substituting $r = r_w$ in Equation (2). It was concluded that as long as the rate of withdrawal of water from storage within a distance r_w of the hypothetical sink is negligible, Equation (2) will accurately give the drawdown at the surface of a well. It was proved, however, by applying other methods [3, 4] that the effect of well radius is not so negligible as it was formerly considered to be.

Analytical Method of Finite-Differences

Let any three successive points L, O and R of radii r_L , r_O and r_R from the well centre (Fig. 1), be chosen through the aquifer within the well influence. It is assumed that the water head at each of these points represents the head of the concentric shell containing the point. The intermediate shell represented by the point O, has an internal radius of $r_{L,O}$ and an external radius of $r_{O,R}$. These two radii are the averages of r_L and r_O , and r_O and r_R respectively.

Considering the point O — or strictly speaking the circle O — then, the problem is to determine the piezometric head h'_O at the end of a given time interval Δt from a knowledge of the piezometric heads h_L , h_O and h_R at L, O and R respectively at the beginning of this time interval. The initial gradient at shell O is assumed to remain constant throughout the assumed time interval Δt which should in practice be small. The smaller the values of both the shell thicknesses and Δt , the more accurate are the results and the more labour will be involved in finding a solution. Since the quantity of water that leaves any shell is larger than that entering into it during a given period by an amount equal to the quantity of water that is released from storage in the same shell due to the decrease of head, then the condition at the well surface can be determined as follows :

The ratio of the discharge q_{w1} (Fig. 1) entering the first shell w defined by r_w and r_{w1} and the discharge q pumped out of the well is given [5] by :

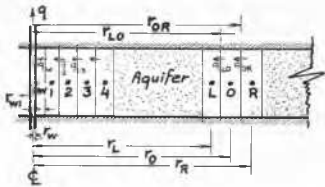


Fig. 1 Divisions of aquifer; finite-difference method.

Les divisions de la nappe; méthode de la différence-finie.

$$q_{w1}/q = e^{-u}, \tag{3}$$

where :

$$u = r_{w1}^2 S/4Tt = r_{w1}^2 S/4T\pi \Delta t, \quad n = 1, 2, 3, \text{ etc.} \tag{4}$$

but :

$$q = q_{w1} + \int_0^t S(\delta h/\delta t)2\pi r dr, \tag{4}$$

or, in a finite difference form :

$$q(1 - e^{-u}) = S\pi(r_{w1}^2 - r_w^2) (h_w - h'_w)/\Delta t, \tag{5}$$

or,

$$s'_w = s_w + q\Delta t (1 - e^{-u})/S\pi(r_{w1}^2 - r_w^2), \tag{5}$$

where h_w and h'_w are the heads at the well surface at the beginning and end of a given time interval Δt . " s'_w " is the drawdown at the end of Δt . Equation (5) can be used to determine the head at the well surface after any time t since the start of pumping.

Considering the finite three consecutive shells L, O and R, then $h'_O = F_L h_L + F_O h_O + F_R h_R$ where :

$$F_L = r_{L,O}/M(r_O - r_L), F_O = 1 - (F_L + F_R), F_R = r_{O,R}/M(r_R - r_O) \tag{6}$$

$M = S A_o/2\pi T \Delta t$, $A_o =$ area of the base of shell O.

Applying the drawdowns $s = h_w - h$ rather than the head values h , Equation (6) reduces to :

$$s'_O = F_L s_L + F_O s_O + F_R s_R \tag{7}$$

The sum of the factors F is equal to unity and M is arbitrarily chosen depending upon the required degree of precision within a certain limiting value :

$$M \geq [r_{O,R}(r_R - r_O) + r_{L,O}(r_O - r_L)] \tag{8}$$

Dividing the aquifer into successive concentric shells $w, 1, 2, 3, \dots, L, O, R, \dots$ etc. starting from the well surface, then the drawdowns within any shell after a certain time interval can be obtained by applying equation (7). In order to start a solution for this equation, the drawdown at the well surface — represented by shell w — at the end of any time interval should be calculated from equation (5).

