Three-dimensional Stress Distribution and Slip Surfaces in Earth Works at Rupture

La Répartition Tridimensionnelle des Contraintes et les Surfaces de Glissement dans les Massifs en Terre, à la Rupture

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Summary

Assuming the validity of statical equilibrium of Coulomb's equation and of the minimum principle at rupture, the three-dimensional differential equations of slip surface and stress distribution in earth works are derived.

Sommaire

A partir de l'hypothèse de la validité de l'équation de Coulomb, et du principe minimum à la rupture, on établit les équations différentielles à trois dimensions pour la surface de glissement, et on en déduit la répartition des contraintes dans les massifs en terre.

Basic Principles

The determination of stress distribution for the calculation of the stability of engineering structures is generally based upon the theory of elastic equilibrium. This means that the internal deformations at any point of the structure are proportional to the corresponding developed stresses. Yet for earth pressure problems, since the time of Coulomb, attempts have been made to compute the internal stress distribution according to the plastic equilibrium of materials. In the plastic range stresses are not proportional to the corresponding deformations although the magnitude of stresses also in this range will be similar to that of the elastic range. In the mass shear surfaces will develop, along which relative slip of particles of material and so-called plastic flow will occur.

In recent years there has been a growing tendency to base stability calculations on the plastic behaviour of materials.

(a) For the determination of the differential equations of internal stress distribution—in the elastic range—the following conditions are necessary and sufficient: (1) the equations of statical equilibrium; (2) Hooke's law, which determines the relation of stresses and strains; and (3) compatibility equations.

According to the previous conditions, characteristic differential equations of stress distribution may be determined. This system of equations together with the corresponding boundary conditions enable us to establish the distribution of stresses in any case of engineering practice.

(b) In the plastic range there is no simple stress–strain relation similar to Hooke's law, therefore the compatibility equations of the elastic range cannot be used. Yet from the equations of equilibrium and other principles such systems of equations may be derived which will give the differential equation of stress distribution and of shear surfaces. The necessary conditions are as follows: (1) the statical equilibrium; (2) the law of rupture, defining the relations of stress components along the slip surface; and (3) the minimum principle: this will be the shear surface causing the stress distribution consistent with the smallest load producing rupture or plastic flow.

This approach has already been applied by several authors (Fellenius, etc.), in the case of a previously assumed shape of slip surface. The aim of this paper is to establish the equation of stress distribution in the case of slip surfaces of unknown shape. Calculations on the minimum principle will be made according to the calculus of variations. The solution of this differential equation to correspond with boundary conditions will determine the slip surface and the corresponding stress distribution.

Assumptions

Rigorous mathematical solution is based upon a set of assumptions which correspond approximately to practical conditions. These are as follows:

(a) The material of the body is homogeneous, isotropic and of constant unit weight γ.

(b) The earth mass is limited by continuous surfaces defined by equations, which have continuous first partial derivatives except along some curves finite in number (Fig. 1).

(c) The external load acting on the surface is a continuous distributed load system limited by closed boundaries.

(d) Plastic behaviour occurs as a consequence of gradual increase of load. This load increases in proportion and without change in direction on the loaded boundaries.

(e) Shear or sliding surfaces are continuous surfaces defined by equations of continuous first partial derivatives. Stress distribution along them is also continuous except along some curves finite in number.

(f) The well known Coulomb equation will be applied as the law governing rupture. Accordingly there is still equilibrium at rupture along the slip surface and the normal stress σ and the tangential stress τ will be related by the following expression

$$\tau = \sigma . \tan \phi + c \qquad \ldots . (1)$$

where ϕ is the angle of internal friction and c the cohesion.

From Coulomb's law it follows that at other points of the earth mass

$$\tau < \sigma . \tan \phi + c$$

which means, that equilibrium exists there. Otherwise equilibrium would be impossible and movement would start.

