Influence of Flexural Rigidity of Superstructure on the Distribution of Contact Pressure and Bending Moments of an Elastic Combined Footing

L'influence de la Rigidité du Bâtiment sur la Répartition de la Pression et le Moment Fléchissant Agissant sur une Semelle de Fondation Élastique

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Summary

The influence of the rigidity of superstructures and of the degree of fixing of the supports in the foundation beam on the distribution of contact pressure and on the bending moment is explored for an elastic, combined footing with three symmetrically arranged single loads. In order to deal with the described conditions arithmetically, a generally applicable approximation method is developed.

A series of comparative computations with presumed varying degrees of rigidity revealed, as the most important result, the particularly great influence of the degree of fixity of the supports on the foundation beam on both the distribution of contact pressure and the bending moments. The influence of the rigidity of the superstructure is not so great. The higher the degree of fixing and the stiffer the soil, the more favourable the distribution of the bending moments in the foundation beam.

Sommaire

Considérant une semelle de fondation élastique avec trois charges disposées symétriquement, on examine l'influence de la rigidité de la construction du bâtiment sur la répartition de la pression exercée sur la semelle et sur les moments fléchissants ainsi que l'influence du degré d'encastrement des supports dans la semelle de fondation. Afin de permettre le calcul numérique des conditions statiques décrites, un procédé de calcul approximatif, de validité générale, sera tout d'abord développé.

Une série de calculs comparatifs, avec différents degrés de rigidité, montre entre autres, comme résultat de plus grande importance, une influence particulièrement grande du degré d'encastrement des supports dans la semelle de fondation sur la répartition de la pression sur la semelle ainsi que sur les moments fléchissants. L'influence de la rigidité de la construction du sommet est moins grande. Plus le degré d'encastrement est grand, plus le terrain est ferme, et plus la répartition des moments fléchissants à la semelle de fondation est favorable.

It is known from experience that the bending moments of combined footings change considerably with the distribution of contact pressure. For this reason the highest possible degree of accuracy should be achieved in all methods for the computation of distribution of contact pressure under combined footings. Apart from the mathematical interpretation of the actual nature of the forces in the subsoil it is especially difficult in computing to evaluate the stiffness of the superstructure. In order to simplify the matter an unrestrained and vertical freedom of movement of the columns in relation to each other is generally assumed, and, further, that they are hinged at their base to the combined footing. In reality, however, the more or less rigid superstructure prevents a free movement of the columns. Apart from this the columns are often rigidly connected with the combined footing. These facts influence the contact pressure and the bending movements. With regard to the economy and the safety of our buildings it seems, however, to be especially important to examine the described relations more closely. Sufficient experience has not been gained up to the present as the corresponding computation methods and measuring data have not been available.

With the help of the following examples the above named relations are represented and evaluated for the first time. To create some mathematical basis a general computation method for the simple case of a combined footing with three symmetrically positioned point loads, taking into consideration the differences in stiffness in the superstructure already mentioned and the all-round elastic properties of the subsoil, will be developed (Fig. 1). It is useful to examine the following limiting cases of stiffness:

(1) Perfectly flexible superstructure (the columns can move freely in a vertical direction with respect to each other). The columns are hinged at their bases to the combined footing (Fig. 2a).

(2) Rigid superstructure (the columns cannot move vertically relative to each other). The columns are hinged at their bases to the combined footing (Fig. 3a).

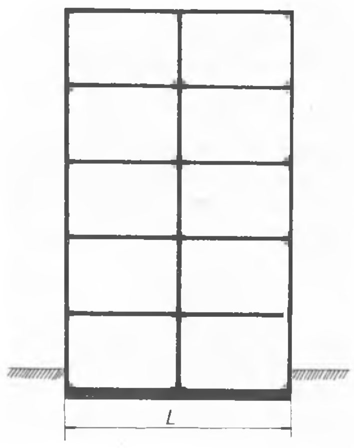

Fig. 1 Symmetrical framed structure with three columns on an elastic combined footing

Une structure symétrique à charpente avec 3 colonnes sur une fondation sur semelle élastique

(3) Perfectly flexible superstructure. The columns are rigidly connected to the combined footing (Fig. 4a).

(4) Rigid superstructure. The columns are rigidly connected to the combined footing (Fig. 5a).