The application of equation (7) should be repeated together with equation (5) during the successive time intervals up till the steady-state condition is reached or the heads at the desired elapsed time since the start of pumping have been found.

A Graphical Solution Based on Finite-Differences

Referring to Fig. 1 and applying Equation (4) in its finite-difference form, then :

$$q_{L,O} = q_{O,R} + S\pi(r_{O,R}^2 - r_{L,O}^2) (h_O - h'_O)/\Delta t \tag{9}$$

Applying equation (3), then :

$$q_{L,O} = qe^{-r_{L,O}^2 S/4Tt} \text{ and } q_{O,R} = qr^{-r_{O,R}^2 S/4Tt} \tag{10}$$

From equations (9) and (10), we get :

$$(s'_O - s_O) = q\Delta t(e^{-r_{L,O}^2 S/4Tt} - e^{-r_{O,R}^2 S/4Tt}) / [S\pi(r_{O,R}^2 - r_{L,O}^2)] \tag{11}$$

where :

$t = n\Delta t =$ time since the start of pumping ($n = 1, 2, 3, 4$, etc.)
At the well surface equation (5) reduces to :

$$(S'_w - S_w) = q\Delta t(1 - e^{-r_{w1}^2 S/4Tt})/S\pi(r_{w1}^2 - r_w^2) \dots \tag{12}$$

Referring to the shells shown in Fig. 1 and noting that s_w at the start of pumping is zero, s'_w at the end of the first time interval can be calculated from equation (12) as far as the constants q, S and T are known and the values of r_w and r_{w1} are fixed. The drawdowns during the successive time intervals ($n = 1, 2, 3, 4, \dots$ etc) can thus be calculated at shell w . This means that in as far as q is constant, these drawdowns can be determined irrespective of the pore water pressures within other shells.

In order to determine the various drawdowns within the shells 1, 2, 3, ... etc., draw a right angled triangle ABC (Fig. 2-i). The horizontal side $AB = q\Delta t/(r_{12}^2 - r_{w1}^2)$ drawn to the same scale of the drawdowns. The vertical side CB is drawn as a dimensionless quantity $S\pi$. Side AC forms ray 1 which corresponds to shell 1 bounded between r_{12} and r_{w1} . Similarly ray 2 corresponds to shell 2 bounded between r_{23} and r_{12} . Distance $a_{2B} = q\Delta t/(r_{23}^2 - r_{12}^2)$. Any other ray O corresponding to shell O can be determined by measuring a_{OB} as equal to $q\Delta t/(r_{OR}^2 - r_{LO}^2)$.

Draw the curve $Y = e^{-u}$ (Fig. 2-ii) where $u = r^2 S/4T\Delta t$, the horizontal scale is the same as that of AB and the vertical scale should be the same as that of $S\pi$ (which is also dimensionless). Plot on this curve the vertical boundaries of the successive shells $w, 1, 2, 3, \dots$ etc. The points of intersection of these vertical boundaries are points I, II, III, etc.

Considering the first time interval ($n = 1$), draw a horizontal line from II to meet the vertical line through I at I_1 . From I, draw a line parallel to ray 1, to meet the horizontal from I at m_1 . The horizontal distance $I_1 m_1$ measured to the

same scale of AB gives the drawdown at shell 1 at the end of the first time interval. The proof is obvious.

Similarly, the drawdown II_{m2} at the end of the first time interval within shell No. 2 is obtained by a similar construction in which II_{1m2} is parallel to ray 2 and so on.

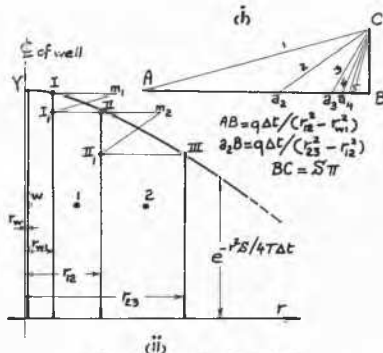


Fig. 2 Graphical method. Méthode graphique.