(g) Equilibrium will be investigated, when the load q has been increased to a value where one slip surface reaches the ground level along its full extent. The common cutting line a–a (Fig. 1) divides the mass into two parts A and B. For any further increase of q the equilibrium ceases and part A slides down or starts moving.

Notations

The following symbols and mathematical expressions will be used in the discussion of the minimum problem:

(a) The slip surface will be defined with rectangular coordinates $(x, y\ z)$ by the equation

$$z = z(x, y)$$

and the surface of the body by the equation

$$z^x = z^x(x, y)$$

In the mathematical discussion the vectorial method will be used for purpose of simplification. Accordingly using rectangular coordinates and referring to unit vectors $\bar{i}$, j, k the equation of the slip surface $z = z(x, y)$ is expressed by the vectorial equation

$$\bar{r}(x, y, z) = x.\bar{i} + yj + z(x, y)k = \begin{cases} x \\ y \\ 2z(x, y) \end{cases} \quad \ldots\ldots (2)$$

The symbolic partial derivatives at any point (x, y, z) are as follows:

$$\bar{r}_x = \begin{cases} 1 \\ 0 \\ z_x \end{cases} \qquad \bar{r}_y = \begin{cases} 1 \\ 0 \\ z_y \end{cases} \quad \ldots\ldots (3)$$

The corresponding normal vector:

$$\bar{n} = \bar{r}_x \times \bar{r}_y = \begin{cases} -z_x \\ -z_y \\ 1 \end{cases} \quad \ldots\ldots (4)$$

The unit normal vector:

$$\bar{n}_0 = \frac{\bar{n}}{(\bar{n}^2)^{\frac{1}{2}}} = \frac{\bar{r}_x \times \bar{r}_y}{\{(\bar{r}_x \times \bar{r}_y)2\}^{\frac{1}{2}}} \quad \ldots\ldots (5)$$

One of the tangential vectors:

$$\bar{t} = \bar{r}_x + \psi\bar{r}_y = \begin{cases} 1 \\ \psi \\ z_x + \psi 2y \end{cases} \quad \ldots\ldots (6)$$

where y is a value changing between $-\infty$ and $+\infty$.

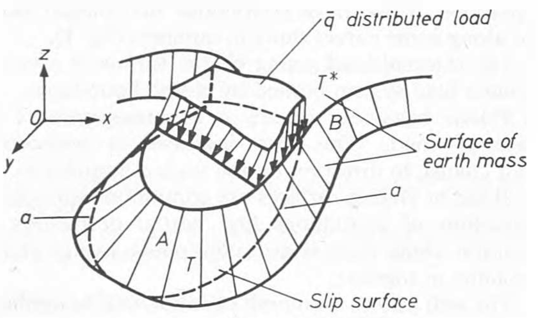


Fig. 1

The unity vector:

$$\bar{t}_0 = \frac{\bar{t}}{(\bar{t}^2)^{\frac{1}{2}}} = \frac{\bar{r}_x + \psi\bar{r}_y}{\{(\bar{r}_x + \psi\bar{r}_y)^2\}^{\frac{1}{2}}} \quad \ldots\ldots (7)$$

The surface element is defined by equation

$$d\tau = (\bar{n}^2)^{\frac{1}{2}}dxdy = \{(\bar{r}_x \times \bar{r}_y)^2\}^{\frac{1}{2}} = (z_x{}^2 + z_y{}^2 + 1)^{\frac{1}{2}}dxdy \quad \ldots\ldots (8)$$

Similar formulae will be used for the terms connected to surface of body $\bar{n}^\star$, $\bar{n}_0{}^\star$, $\bar{t}^\star$, $\bar{t}_0{}'$ and $dT^\star$.

