Visco-Plastic Flow of Soils

Les écoulements visco-plastiques des sols

by A. S. STROGANOV, Institute of Mechanics U.S.S.R., Academy of Sciences

Summary

The author gives the invariable characteristics of the mechanical properties and mechanical equations of state of the visco-plastic flow of soils based on the generalisation of the Hencky and Bingham conception. The author also gives the solutions of problems on visco-plastic flow of a soil layer along a rough inclined plane and between two rough surfaces which have practical significance in an estimate of the stability of certain types of landslides ans slopes. The author solves the problem of visco-plastic flow of a hollow circular cylinder that can be used to compute the strength of frozen soils formed during shaft sinking with the aid of the freezing process. In addition, the principal equations concerning the plane problem of visco-plastic flow of soils are given. Their further application will facilitate the solution of many significant problems in soil mechanics.

Sommaire

La communication donne les caractéristiques mécaniques invariables et les équations d'état de l'écoulement visco-plastique du sol ; les équations résultantes sont une généralisation de celles de Hencky et de Bingham.

On y trouve également la solution du problème de l'écoulement d'une couche de sol le long d'une surface rugueuse, inclinée vers l'horizontale et de la couche comprise entre deux parois rugueuses ; ces solutions présentent un intérêt pratique au point de vue du calcul de la stabilité de certains types de talus et de glissements de terrain. La communication contient aussi la solution du problème de l'écoulement visco-plastique dans le cas d'un cylindre creux, ce qui correspond au cas de fonçage des puits avec application de la congélation du sol. Enfin la communication comporte les équations fondamentales de l'écoulement visco-plastique plan des sols, à l'aide desquelles on peut résoudre différents problèmes de la mécanique du sol.

1. Mechanical Equations of State of the Visco-Plastic Flow of Soils

Mechanical properties of soils depend on the time factor to a greater or less extent. The author's tests on soils to discover their viscosity and creep, although not numerous, have led to the conclusion that a plastic state can be given by the following expressions :

$$T = (H + \sigma)\tan\psi + \mu \cdot S \qquad (1)$$

where T — is intensity of tangential stresses, H — cohesion, σ — mean normal stress, $\tan\psi$ — friction coefficient for an octahedral area or "Botkin constant", μ — coefficient of plastic viscosity, S — intensity of shear strain velocities.

This expression is shown diagrammatically in Fig. 1 and is a modification of the H. Hencky expression (R. 1) that in its turn summarises the conception of a visco-plastic body given by Bingham (R. 2).

The mechanical equations of visco-plastic flow of soils resulting from condition (1), imcompressibility condition and tensor ratios are expressed in components as follows :

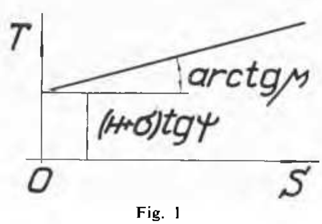

Fig. 1

$$\sigma_x - \sigma = 2\left[\mu + \frac{(H+\sigma)\tan\psi}{S}\right]\xi_x ,$$
$$\tau_{xy} = \left[\mu + \frac{(H+\sigma)\tan\psi}{S}\right]\eta_{xy} \qquad (2)$$

etc...,

The use of equations of state (2) in some problems on the visco-plastic flow of soils, which are of practical interest, is given below as solutions of these problems with appropriate major premises.

2. Visco-Plastic Flow of a Soil Layer along an Inclined Plane

Let us consider the plane visco-plastic flow along an inclined rough plane of an heavy soil layer of infinite extent (Fig. 2) under evenly distributed load.

The mechanical equations of state (2) assume the form :

$$\left.\begin{aligned} \sigma_v - \sigma &= 2\left[\mu + \frac{(H+\sigma)\tan\psi}{S}\right]\xi_v , \\ \sigma_u - \sigma &= 2\left[\mu + \frac{(H+\sigma)\tan\psi}{S}\right]\xi_u , \\ \tau_{uv} &= \left[\mu + \frac{(H+\sigma)\tan\psi}{S}\right]\eta_{uv} , \end{aligned}\right\} \qquad (3)$$

where ξ_v, ξ_u, η_{uv} — are strain velocity components. Let the soil in a state of the visco-plastic flow be imcompressible.