The actual rigidity, which influences the magnitude of the column loads and the distribution of contact pressure, will lie between these limiting cases. An exact computation of the elastic combined footing is prevented by the complete statical

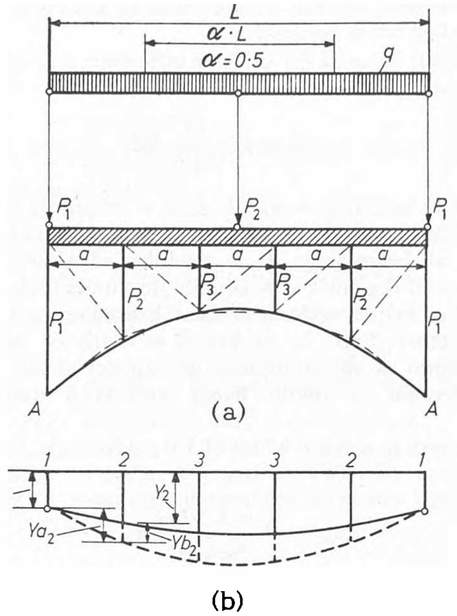

Fig. 2 Soil pressure and settlements resulting from flexural rigidity, Case No. 1

La pression de contact et les tassements dans le cas de rigidité No. 1

indeterminacy due to the continuous bedding on the compressible subsoil. Therefore the solution must be found by an approximate method, the mathematical treatment of which must still be within reasonable limits.

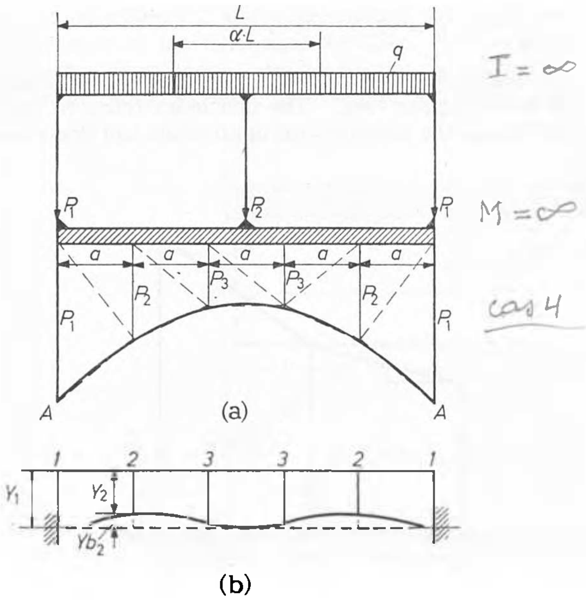

Fig. 3 Soil pressure and settlements resulting from flexural rigidity, Case No. 2

La pression de contact et les tassements dans le cas de rigidité No. 2

The general computation method is derived as follows: Figs. 2a to 5a each show an elastic beam of finite length L with 3 single loads in symmetrical position on a compressible bedding. In proportion to the length, the width of the beam will be so small that it can be treated with sufficient accuracy as if dealing with a plane problem. The symmetrical curve A–A shows the distribution of contact pressure under the beam which is unknown at first. If, for instance, this beam is

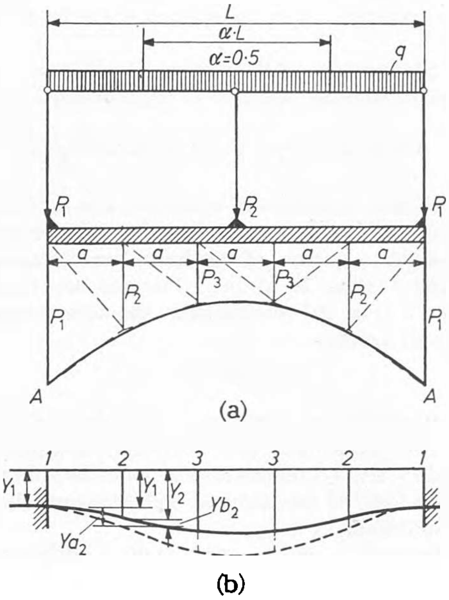

Fig. 4 Soil pressure and settlements resulting from flexural rigidity, Case No. 3

La pression de contact et les tassements dans le case de rigidité No. 3

divided into 5 equal sections of length a, then the soil stress curve can approximately be substituted by a trace polygonal with the ordinates p_1 to p_3. The area of contact pressure