The following vectorial identities will be used:

$$\bar{a}(\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b})\bar{c} \quad \ldots\ldots (a)$$

$$(\bar{a} \times \bar{b})(\bar{b} \times c) = (\bar{a}\bar{b})(\bar{b}\bar{c}) - (\bar{a}\bar{c})\bar{b}^2 \quad \ldots\ldots (b)$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\bar{c})\bar{b} - (\bar{a}\bar{b})\bar{c} \quad \ldots\ldots (c)$$

$$(\bar{a} \times \bar{b})^2 = \bar{a}^2\bar{b}^2 - (\bar{a}\bar{b})^2 \quad \ldots\ldots (d)$$

$$\bar{a} \times \bar{b} = -\bar{b} \times \bar{a} \quad \ldots\ldots (e)$$

$$\bar{i} = \bar{j} \times \bar{k}, \quad \bar{j} = \bar{k} \times \bar{i}, \quad \bar{k} = \bar{i} \times \bar{j} \quad \ldots\ldots (f)$$

$$\bar{i}\bar{j} = \bar{j}\bar{k} = \bar{k}\bar{i} = 0 \quad \ldots\ldots (g)$$

(b) Stresses at the slip surface $\bar{p} = \bar{p}(x, y)$ will be resolved into two components as scalar functions of the coordinates x, y conforming to Coulomb's equation. One of these will be the normal stress

$$\sigma = \sigma(x, y)$$

the second will be the tangential stress

$$\tau = \tau(x, y)$$

As the resultant stress we obtain

$$\bar{p} = \sigma\bar{n}_0 + \tau\bar{t}_0 \quad \ldots\ldots (9)$$

The distributed load will be

$$\bar{q} = \bar{q}(x, y)$$

(c) The surface and body forces acting on the elementary prism along the sliding surface and body surface are as follows:

On the surface of sliding

$$d\bar{P} = \bar{p}d\tau = (\sigma\bar{n}_0 + \tau\bar{t}_0)(\bar{n}^2)^{\frac{1}{2}}dxdy = \left(\sigma\bar{n} + \tau\frac{(\bar{n}^2)^{\frac{1}{2}}}{(\bar{t}^2)^{\frac{1}{2}}}\bar{t}\right)dxdy$$

On the surface of body

$$d\bar{Q} = \bar{q}d\tau^x = \bar{q}(\bar{n}^{xz})^{\frac{1}{2}}dxdy$$

The body force

$$d\bar{G} = \gamma(\bar{r}^x - \bar{r})dxdy$$

The Extremum Problem

Now the extremum problem may be expressed mathematically as follows:

What will be the slip surface $z = z(x, y)$ and the normal stress distribution $\sigma = \sigma(x, y)$ along it, and further the direction of tangential stress $\psi(x, y)$ caused by the smallest resultant Q of the distribution load q system acting on body part A in the case of equilibrium?

Force Q will be a minimum when its components are smallest, because the direction of the q components does not change. Therefore the minimum may be referred to the component Q_3 parallel to the z axis as follows:

$$Q_3 = \iint_T q_3 d\tau^x = \min$$

The equations of conditions are as follows:

Summation of stresses gives the vectorial equation:

$$\bar{I}_1 = \iint_T [(\sigma\bar{n}_0 + \tau\bar{t}_0)(\bar{n}^2)^{\frac{1}{2}} + \gamma(\bar{r}^x - \bar{r}) + \bar{q}(\bar{n}^x)^{\frac{1}{2}}]dxdy = 0$$

The integration range being the projection of the surfaces limited by line a–a (Fig. 1) at the coordinate plane x, y.

This vector equation includes three scalar equations: the moment of stresses at the point 0 (Fig. 2) is equal to zero, i.e.

$$\bar{I}_2 = \iint_T [\bar{r} \times (\sigma\bar{n}_0 + \tau\bar{t}_0)(\bar{n}^2)^{\frac{1}{2}} + \bar{r} \times \gamma(\bar{r}^x - \bar{r}) + \bar{r}^x \times \bar{q}(n^{x2})^{\frac{1}{2}}]dxdy = 0$$

The integrals Q_3, $\bar{I}$ and $\bar{I}_2$ refer to the same range T and their functions depend on functions $\bar{r}(x, y, z)$, $\sigma(x, y)$ and $\psi(x, y)$. Therefore the extremum problem is a problem of the calculus of variations. According to Lagrange the conditional extremum is defined by the general extremum of function

$$\Omega = Q_3 + \bar{\lambda}\bar{I}_1 + \bar{\mu}\bar{I}_2 = \min.$$

where Lagrange's constant factors are regarded as components of vectors and will be chosen conforming to the boundary conditions.