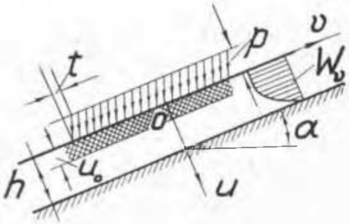

Fig. 2

The imcompressibility condition of the material yields the equation :

$$\xi_v + \xi_u = 0, \quad (4)$$

where $\xi_v = \frac{\partial W_v}{\partial v} = 0$ owing to the independence of the strained state on coordinate v, hence displacement velocity W_v can depend only on u :

$$W_v = W_v(u). \quad \ldots. \quad (5)$$

As $\xi_u = \frac{\partial W_u}{\partial u}$ and does not depend on coordinate v either, equation (4) assumes the form :

$$\frac{dW_u}{du} = 0, \quad \ldots. \quad (6)$$

i.e. velocities W_u are constant in the layer thickness.

The ratios obtained lead to the following :

$$\eta_{uv} = \frac{\partial W_u}{\partial v} + \frac{\partial W_v}{\partial u} = \frac{dW_v}{du}. \quad (7)$$

Substituting $\xi_v = \xi_u = 0$ in the first and second mechanical equations of state (3), we obtain :

$$\sigma_v = \sigma_u = \sigma. \quad (8)$$

Substituting (7) in the third equation (3) and taking into account :

$$S = |\sqrt{(\xi_v - \xi_u)^2 + \eta_{uv}{}^2}| = \left|\frac{dW_v}{du}\right|, \quad \ldots. \quad (9)$$

we get :

$$-\tau_{uv} = \mu\frac{dW_v}{du} + (H + \sigma)\tan\psi \cdot \quad (10)$$

Considering that the state of stress does not depend on co-ordinate v, the equilibrium equations take the form :

$$\frac{d\sigma_u}{du} = \gamma \cdot \cos\alpha, \quad \ldots. \quad (11)$$

$$\frac{d\tau_{uv}}{du} = -\gamma \cdot \sin\alpha, \quad (12)$$

where α is an angle of slope of the plane and the turn of the co-ordinate axes.

The direct integration of equations (11) and (12) yields :

$$\sigma = \gamma \cdot u \cdot \cos\alpha + p, \quad (13)$$

$$\tau_{uv} = -\gamma \cdot u \cdot \sin\alpha - t, \quad \ldots. \quad (14)$$

where p and t are normal and tangential components of the surface load respectively. Equalizing (10) and (14) and simultaneously introducing value (13), we get :

$$-\mu\frac{dW_v}{du} + \gamma(\sin\alpha - \cos\alpha \cdot \tan\psi)u = (H + p)\tan\psi - t \cdot \quad \ldots. \quad (15)$$

The integration of this equation on the assumption that the displacement velocity in the contact point ($u = h$) of soil and rough inclined plane is equal to zero ($W_v = 0$) results in :

$$W_v = \frac{1}{\mu}\left\{\frac{\gamma}{2}(\sin\alpha - \cos\alpha \cdot \tan\psi)(h^2 - u^2) - [(H + p)\tan\psi - t](h - u)\right\} \quad \ldots. \quad (16)$$

The boundary of "rigid layer" with the equal strain velocity $\left(\frac{dW_v}{du} = 0\right)$ within its thickness can be determined from the condition that $\frac{dW_v}{du} = 0$ will also take place in the contact point ($u = u_0$). The introduction of this condition in (15) gives :

$$u_0 = \frac{(H + p)\tan\psi - t}{\gamma(\sin\alpha - \cos\alpha \cdot \tan\psi)} - \quad (17)$$

— the thickness of the rigid layer.

Thus, the visco-plastic flow of soil is accurring within $h \geqslant u \geqslant u_0$ where it is described by formula (16). The diagram of the displacement velocities constructed from formula (16) is shown in Fig. 2.

The condition of the absence of visco-plastic flow when the rigid layer spreads to the contact point with the inclined plane ($u_0 = h$) enables the correct angle of slope of the plane to be determined by :

$$\cos\alpha_0 = \frac{(H + p)\tan\psi - t}{\gamma \cdot h(1 + \tan^2\psi)} \times \left\{\sqrt{\left[\frac{\gamma \cdot h}{(H + p\tan\psi - t}\right]^2 (1 + \tan^2\psi) - 1} - \tan\psi\right\}, \quad \ldots. \quad (18)$$

which is a direct resultant from ratio (17).