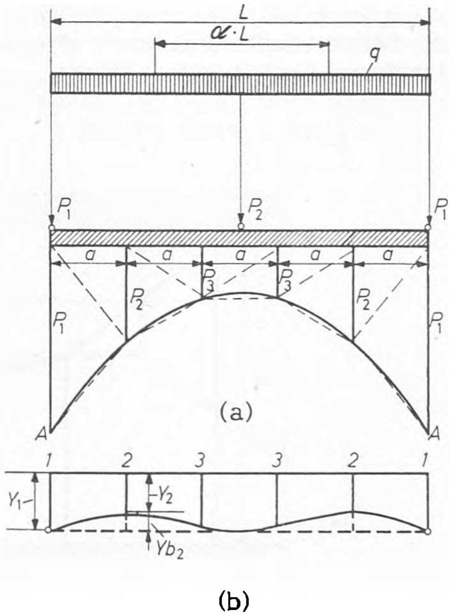

Fig. 5 Soil pressure and settlements resulting from flexural rigidity, Case No. 4

La pression de contact et les tassements dans le cas de rigidité No. 4

which is now bounded by finite lines is composed of 10 stress triangles. Consequently the foundation beam can be looked upon as a beam in equilibrium on two compressible end supports. It is loaded on its upper surface by the three symmetrically positioned single loads and on its lower surface by the symmetrically acting 10 soil stress triangles.

In Figs. 2a and 3a the end supports can move freely, and in Figs. 4a and 5a they are rigidly fixed. In the Figs. 3a and 5a the central load P_2 cannot produce deflections because of the rigid superstructure. It is maintained at the height of the end supports.

For the computation of the required 3 soil stress ordinates p_1 to p_3 we begin with the equation of equilibrium $\Sigma V = 0$:

$$0{\cdot}2.p_1 + 0{\cdot}4.p_2 + 0{\cdot}4.p_3 = \frac{2.P_1 + P_2}{L} \quad \ldots. \; (1)$$

The other two equations of condition are derived from the deformations of the beam and the subsoil. For this purpose the settlements y_2 and y_3 of the beam are determined at the points 2 and 3 (Figs. 2b to 5b). For instance, the settlement y_2 at point 2 (Fig. 2b) results from the superposition of the following part values:

$$y_2 = y_1 + y_{a2} - y_{b2} \quad \ldots. \; (2)$$

y_1 is the settlement at the beam end, y_{a2} is free deflection of the beam on two immovable end supports, resulting from the central load P_2, and y_{b2} represents the opposite deflection resulting from the load of the required soil pressure, always under the same conditions as in y_{a2}.

The settlements y_1 and y_2 result from a settlement analysis with the required soil pressure as a perfectly flexible load (Fig. 6a). To simplify the computation the stress intervals are graduated according to Fig. 6. Thus uniformly distributed perfect flexible rectangular loads are formed whose shares in the settlement must be superimposed correspondingly. To make matters easier, the slim beam will be dealt with as a plane problem; consequently its cross direction must be looked upon as being rigid. Therefore the settlement analysis of the perfectly flexible load is executed at the so-called 'characteristic cross-section' (1) (Fig. 6 b), i.e. at the point, where the settlement of the load, which also in the cross-direction is supposed to be perfectly flexible, equals the settlement of the rigid load. The general settlement formula reads as follows:

$$y_i = \frac{L}{K}(p_1\phi_{i,\,1} + p_2\phi_{i,\,2} + p_3\phi_{i,\,3}) \quad \ldots. \; (3)$$

L = length of beam (cm), K = constant coefficient of compressibility of the subsoil (kg/cm²), ϕ are dimensionless influence coefficients of settlement which result from the prevailing settlement analysis. The first index refers to the point of the settlement and the second index to the influence of the corresponding stress ordinate.

The general formula for the free deflection as a result of the central load P_2 is as follows:

$$y_{a2} = P_2.\frac{L^3}{E_b.I}.\delta_2 \quad \ldots. \; (4)$$

P_2 = central load (kg) = $q.\alpha.L.B$, L = length of beam (cm), $I = (B.d^3)/12$, the moment of inertia of the beam (cm⁴), d = thickness of beam (cm), E_b = modulus of elasticity of the beam material (kg/cm²), q = equally distributed load (kg/cm²) (Figs. 2a to 5a), α = share of load belonging to P_2 (dimensionless factor) (Figs. 2a to 5a), B = width of beam (cm), δ is a function of the conditions of support at the beam end (freely movable or rigidly fixed) and is a dimensionless factor.