The notations $\bar{\lambda}$ and $\bar{\mu}$ are the vectors of components

$$\bar{\lambda} = \begin{cases} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{cases} \qquad \bar{\mu} = \begin{cases} \mu_1 \\ \mu_2 \\ \mu_3 \end{cases}$$

Written in detail according to equations 5 and 7 we find

$$\Omega = \iint_{\tau}\Big[(\bar\mu \times \bar r + \bar\lambda)\Big(\sigma\bar n + \tau\frac{(\bar n^2)^{\frac12}}{(\bar t^2)^{\frac12}}\bar t\Big) + \gamma(\bar\mu \times \bar r + \bar\lambda)(\bar r^x - \bar r)$$

$$+ (\bar\mu \times \bar r^x + \bar\lambda)\bar q(\bar n^{x2})^{\frac12} + q_3(\bar n^{x2})^{\frac12}\Big]dxdy = \min$$

The expression behind the integration symbol denoted by F is

$$F(\psi, \sigma, z, z_x, z_y, x, y) = \bar\omega\Big(\sigma\bar n + \tau\frac{(\bar n^2)^{\frac12}}{(\bar t^2)^{\frac12}}\bar t\Big) + \gamma\bar\omega(\bar r^x - \bar r)$$

$$+ (\bar\mu \times \bar r^x + \bar\lambda)\bar q(\bar n^{x2})^{\frac12} + q_3(\bar n^{x2})^{\frac12} \quad \ldots . \ (10)$$

If we introduce the notation

$$\bar\omega = \bar\mu \times \bar r + \bar\lambda = \begin{cases}\mu_2 z - \mu_3 y + \lambda_1\\ \mu_3 x - \mu_1 z + \lambda_2\\ \mu_1 y - \mu_2 x + \lambda_3\end{cases} \quad \ldots . \ (11)$$

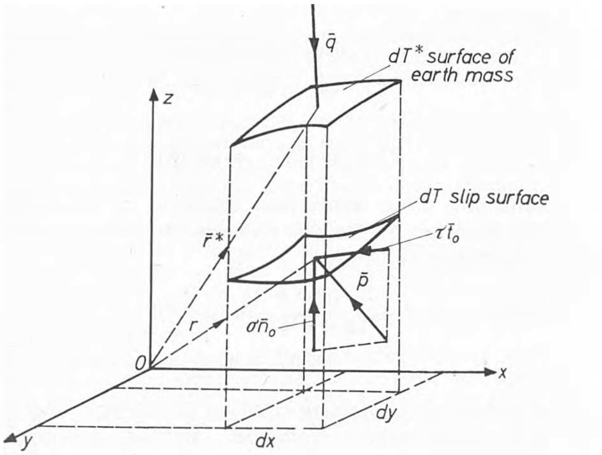


Fig. 2

According to the calculus of variations the function $\psi(x, y)$, $\sigma(x, y)$ and $z(x, y, z)$ have to satisfy Euler's partial differential equations:

$$F_\psi - \frac{\partial}{\partial x}F_{\psi x} - \frac{\partial}{\partial_y}F_{\psi y} = 0 \qquad \text{(I)}$$

$$F_\sigma - \frac{\partial}{\partial x}F_{\sigma x} - \frac{\partial}{\partial_y}F_{\sigma y} = 0 \qquad \text{(II)}$$

$$F_{\bar z} - \frac{\partial}{\partial x}F_{\bar z\bar x} - \frac{\partial}{\partial_y}F_{\bar z y} = 0 \quad \ldots . \ \text{(III)}$$