At the end of the second time interval ($n = 2$), another curve should be constructed in which $u = r^2 S / 8T \Delta t$. However, the same curve used for the first time interval can be used for the second time interval on condition that the vertical lines binding the various shells are shifted to the left by reducing r to $r/\sqrt{2}$. By doing this, the same construction can be followed without making any changes either in Fig. 2-i or in the curve e^{-u} . The reason for this is simply because :

$$\left(r/\sqrt{2} \right)^2 S / 4T \Delta t = r^2 S / 4T (2 \Delta t)$$

After the end of any other time $t = n \Delta t$ the vertical lines are shifted so as to reduce r to r/\sqrt{n} .

The drawdowns corresponding to any time interval can thus be plotted for the various shells.

A Suggested Semi-Graphical Solution

The main drawdown of numerical and graphical methods just presented is that they are laborious. The following semi-graphical method is therefore suggested.

Substituting [5] in equation (2), $u = br^2$, where $b = S/4Tt$, then $du = 2br dr$, and

$$s = (q/4\pi T) \int_{br^2}^{\infty} [e^{-br^2}/br^2 d(br^2)]$$

Therefore $ds/dr = (q/2\pi T) e^{-u}$ (13)

Transforming equation (13) to its finite difference form, then :

$$\Delta s = (\Delta r q e^{-r^2 S / 4T n \Delta t}) / 2\pi r \cdot T \cdot \quad (14)$$

$$\text{and } s = \sum_{r=r_w}^{\infty} \Delta s \quad (15)$$

In order to follow up the semi-graphical solution of equation (14), an example is solved which has the same data as an example previously solved by Theis and [1] resolved by the

finite-difference method [3]. The aquifer of the artesian well is divided to the same shell sizes [3]. The time interval Δt is taken equal to 6 hours = 360 minutes (rather than 27 minutes [3]). The constants are : $q = 540$ gallons per minute = 72.1872 cu.ft. per minute, $T = 8.866$ ft.² per minute, $S = 0.217$, $r_w = 1.0$ ft.

Considering the first time interval ($n = 1$) and plotting the e^{-u} diagram and the various shell boundaries (Fig. 3), " Δs " within each shell lower than the next shell to its right can thus be calculated from equation (14). The accumulative values of Δs , shown in the table, are those corresponding to the first time interval (6 hours).

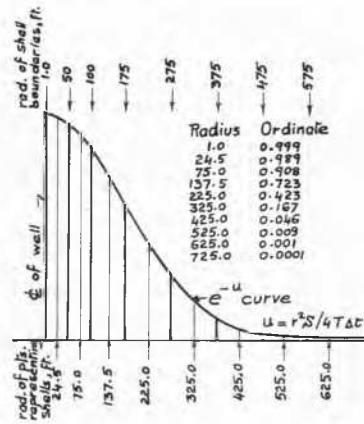


Fig. 3 Computation of Δs ; semi-graphical method. Computation de Δs ; méthode semi-graphique.

After the second time interval, or after $t = 2 \Delta t = 12$ hours, the same Δs values hold true but for new radii = $\sqrt{2} r$ because

$$\begin{aligned} \Delta s &= q \left(\Delta r \sqrt{2} e^{-(r\sqrt{2})^2 S / 4T \cdot 2 \Delta t} \right) / 2\pi T \cdot r \sqrt{2} \\ &= q \left(\Delta r e^{-r^2 S / 4T \Delta t} \right) / 2\pi T \cdot r \end{aligned}$$