With respect to a beam width of 1·0 cm formula 2 changes as follows:

$$y_{a2} = 9.\frac{\alpha.L}{E_b}.\left(\frac{L}{d}\right)^3.\epsilon_2 \quad \ldots. \; (5)$$

$$\epsilon_2 = 12.\delta_2$$

The general formula for the opposite deflections resulting from the triangular loads of the soil pressure, relating with reference to a beam width of 1·0 cm, is as follows:

$$y_{b2} = \frac{L^4}{E_b.I}(p_1.\theta_{2,\,1} + p_2.\theta_{2,\,2} + p_3.\theta_{2,\,3}) \quad \ldots. \; (6)$$

$$y_{b2} = \frac{L}{E_b}.\left(\frac{L}{d}\right)^3(p_1.\eta_{2,\,1} + p_2.\eta_{2,\,2} + p_3.\eta_{2,\,3}) \quad \ldots. \; (7)$$

$$\eta = 12.\theta$$

θ = dimensionless factor dependent on the prevailing position of the triangular load. The first index refers to the point of settlement, the second to the appropriate soil stress ordinate.

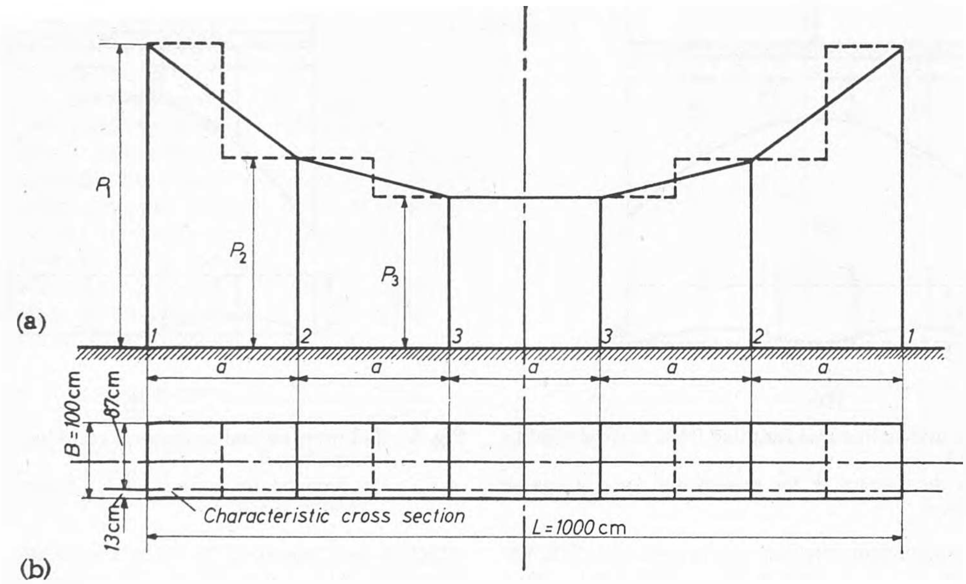

Fig. 6 Stepped curve of soil pressure simplifying the settlement analysis
La courbe représentant pression de contact modifiée en forme d'escalier pour faciliter le calcul du tassement

After inclusion in formula 2 the result is:

$$\frac{L}{K}.(p_1.\phi_{2\cdot1} + p_2.\phi_{2\cdot2} + p_3.\phi_{2\cdot3})$$

$$= \frac{L}{K}.(p_1.\phi_{1\cdot1} + p_2.\phi_{1\cdot2} + p_3.\phi_{1\cdot3}) + 9.\frac{\alpha.L}{E_b}\left(\frac{L}{d}\right)^3.\epsilon_2 -$$

$$\frac{L}{E_b}.\left(\frac{L}{d}\right)^3(p_1.\eta_{2\cdot1} + p_2.\eta_{2\cdot2} + p_3.\eta_{2\cdot3})$$

Introducing $C = \frac{K}{E_b}.\left(\frac{L}{d}\right)^3$ as a simplifying factor the above formula changes as follows:

$$(\eta_{2\cdot1}.C + \phi_{2\cdot1} - \phi_{1\cdot1}).p_1 + (\eta_{2\cdot2}.C + \phi_{2\cdot2} - \phi_{1\cdot2}).p_2 + (\eta_{2\cdot3}.C + \phi_{2\cdot3} - \phi_{1\cdot3}).p_3 = 9.\alpha.\epsilon_2.C \quad \ldots . \ (8)$$