3. The Visco-Plastic Flow of a Soil Layer between two Rough Surfaces

Let us consider the plane visco-plastic flow of a soil layer between two rough surfaces (Fig. 3).

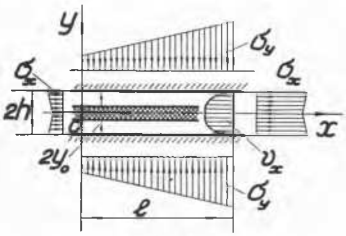

Fig. 3

We assume that at some distance from the adges of the layer the displacement velocities do not depend on co-ordinates and that the motion is steady.

The imcompressibility condition of the material :

$$\xi_x + \xi_y = 0 \quad (19)$$

and corresponding deductions like those above give the following results :

$$\xi_x = \xi_y = 0, \quad (20)$$

$$\eta_{xy} = S = \frac{dv_x}{dy}. \quad (21)$$

Placing (20) and (21) into equations (2) we obtain :

$$\sigma_x = \sigma_y = \sigma, \quad \dots. \quad (22)$$

$$-\tau_{xy} = \mu\frac{dv_x}{dy} + (H + \sigma)\tan\psi. \quad \dots. \quad (23)$$

The equilibrium equations take the form :

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma}{\partial y} = 0, \quad (24)$$

$$\frac{\partial\sigma}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0. \quad (25)$$

Introducing (23) into (24) we get :

$$-\frac{\partial\sigma}{\partial x}\tan\psi + \frac{\partial\sigma}{\partial y} = 0. \quad \dots. \quad (26)$$

Solving the Cauchy problem for this equation by the characteristic method and assuming that in the middle of the layer ($y = 0$) average normal stress is distributed by the linear law, we get :

$$\sigma = \sigma_0 + I(x + y\cdot\tan\psi), \quad (27)$$

where σ_0 — is a given value and I — a given gradient of the mean normal stress.

Substituting (23) into (25) and taking account of (27) we obtain :

$$-\mu\frac{d^2v_x}{dy^2} + I(1 - \tan^2\psi) = 0. \quad \dots. \quad (28)$$

The integration of this equation in boundary conditions corresponding to no displacement ($v_x = 0$) in the contact point with rough surfaces ($y = \pm h$) and to the constant velocity $\left(\frac{dv_x}{dy} = 0\right)$ within the "rigid layer" and on its boundary ($y = \pm y_0$) leads to formula :

$$v_x = \frac{I}{2\mu}(1 - \tan^2\psi)[(y^2 - h^2) - 2(y - h)y_0], \quad \dots. \quad (29)$$

which is valid at $y \geqslant y_0$.

The thickness ($2y_0$) of the rigid layer is obtained from the equilibrium condition of a plot of the soil with length l at the already known stresses on its contour determined from the above formulae. As a result we get formula :

$$y_0 = \frac{I\cdot l + 2(H + \sigma_0)}{2I(1 - \tan^2\psi)}\tan\psi. \quad \dots. \quad (30)$$

Substituting (30) in (29) we get the final expression of the displacement velocities :

$$v_x = \frac{1}{2\mu}\left\{I(1 - \tan^2\psi)(y^2 - h^2) - [I\cdot l + 2(H + \sigma_0)](y - h)\tan\psi\right\} \quad \dots. \quad (31)$$

It is expedient to find an expression for critical pressure gradient I_0 at which the rigid layer occupies the whole of the space ($y_0 = \pm h$) between the rough surfaces and no visco-plastic flow takes place. Using the given condition, we get directly from (30) :

$$I_0 = \frac{2(H + \sigma_0)\tan\psi}{2(1 - \tan^2\psi)h - l\cdot\tan\psi} \quad \dots. \quad (32)$$

The diagrams of stresses and displacement velocities constructed by the given solution are given in Fig. 3.

The solution obtained can be used to estimate the stability of slopes in foundations of which a visco-plastic layer of soil is liable to be compressed by the weight of a sloping mass.