Differential Equations

The solutions of Euler's differential equations give the differential equations of stresses and slip surfaces. Therefore equation 10 for F will be substituted into Euler's equations to obtain

$$F_{\psi x} = F_{\psi y} = F_{\sigma x} = F_{\sigma y} = 0$$

(a) After substituting equation I the term will be:

$$F_\psi = \bar\omega\tau(\bar n^2)^{\frac12}\frac{\partial}{\partial\psi}\frac{\bar t}{(\bar t^2)^{\frac12}} = 0$$

As the term $\tau(\bar n^2)^{\frac12}$ cannot be identically equal to zero it follows that

$$\bar\omega\frac{\partial}{\partial\psi}\frac{\bar t}{\bar t^{\frac12}} = 0$$

After partial differentiation it may be written

$$\bar\omega\frac{\bar t_\psi(\bar t^2)^{\frac12} - \Big(\frac{2\bar t}{\bar t^{\frac12}}.\bar t_\psi\Big)\bar t}{\bar t^2} = 0$$

From equation 6

$$\bar t_\psi = \bar r_y$$

After substitution and multiplication by $(\bar t^2)^{\frac12}$:

$$\bar\omega.\bar r_y\bar t^2 - (\bar t.\bar r_y)(\bar\omega\bar t) = 0$$

Putting in this the expression for $\bar t$ from 6 we obtain:

$$(\bar\omega\bar r_y)(\bar r_x + \psi\bar r_y)^2 - [(\bar r_x + \psi\bar r_y)\bar r_y][\bar\omega(\bar r_x + \psi\bar r_y)] = 0$$

Therefore the expression of ψ will be:

$$\psi = \frac{(\bar\omega\bar r_x)(\bar r_x\bar r_y) - (\bar\omega\bar r_y)\bar r_x{}^2}{(\bar\omega\bar r_y)(\bar r_x\bar r_y) - (\bar\omega\bar r_x)\bar r_y{}^2} = \frac{(\bar\omega \times \bar r_x)(\bar r_x \times \bar r_y)}{(\bar\omega \times \bar r_y)(\bar r_x \times \bar r_y)} = \frac{\bar\omega\bar r_x\bar n}{\bar\omega\bar r_y\bar n} \quad \ldots . \ (12)$$

Thus from 7

$$\frac{\bar t}{(\bar t^2)^{\frac12}} = \frac{\bar r_x + \psi\bar r_y}{\{(\bar r_x + \psi\bar r_y)^2\}^{\frac12}} = \frac{(\bar n\bar\omega)\bar n - \bar n^2\bar\omega}{\{\bar n^2\bar\omega^2 - (\bar n\bar\omega)^2\}^{\frac12}(\bar n^2)^{\frac12}} = \frac{(\bar\omega \times \bar n) \times \bar n}{(\bar\omega \times \bar n)^{\frac12}.(\bar n^2)^{\frac12}}$$

when considering the vector identity c and d, furthermore the term 4 of $\bar n$ and expression

$$\frac{\bar\omega\bar t}{(\bar t^2)^{\frac12}} = -\frac{\{(\bar n\bar\omega)^2 - \bar n^2\bar\omega^2\}^{\frac12}}{(\bar n^2)^{\frac12}} = -\frac{\{(\bar\omega \times \bar n)^2\}^{\frac12}}{(\bar n^2)^{\frac12}}$$

Substituting the latter expression into equation 10 for F we obtain

$$F = \sigma\bar\omega\bar n - \tau\{\bar\omega^2\bar n^2 - (\bar\omega\bar n)^2\}^{\frac12} - \gamma(\bar\omega^x\bar r - \bar\lambda\bar r^x) + \bar\omega^x q(\bar n^{x2})^{\frac12}$$

$$+ q_3(\bar n^{x2})^{\frac12} \quad \ldots . \ (13)$$

where

$$\bar\omega^x = \bar\mu \times \bar r^x + \bar\lambda$$

(b) By substituting expression 13 for F into Euler's equation II we obtain

$$\bar\omega\bar n - \tan\phi\{\bar\omega^2\bar n^2 - (\bar\omega\bar n)^2\}^{\frac12} = 0 \quad \ldots . \ (14)$$

or considering the identity d

$$\bar\omega\bar n - \tan\phi\{(\bar\omega \times \bar n)^2\}^{\frac12} = 0 \quad \ldots . \ (15)$$