In the same way, the third equation of condition is derived at point 3:

$$(\eta_{3\cdot1}.C + \phi_{3\cdot1} - \phi_{1\cdot1})p_1 + (\eta_{3\cdot2}.C + \phi_{3\cdot2} - \phi_{1\cdot2}).p_2 + (\eta_{3\cdot3}.C + \phi_{3\cdot3} - \phi_{1\cdot3}).p_3 = 9.\alpha.\epsilon_3.C \quad \ldots . \ (9)$$

For the cases of stiffness 1 and 3 (Figs. 2a and 4a) the bending factors η and ϵ are determined in Tables 1 and 3:

Table 1

Case of rigidity No. 1	
$\eta_{2,\,1} = 0{\cdot}00624$	$\eta_{3,\,1} = 0{\cdot}00944$
$\eta_{2,\,2} = 0{\cdot}03376$	$\eta_{3,\,2} = 0{\cdot}05280$
$\eta_{2,\,3} = 0{\cdot}05280$	$\eta_{3,\,3} = 0{\cdot}08656$
$\epsilon_2 = 0{\cdot}142$	$\epsilon_3 = 0{\cdot}236$

In the cases of stiffness 2 and 4 the central column P_2 is immovable because of the rigid superstructure. It therefore cannot create the deflections y_{a2} and y_{a3} so that the deflection ordinates ϵ become equal to zero. But the opposite deflections y_{b2} and y_{b3} due to the soil stress triangles are computed so as to allow the load P_2 to act as an immovable central support.

The bending factors computed with this assumption are found in Tables 2 and 4.

Table 2

Case of rigidity No. 2	
$\eta_{2,\,1} = 0{\cdot}00065088$	$\eta_{3,\,1} = 0{\cdot}00015104$
$\eta_{2,\,2} = 0{\cdot}00240640$	$\eta_{3,\,2} = 0{\cdot}00069120$
$\eta_{2,\,3} = 0{\cdot}00099272$	$\eta_{3,\,3} = 0{\cdot}00045776$
$\epsilon_2 = 0$	$\epsilon_3 = 0$

Table 3

Case of rigidity No. 3	
$\eta_{2,\,1} = 0{\cdot}00048$	$\eta_{3,\,1} = 0{\cdot}00080$
$\eta_{2,\,2} = 0{\cdot}00432$	$\eta_{3,\,2} = 0{\cdot}00864$
$\eta_{2,\,3} = 0{\cdot}00800$	$\eta_{3,\,3} = 0{\cdot}01936$
$\epsilon_2 = 0{\cdot}022$	$\epsilon_3 = 0{\cdot}056$

Table 4

Case of rigidity No. 4	
$\eta_{2,\,1} = 0{\cdot}00018432$	$\eta_{3,\,1} = 0{\cdot}00004736$
$\eta_{2,\,2} = 0{\cdot}00108160$	$\eta_{3,\,2} = 0{\cdot}00039680$
$\eta_{2,\,3} = 0{\cdot}00053408$	$\eta_{3,\,3} = 0{\cdot}00035584$
$\epsilon_2 = 0$	$\epsilon_3 = 0$

The sometimes tedious calculation of the influence coefficients of settlement by means of a settlement analysis can be avoided