Thus the same drawdowns calculated for the first time interval are valid for the second time interval but under $r\sqrt{2}$. The drawdown curve after time $t = 2 \Delta t$ can thus be obtained except for the shell lying between $r = r_w$ and $r = r_w \sqrt{2}$. Similarly after any other time $t = n \Delta t$, the same drawdowns Δs calculated for the first time interval express those corresponding to radii = $r\sqrt{n}$ except for a distance between $r = r_w$ and $r = r_w \sqrt{n}$ for which Δs has to be calculated separately and added to that of shell w as follows :

$$\Delta s \cong q(\sqrt{n} - 1) / \pi T (\sqrt{n} + 1), \quad (16)$$

because $e^{-u} \cong 1.0$ in the vicinity of the well. In the proposed solution the curve (or polygon) corresponding to the first time intervals was drawn once and for all and the radii were shortened to $r/\sqrt{2}$, $r/\sqrt{3}$, $r/\sqrt{4}$,... etc. to measure the drawdowns corresponding to the second, third, fourth, ... etc. time intervals respectively (Fig. 4). This is much more simpler than shifting the curve itself. The results are tabulated in the given table in which the adjustments near the well surface

Table
Results of Example Solved by the Semi-Graphical Method

		Shell Numbers								
		w			1	2	3	4	5	6
		Measured value	Correc-tion near Well (Eq. 16)	Actual Value						
r_o , ft.				24.5	75.0	137.5	225.0	325.0	425.0	525
r_{OR} , ft.				50	100.0	175.0	275.0	375.0	475.0	575.0
r_{LO} , ft.				1.0	50.0	100.0	175.0	275.0	375.0	475.0
s , ft. Eq. (16)				2.616	0.784	0.511	0.164	0.066	0.014	0.003
Drawdown ft. after	6 hours	—	0.000	4.158 (4.243)	1.542 (1.741)	0.758 (0.892)	0.247 (0.334)	0.083 (0.080)	0.017 (0.015)	0.003 (0.002)
	12 hours	4.550	0.444	4.994 (4.758)	2.820 (2.238)	1.270 (1.346)	0.625 (0.687)	0.235 (0.277)	0.120 (0.104)	0.050 (0.036)
	18 hours	4.577	0.694	5.271 (5.052)	3.180 (2.527)	1.465 (1.620)	0.830 (0.925)	0.440 (0.450)	0.210 (0.211)	0.115 (0.095)
	24 hours	4.950	0.863	5.813 (5.256)	3.500 (2.729)	1.920 (1.815)	1.070 (1.102)	0.610 (0.591)	0.315 (0.314)	0.185 (0.162)

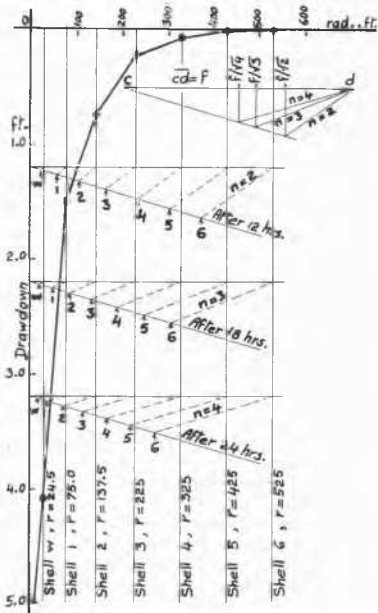


Fig. 4 Solution by semi-graphical method.
Solution par la méthode semi-graphique.

were calculated according to equation (16). In this table, the shown figures in brackets are those calculated by the numerical method [3]. The present method allowed the selection of a time interval about 13 times as much as that used in the numerical method. It is recommended that there should be small shells near the well so that the drawdown curve approximates to a curve rather than to a polygon. Any other drawdown curve may be drawn and applied for the entire solution and not necessarily that corresponding to the first time interval. The radii are thus increased or decreased accordingly. It is obvious that this method largely eliminates the laboriousness of numerical methods and it competes reasonably well with any mathematical or analytical method.

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