4. Visco-Plastic Flow of a Hollow Circular Cylinder

Let us consider a plane axial symmetrical problem of visco-plastic flow of a hollow circular cylinder (Fig. 4) subject to compression or rupture by outer or inner radial pressure respectively.

The imcompressileility condition of a medium yields :

$$\xi_r + \xi_\varphi = 0, \quad (33)$$

where $\xi_r = \frac{dv}{dr}, \xi_\varphi = \frac{v}{r}$

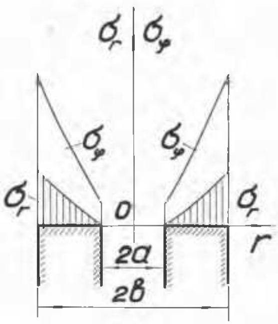

Fig. 4

hence the equation :

$$\frac{dv}{dr} + \frac{v}{r} = 0, \qquad \ldots. \quad (34)$$

the solution of which is obtained at once :

$$v = \frac{c_1}{r}. \qquad (35)$$

The arbitrary constant is determined from the conditions : (1) at $r = b$ $v|_{r=b} = v_b$ (cylinder compression) and (2) at $r = a$ $v|_{r=a} = -v_a$ (cylinder rupture) which result in :

$$c_1 = + v_b, b, \qquad (36)$$

or

$$c_1 = - v_a \cdot a. \qquad (37)$$

The strain velocities will be now expressed by formulae :

$$\xi_r = -\frac{c_1}{r^2}, \quad \xi_\varphi = \frac{c_1}{r^2}, \quad \eta_{r\varphi} = 0, \qquad \ldots. \quad (38)$$

the introduction of which into the equations of state taking into account

$$S = \sqrt{(\xi_r - \xi_\varphi)^2} = 2\left|\frac{c_1}{r^2}\right| \qquad \ldots. \quad (39)$$

and the change of indices yields *) :

$$\left.\begin{aligned} \sigma_r - \sigma &= -2\mu\frac{c_1}{r^2} - \operatorname{sign} v\cdot(H + \sigma)\tan\psi_1 , \\ \sigma_u - \sigma &= 2\mu\frac{c_1}{r^2} + \operatorname{sign} v\cdot(H + \sigma)\tan\psi_1 , \end{aligned}\right\} \qquad (40)$$

The equilibrium equation has the form :

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0. \qquad (41)$$

* Function sign v is determined by equalities : sign $v = +1$ at $v > 0$ and sign $v = -1$ at $v < 0$.

Substituting corresponding expressions resulting from ratios (40) into equilibrium equation (41), we finally have :

$$(1 - \operatorname{sign} v\cdot\tan\psi)\frac{d\sigma}{dr} - \operatorname{sign} v\,\frac{2(H + \sigma)\tan\psi}{r} = 0. \qquad \ldots. \quad (42)$$

Integration (42) yields :

$$\left[(H + \sigma)\tan\psi\right]^{\frac{1-\operatorname{sign} v\cdot\tan\psi}{2\tan\psi}} = c_2\cdot r^{\operatorname{sign} v} \qquad (34]$$

Let us further confine the problem to a consideration of the crushing of the cylinder by outward radial pressure. It is clear that the inner surface $r = a$ has an average normal stress $\sigma|_{r=a} = \sigma_a$.

Then, substituting this boundary condition into (43) and taking into account (36) we obtain the ratio :

$$H + \sigma = (H + \sigma_a)\left(\frac{r}{a}\right)^{\frac{2\tan\psi}{1-\tan\psi}} \qquad 44)$$

Substituting the result into the first equation of state (40), taking into account again (36) and expressing pressure p_a we finally get :

$$\left.\begin{aligned} \sigma_r &= 2\mu\cdot v_b\,\frac{b}{a^2}\left[\left(\frac{r}{a}\right)^{\frac{2\tan\psi}{1-\tan\psi}} - \left(\frac{a}{r}\right)^2\right] + \\ &+ (H + p_a)\left(\frac{r}{a}\right)^{\frac{2\tan\psi}{1-\tan\psi}} - H. \end{aligned}\right\} \qquad (45)$$