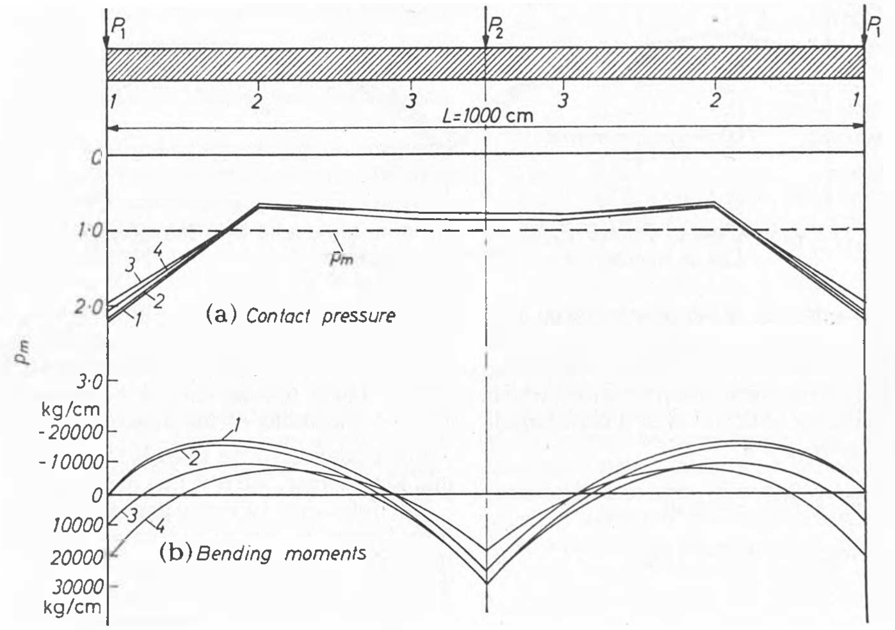

Fig. 7 Cases of flexural rigidity 1–4 based on soft clay $K = 50$ kg/cm²
Cas de rigidité 1–4, sur argile molle

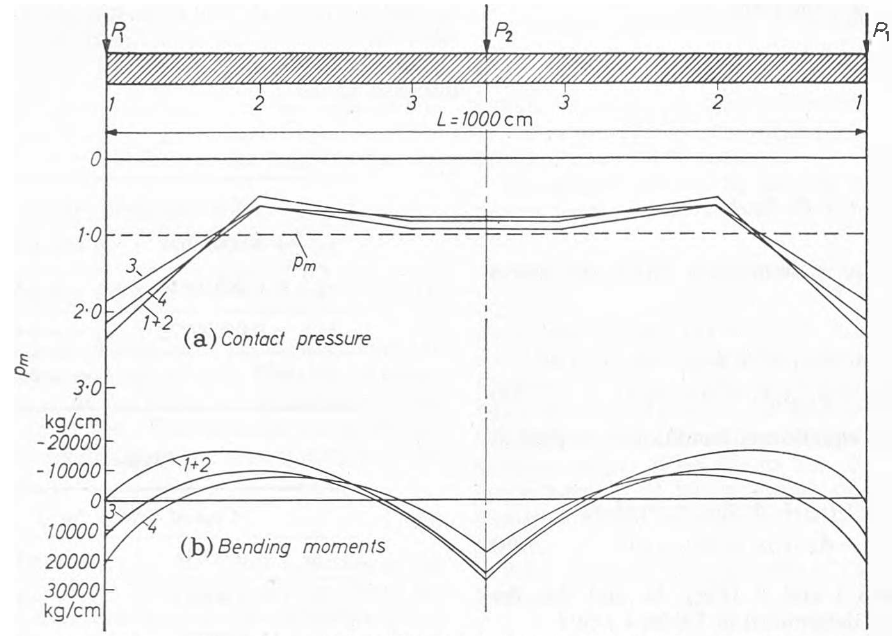

Fig. 8 Cases of flexural rigidity 1–4 based on stiff clay $K = 134{\cdot}4$ kg/cm²
Cas de rigidité 1–4 sur argile raide

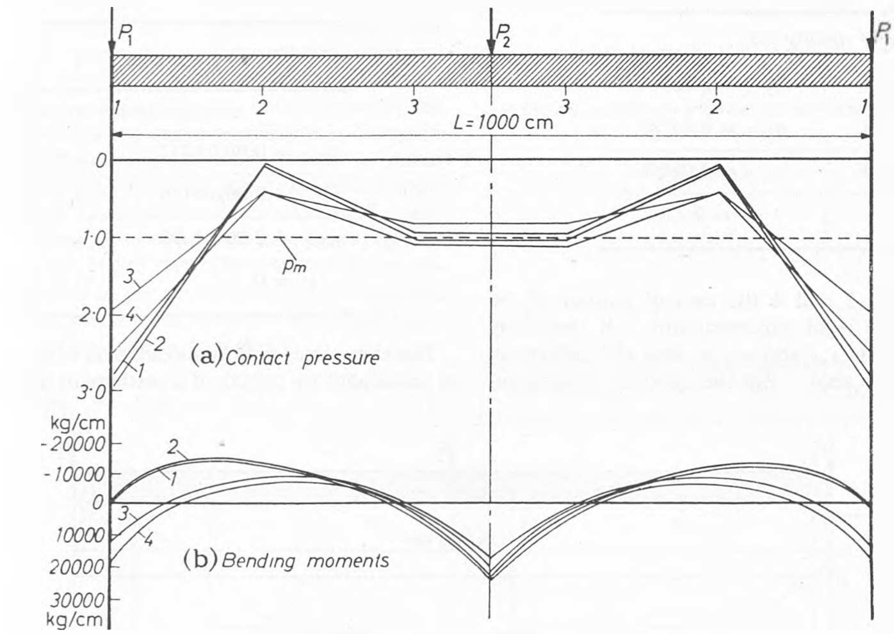

Fig. 9 Cases of flexural rigidity 1–4 based on loose sand $K = 500$ kg/cm²
Cas de rigidité 1–4 sur sable meuble