14 can be written

$$\Phi = (\bar\omega\bar n)^2 - \frac{\tan^2\phi}{\tan^2\phi + 1}\bar\omega^2\bar n^2 = 0 \quad \ldots . \ (16)$$

This is the three-dimensional differential equation of the slip surface of which, corresponding to terms 4 and 11, the scalar form will be:

$$\Phi = [(\mu_2 z - \mu_3 y + \lambda_1)z_x + (\mu_3 x - \mu_1 z + \lambda_2)z_y$$

$$- (\mu_1 y - \mu_2 x + \lambda_3)]^2 - \frac{\tan^2\phi}{\tan^2\phi + 1}[(\mu_2 z - \mu_3 y + \lambda_1)^2 z_x$$

$$+ (\mu_3 x - \mu_1 z + \lambda_2)^2 + (\mu_1 y - \mu_2 x + \lambda_3)^2]$$

$$(z_x{}^2 + z^2{}_y + 1) = 0 \quad \ldots . \ (17)$$

This complicated equation may be simplified by a transformation of the coordinate system. If point O of the coordinate axes is transposed into the end point of vector $(\bar\mu \times \bar\lambda)/\bar\mu^2$ (Fig. 3a), i.e. let:

$$\bar r = \bar\rho + \frac{\bar\mu \times \bar\lambda}{\bar\mu^2} \quad \ldots . \ (18)$$

then the new positional vector will be:

$$\bar\rho = \begin{cases}\xi\\ \eta\\ \zeta(\xi, \eta)\end{cases} = \bar r - \frac{\bar\mu \times \bar\lambda}{\bar\mu^2} \quad \ldots . \ (19)$$

(c) The vector $\bar{\omega}$, considering identity c, will be:

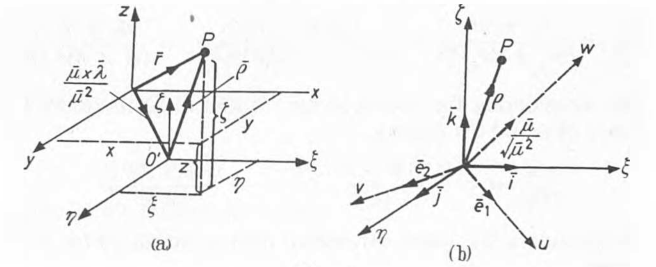
$$\bar{\omega} = \bar{\mu} \times \bar{r} + \bar{\lambda} - \bar{\mu} \times \bar{\rho} - \bar{\mu} \times \frac{\bar{\mu} \times \bar{\lambda}}{\bar{\mu}^2} + \bar{\lambda}$$

$$= \bar{\mu} \times \bar{\rho} + \frac{\bar{\mu}\bar{\lambda}}{\bar{\mu}^2}\bar{\mu} \quad \ldots . \; (20)$$

Now the coordinate systems are rotated in such a way that, first, the w axis is made parallel to the unit vector $\bar{\mu}/(\bar{\mu}^2)^{\frac{1}{2}}$ and secondly, that the $\mathring{\mu}$ and v axes are also brought parallel to the unit vectors $\bar{e}_1$ and $\bar{e}_2$ (Fig. 3b)

Then:

$$\bar{\rho} = \begin{cases} \bar{\rho}\bar{e}_1 = \mathring{\mu} \\ \bar{\rho}\bar{e}_2 = v \\ \rho\frac{\bar{\mu}}{(\bar{\mu}^2)^{\frac{1}{2}}} = \bar{\omega}(\mathring{\mu}_1 v) \end{cases} \quad \ldots . \; (21)$$