The circular stress will be expressed as follows :

$$\left.\begin{aligned} \sigma_\varphi &= 2\mu\cdot v_b\,\frac{b}{a^2}\left[\frac{1 + \tan\psi}{1 - \tan\psi}\left(\frac{z}{a}\right)^{\frac{2\tan\psi}{1-\tan\psi}} + \left(\frac{a}{r}\right)^2\right] + \\ &+ \frac{1 + \tan\psi}{1 - \tan\psi}(H + p_a)\left(\frac{r}{a}\right)^{\frac{2\tan\psi}{1-\tan\psi}} - H. \end{aligned}\right\} \qquad (46)$$

The stress diagrams from these formulae are given in Fig. 4.

The solution can be used to compute the strength of a protective jacket made of frozen soil when sgafts are sunk by the freezing method.

5. Main Equations of the Plane Problem of Visco-Plastic Flow of Soils

In a more general case, when the medium is thought to be imcompressible the mechanical equations of state (2) take the form :

$$\left.\begin{aligned} \sigma_x - \sigma &= 2\left[\mu + \frac{(H + \sigma)\tan\psi}{S}\right]\cdot\left(\xi_x - \frac{1}{3}\xi\right), \\ \sigma_y - \sigma &= 2\left[\mu + \frac{(H + \sigma)\tan\psi}{S}\right]\cdot\left(\xi_y - \frac{1}{3}\xi\right), \\ \tau_{xy} &= \left[\mu + \frac{(H + \sigma)\tan\psi}{S}\right]\eta_{xy} . \end{aligned}\right\} \qquad (47)$$

where ξ — is the cubic strain velocity related to the average normal stress by the ratio :

$$\sigma = k\cdot\xi, \qquad (48)$$

where k is the modulus of cubic viscosity.

Expressing the stress components in equations (47) through strain components and introducing values $\nu = \frac{H}{k}$ and $\bar{\mu} = \frac{\mu}{k}$ we make some transformations and finally get :

$$\left.\begin{aligned}\sigma_x &= k\left\{2\,[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left[\left(\frac{\xi_x}{S}\right)-\frac{1}{3}\left(\frac{\xi}{S}\right)\right]+\xi\right\},\\ \sigma_y &= k\left\{2\,[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left[\left(\frac{\xi_y}{S}\right)-\frac{1}{3}\left(\frac{\xi}{S}\right)\right]+\xi\right\},\\ \tau_{xy} &= k\,[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left(\frac{\eta_{xy}}{S}\right).\end{aligned}\right\} \quad (49)$$

The equilibrium equations for an inponderable medium in the Cartesian co-ordinates take the form :

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y} &= 0,\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y} &= 0.\end{aligned}\right\} \quad (50)$$

Substituting ratios (49) for these equations and making some differentiation, we shall finally have :

$$\left.\begin{aligned}&\frac{\partial}{\partial x}[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left[\left(\frac{\xi_x}{S}\right)-\left(\frac{\xi}{3S}\right)\right]+\\ &+\frac{\partial}{\partial y}[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left(\frac{\eta_{xy}}{2S}\right)+\frac{1}{2}\frac{\partial\xi}{\partial x}=0,\\ &\frac{\partial}{\partial y}[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left[\left(\frac{\xi_y}{S}\right)-\left(\frac{\xi}{3S}\right)\right]+\\ &+\frac{\partial}{\partial x}[\bar{\mu}\cdot S+(\nu+\xi)\tan\psi]\cdot\left(\frac{\eta_{xy}}{2S}\right)+\frac{1}{2}\frac{\partial\xi}{\partial y}=0.\end{aligned}\right\} \quad (51)$$

The equations are equilibrium equations in displacement velocities for a compressible visco-plastic medium.

Analogous equations for a plastic medium were first mentioned by H. Hencky (R. 1). Excluding the strain velocity components from the system of equations (51) by means of the Cauchu ratios

$$\left.\begin{aligned}\xi_x = \frac{\partial v_x}{\partial x},\quad \xi_y = \frac{\partial v_y}{\partial y},\quad \xi = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\\ \eta_{xy} = \frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x},\end{aligned}\right\} \quad (52)$$

it is possible to obtain a system of two differential equations for two unknown functions $v_x(x, y)$ and $v_y(x, y)$. The integration of these equations can be effected by numerical methods.