when the subsoil down to a sufficient depth may be taken as a perfectly elastic half space. Using the settlement formula 10 by Boussinesq for the corner of a perfectly flexible, uniformly distributed rectangular load (2) complete algebraic terms, which can be calculated numerically, are obtained after a corresponding superposition as per Fig. 6a and b.

$$y = p.\frac{L}{K}.\frac{\beta}{\pi}[\alpha\{\log_e(1 + (\alpha^2 + 1)^{\frac{1}{2}}) - \log_e \alpha\} + \log_e\{\alpha + (\alpha^2 + 1)^{\frac{1}{2}}\}] \quad \ldots. \ (10)$$

$$\beta = \frac{b'}{L}$$

$$\alpha = \frac{b'}{l'}$$

b' = width of the prevailing rectangular area (cm), l' = length of the prevailing rectangular area (cm), p = uniformly distributed load (kg/cm²), L = length of combined footing (cm), K = constant coefficient of compressibility of the subsoil (kg/cm²).

For a relation of the sides $B/L = 1/10$ of the combined footing, for instance, the ϕ values are compiled in Table 5.

The following 16 calculated examples are based on a reinforced concrete combined footing ($E_b = 210{,}000$ kg/cm²) with the dimensions $L = 1000$ cm, $B = 100$ cm and $d = 40$ cm. The examples are divided as follows:

Examples 1–4: Cases of flexural rigidity 1 to 4, coefficient of compressibility of the subsoil $K = 50$ kg/cm² (for instance: soft clay), $C = 3{\cdot}72024$.

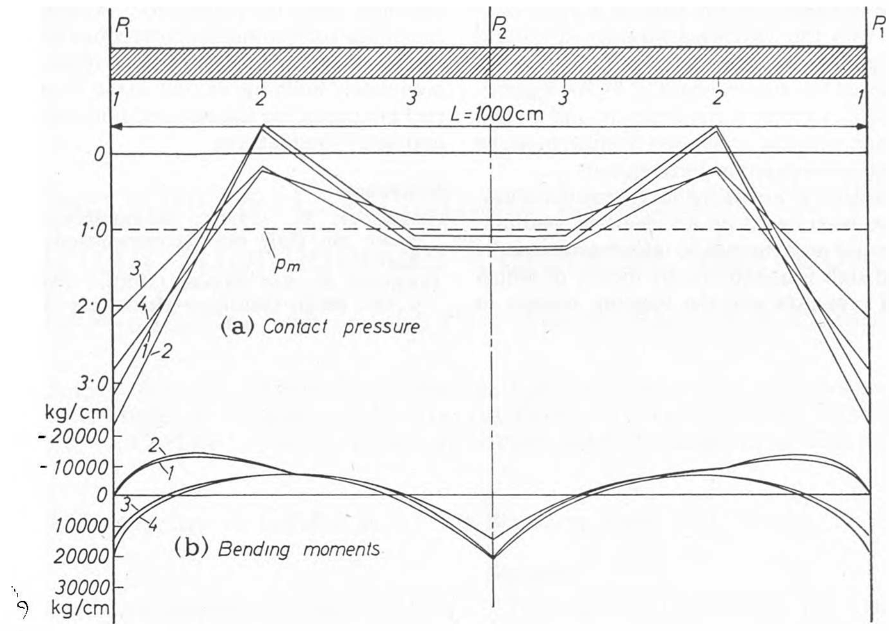

Fig. 10 Cases of flexural rigidity 1–4 based on firm sand K = 1000 kg/cm²
Cas de rigidité 1–4 sur sable compact

Table 5

Point 1	Point 2	Point 3
$\phi_{1,1} = 0{\cdot}071064$	$\phi_{2,1} = 0{\cdot}025205$	$\phi_{3,1} = 0{\cdot}014856$
$\phi_{1,2} = 0{\cdot}041626$	$\phi_{2,2} = 0{\cdot}146107$	$\phi_{3,2} = 0{\cdot}049778$
$\phi_{1,3} = 0{\cdot}026815$	$\phi_{2,3} = 0{\cdot}049778$	$\phi_{3,3} = 0{\cdot}169070$

Examples 5–8: Cases of flexural rigidity 1 to 4, coefficient of compressibility of the subsoil K = 134·4 kg/cm² (for instance: stiff loam), C = 10.