(a) (b)

Fig. 3

Considering the identity a for the component of vector $\bar{\omega}$ in direction of u we get:

$$\mathring{U} = \bar{\omega}\bar{e}_2 = \left(\bar{\mu} \times \bar{r} + \frac{\bar{\mu}\bar{\lambda}}{\bar{\mu}z}\bar{\mu}\right)\bar{e}_1 = (\bar{\mu} \times \bar{\rho})\bar{e}_1 = -\,\bar{\rho}(\bar{\mu} \times \bar{e}_1)$$

$$= -\,\bar{\rho}\bar{e}_2(\bar{\mu}^2)^{\frac{1}{2}} - v(\bar{\mu}^2)^{\frac{1}{2}}$$

Similarly

$$V = \bar{\omega}\bar{e}_2 = \mathring{\mu}(\bar{\mu}^2)^{\frac{1}{2}} \quad \text{and} \quad W = \bar{\omega}\frac{\bar{\mu}}{(\bar{\mu}^2)^{\frac{1}{2}}} = \frac{\bar{\mu}\bar{\lambda}}{(\bar{\mu}^2)^{\frac{1}{2}}} = a(\bar{\mu}^2)^{\frac{1}{2}}$$

where

$$a = \frac{\bar{\mu}\bar{\lambda}}{\bar{\mu}^2} \quad \ldots . \; (22)$$

and finally

$$\bar{\omega} = (\bar{\mu}^2)^{\frac{1}{2}}\begin{cases} -\,v \\ \mathring{\mu} \\ a \end{cases} \quad \ldots . \; (23)$$

And last when introducing cylinder coordinates then

$$\bar{\rho} = \begin{cases} \rho\cos\vartheta \\ \rho\sin\vartheta \\ \omega(\rho_1\vartheta) \end{cases} \quad \bar{\rho}_\rho = \begin{cases} \cos\vartheta \\ \sin\vartheta \\ \omega_\rho \end{cases} \quad \bar{\rho}_v = \begin{cases} -\,\rho\sin\vartheta \\ \rho\cos\vartheta \\ \omega_\vartheta \end{cases} \quad \ldots . \; (24)$$

$$\mathring{\mu} = \bar{\rho}_\rho \times \bar{\rho}_\vartheta = \begin{cases} -\,\omega_\rho\rho\cos\vartheta = \omega_\vartheta\sin\vartheta \\ -\,(\omega_\rho\rho\sin\vartheta = \omega_\vartheta\cos\vartheta \\ \rho \end{cases}$$

and

$$\bar{\omega} = \begin{cases} -\,\rho\sin\vartheta \\ \rho\cos\vartheta \\ a \end{cases} \quad \ldots . \; (25)$$

From which

$$\bar{\omega}\bar{\mu} = \rho(a - \omega_\vartheta) \quad \ldots . \; (25a)$$

$$\bar{\omega}^2 = a^2 + \rho^2 \quad \ldots . \; (26)$$

and

$$\bar{\mu}^2 = 1 + \omega_\rho{}^2 + (\omega_v{}^2)/(\rho^2) \quad \ldots . \; (27)$$

Substituting these terms into equation 16 the scalar differential equation of the slip surfaces in cylinder coordinates will be obtained as follows:

$$\Phi = \rho^2(a - \omega_\theta)^2 - \frac{\tan^2\phi}{\tan^2\phi + 1}(a^2 + \rho^2)$$

$$[\rho^2(\omega_\rho{}^2 + 1) + \omega_\theta{}^2] = 0 \quad \ldots . \; (28)$$

The general and complete solution of this equation is to be found in mathematical literature. Without geometrical or mathematical interpretation it may be mentioned that it generally defines a screw surface with an axis parallel to the direction of vector $\bar{\mu}$, having the characteristics of the logarithmic spiral.

(c) Substituting the expression of F into the Euler's differential equation III the differential equation of stress distribution on slip surface may be derived.