It is not difficult to determine the state of stress directly from formulae (49) after the strain velocities had been determined by solving appropriate systems of equations.

It is of considerable interest to obtain equations analogous to (51) under condition of the medium imcompressibility usually assumed in the plasticity theory. Instead of (48) this condition yields the equation :

$$\xi_x + \xi_y = 0, \quad (53)$$

which is identically satisfied if the function of flow $\varphi(x, y)$ is introduced by means of ratios :

$$v_x = \frac{\partial\varphi}{\partial y},\quad v_y = -\frac{\partial\varphi}{\partial x}. \quad (54)$$

After the substitution of equation of state (2) into the equilibrium equations (50) and taking into account (54) and (52), we obtain the following system of equations :

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left\{\left[\mu+\frac{(H+\sigma)\tan\psi}{S}\right]\frac{\partial^2\varphi}{\partial x\,\partial y}\right\}+\\ &+\frac{1}{2}\frac{\partial}{\partial y}\left\{\left[\mu+\frac{(H+\sigma)\tan\psi}{S}\right]\cdot\left(\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial x^2}\right)\right\}+\\ &\frac{1}{2}\frac{\partial}{\partial x}(H+\sigma)=0,\\ &-\frac{\partial}{\partial y}\left\{\left[\mu+\frac{(H+\sigma)\tan\psi}{S}\right]\frac{\partial^2\varphi}{\partial x\,\partial y}\right\}+\\ &+\frac{1}{2}\frac{\partial}{\partial x}\left\{\left[\mu+\frac{(H+\sigma)\tan\psi}{S}\right]\cdot\left(\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial x^2}\right)\right\}-\\ &\frac{1}{2}\frac{\partial}{\partial y}(H+\sigma)=0,\end{aligned}\right\} \quad (55)$$

which contains two functions unknown $\varphi(x, y)$ and $\sigma(x, y)$. The integration of this system can be also carried out by numerical methods.

If we assume the medium to be non-viscous ($\mu = 0$) and introduce the new function $\chi(x, y)$ by means of ratios :

$$\left.\begin{aligned}\sin\chi &= \frac{2\dfrac{\partial^2\varphi}{\partial x\,\partial y}}{\sqrt{\left(2\dfrac{\partial^2\varphi}{\partial x\,\partial y}\right)^2+\left(\dfrac{\partial^2\varphi}{\partial y^2}-\dfrac{\partial^2\varphi}{\partial x^2}\right)^2}},\\ \cos\chi &= \frac{\dfrac{\partial^2\varphi}{\partial y^2}-\dfrac{\partial^2\varphi}{\partial x^2}}{\sqrt{\left(2\dfrac{\partial^2\varphi}{\partial x\,\partial y}\right)^2+\left(\dfrac{\partial^2\varphi}{\partial y^2}-\dfrac{\partial^2\varphi}{\partial x^2}\right)^2}},\end{aligned}\right\} \quad (56)$$

the system of equations (55) will take a simpler form :

$$\left.\begin{aligned}&\frac{\partial\chi}{\partial x}\cos\chi-\frac{\partial\chi}{\partial y}\sin\chi+\frac{\partial\sigma^*}{\partial x}\sin\chi+\\ &+\frac{\partial\sigma^*}{\partial y}\cos\chi+\frac{1}{\tan\psi}\cdot\frac{\partial\sigma^*}{\partial x}=0,\\ &-\frac{\partial\chi}{\partial x}\sin\chi-\frac{\partial\chi}{\partial y}\cos\chi+\frac{\partial\sigma^*}{\partial x}\cos\chi-\\ &-\frac{\partial\sigma^*}{\partial y}\sin\chi+\frac{1}{\tan\psi}\cdot\frac{\partial\sigma^*}{\partial y}=0,\end{aligned}\right\} \quad (57)$$

where σ^* is determined by formula :

$$\sigma^* = \log e\,(H+\sigma). \quad (58)$$

The quasi linear equation system obtained can be integrated out by the method of characteristics.

In the cases of incompressible medium the state of stress is determined directly by formula (2).

The equations can be applied for solving a nimber of important practical problems each of which represents a special investigation beyond the limits of the present paper.

References

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