Examples 9–12: Cases of flexural rigidity 1 to 4, coefficient of compressibility of the subsoil K = 500 kg/cm² (for instance: loose sand), C = 37·20238.

Examples 13–16: Cases of flexural rigidity 1 to 4, coefficient of compressibility of the subsoil K = 1000 kg/cm² (for instance: firm sand), C = 74·40476

The computation results (distribution of contact pressure and bending moments) have been compiled in two different ways. In Figs. 7–10 for each kind of soil the 4 different cases of flexural rigidity have been superimposed.

It is remarkable that all contact pressure curves show comparatively little differences. The differences are less marked on soft soil (Fig. 7). They increase slowly with the increasing modulus of compressibility of the soil and this in such a way that the contact pressure concentrates under the point loads. The differences are, however, so small that they are still likely to remain in that region in which displacements of the contact pressure in consequence of plastic deformations of the subsoil make themselves felt. The increasing rigidity of the superstructure and the rigid connection of the columns with the combined footing exercise an equalizing effect on the contact pressure.

The distribution of the bending moments, which was computed for a medium soil pressure p_m = 1·0 kg/cm², corresponds to the contact pressure. In the cases of stiffness 1 and 2 (the columns are standing with a hinge on the combined footing) hardly any difference can be noticed. Also the cases of stiffness 3 and 4 (the columns are rigidly connected with the combined footing) show between themselves only slight differences. In the case of stiffness 4 (rigid superstructure) the most advantageous compensation of the bending moments is obtained. Naturally, in the cases of stiffness 3 and 4 a better distribution appears, as compared with the cases of stiffness 1 and 2, because fixed point moments are effective at the end columns. Therefore this arrangement has advantages in construction.

Table 6

	Case of rigidity	y_1 cm	y_2 cm	y_3 cm		Case of rigidity	y_1 cm	y_2 cm	y_3 cm
Soil No. 1 K = 50 kg/cm²	1	4·07	3·72	3·88	*Soil No. 3* K = 500 kg/cm²	1	0·46	0·26	0·43
	2	4·01	3·74	3·93		2	0·48	0·26	0·41
	3	3·79	3·82	4·11		3	0·37	0·33	0·48
	4	3·95	3·83	3·90		4	0·44	0·33	0·39
Soil No. 2 K = 134·4 kg/cm²	1	1·57	1·29	1·47	*Soil No. 4* K = 1000 kg/cm²	1	0·25	0·09	0·24
	2	1·56	1·30	1·48		2	0·27	0·09	0·22
	3	1·38	1·38	1·61		3	0·19	0·14	0·25
	4	1·50	1·39	1·45		4	0·24	0·14	0·20

The settlements at the computation points 1 to 3 are compiled for a medium soil pressure p_m = 1·0 kg/cm² in Table 6. Very small differences in settlement result and they quickly decrease as the subsoil becomes more firm. Especially remarkable are the small uneven settlements between a perfectly flexible and a rigid superstructure. From this it might be concluded that the growing rigidity of the superstructure has no important beneficial influence on the stresses of the combined footing. This, however, is only valid for an ideal homogeneous subsoil, such as was taken as a basis for the above examples. Actually, subsoil is usually of a heterogeneous nature and shows local differences in firmness, and because of this the differential settlements may increase if the superstructure is perfectly flexible. For the above example—in the case of a rigid connection of the columns with the combined footing—a mutual displacement of two neighbouring columns of only 2·0 cm for instance causes an additional bending moment of 53,700 kg/cm^2. Therefore, the utmost rigidity of the superstructure and a rigid connection of the columns with the combined footing must be looked upon as the most advantageous construction.

In order to be able critically to assess the foregoing computation methods which have been based on an idealized medium, it seems to become more and more urgent to take measurements on completed beam and slab foundations, by means of which the settlements, the soil pressures and the bending stresses in the slab may be computed. Attempts should be made to introduce such empirical corrections and additions in the computation methods by means of observations obtained on the completed building as will make it possible to approach the real picture of the stresses and thus secure still more economical and safer foundations.

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