The Numerical Solution for Stresses and Displacements in a Three-layer Soil System

Solution Numérique des Contraintes et des Déplacements dans un Sol à Trois Couches

by R. L. SCHIFFMAN, Assistant Professor of Civil Engineering, Lehigh University, U.S.A.

Summary

This paper presents methods for numerical solutions of influence values, for certain stresses and displacements in the Burmister problem of a three-layer soil system. The theoretical equations are arranged and presented in their form for computation. The majority of the text is concerned with the method and technique of the numerical solution to this problem, including details of the numerical analysis necessary, and the error analysis required to establish a maximum absolute upper bound on the errors of computation. The paper also describes computing procedures applicable for use with high speed, modern computing machines. The last section presents alternate procedures that may be used to solve the problem computationally.

Sommaire

Cette communication présente des méthodes de calcul, des contraintes et des déplacements dans un sol à trois couches (problème de Burmister).

Les équations obtenues sont présentées sous une forme permettant d'aboutir à des solutions numériques. La majeure partie de l'exposé est consacrée à cette méthode de solution numérique, et comprend certains détails de l'analyse numérique, ainsi que les calculs d'erreur nécessaires pour obtenir la valeur absolue de l'erreur maximum du calcul. Cette communication décrit aussi, certaines méthodes de calcul qui peuvent être utilisées avec des machines à calculer modernes à grande rapidité. La dernière partie expose d'autres méthodes de calcul pour obtenir aussi une solution numérique.

Introduction

With the increasing importance and complexity of numerical solutions for the stresses and displacements in soil systems, new methods and techniques take on added importance. This paper presents certain numerical solutions of the Burmister three-layer problem, and details the methods and techniques used in connection with the computation of this problem, by means of high speed computing machines. The research described was performed at Columbia University, under the sponsorship of the U.S. Corp of Engineers, Waterways Experiment Station, Vicksburg, Mississippi, and was directed by Professor D. M. Burmister of Columbia University.

Theoretical Problem

The general solution of the stresses and displacements in a three-layer system was first solved by Burmister in 1943 (BURMISTER, 1943); the writer, in 1954, showed the real significance and practical use of integral transforms involving the Bessel functions in the solution of the three-layer problem, and extended the solutions to general types of surface loadings. The solution was made for a semi-infinite elastic layered solid, under imposed axially symmetric, boundary tractions, normal to the free surface. Each layer differed from the others in terms of the elastic modulus and Poisson's ratio. The two upper layers were of varying thickness, while the bottom layer was infinite in depth. A diagrammatic representation of this system is shown in Fig. 1.

The particular problem solved in this paper was for the following set of values for the basic three-layer system parameters:

Three-layer system parameters

$E_1/E_2 = 1{\cdot}0$	$\mu_3 = 0{\cdot}4$
$E_2/E_3 = 10{\cdot}0$	$H/h = 0{\cdot}2,\ 1{\cdot}0$
$\mu_1 = 0{\cdot}4$	$r/h = 0{\cdot}1(0{\cdot}1)1{\cdot}0(0{\cdot}5)2{\cdot}0(1{\cdot}0)6{\cdot}0$
$\mu_2 = 0{\cdot}2$	

For the preceding set of parameters the following stresses and displacements were computed, under the indicated boundary tractions:

(1) Settlement (W_0) at the surface, beneath the centre of a uniformly loaded circular area of radius (a), and intensity of load (p).

(2) Vertical stress (σ_z) at the first interface, and beneath the centre of a uniformly circular area.

(3) Shear stress (τ_{rz}) at the first interface, due to a single force at the surface.

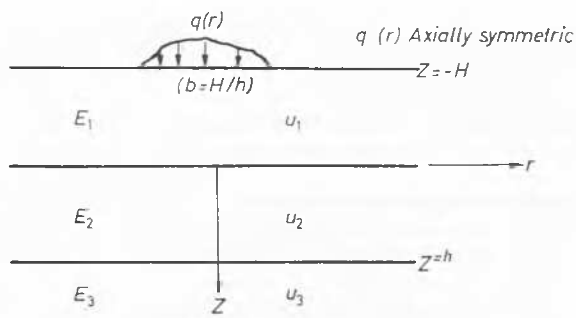

Fig. 1 Three-layer system
Sol composé de trois couches distinctes

The Burmister equations to be solved for the above cases are as follows:

Strength ratios

$$K = \frac{E_2(1 + \mu_1)}{E_1(1 + \mu_2)} \quad \ldots\ (1a)$$

$$K = \frac{1 - K}{1 + (3 - 4\mu_1)K} \quad \ldots\ (1b)$$

$$J = \frac{(3 - 4\mu_2) - (3 - 4\mu_1)K}{(3 - 4\mu_2) + K} \quad \ldots\ (1c)$$

$$M = \frac{E_3(1 + \mu_2)}{E_2(1 + \mu_3)} \quad \ldots\ldots (1d)$$

$$N = \frac{1 - M}{1 + (3 - 4\mu_2)M} \quad \ldots\ldots (1e)$$

$$L = \frac{(3 - 4\mu_3) - (3 - 4\mu_2)M}{(3 - 4\mu_3) + M} \quad \ldots\ldots (1f)$$

Three-layer strength coefficients

$$[A] = 1 + \left(\frac{1-K}{1-J}\right)JNe^{-2x} + \left(\frac{1-J}{1-K}\right)KLe^{-2x} + 4\left(\frac{1-J}{1-K}\right)KNx^2e^{-2x} + JKLNe^{-4x} \quad \ldots\ldots (2a)$$

$$[B] = J + \left(\frac{1-K}{1-J}\right)J^2Ne^{-2x} + \left(\frac{1-J}{1-K}\right)Le^{-2x} + 4\left(\frac{1-J}{1-K}\right)Nx^2e^{-2x} + JLNe^{-4x} \quad \ldots\ldots (2b)$$

$$[C] = K + \left(\frac{1-K}{1-J}\right)Ne^{-2x} + \left(\frac{1-J}{1-K}\right)K^2Le^{-2x} + 4\left(\frac{1-J}{1-K}\right)K^2Nx^2e^{-2x} + KLNe^{-4x} \quad \ldots\ldots (2c)$$

$$[D] = 2N(1 - JK)xe^{-2x}$$

$$[E] = JK - 1 + [A] \quad \ldots\ldots (2d)$$

First three-layer denominator

$$\Delta_3(x) = [A] - [B]e^{-2bx} - [C](1 + 4b^2x^2)e^{-2bx} - 4[D]b \times e^{-2bx} + [E]e^{-4bx} \quad \ldots\ldots (3)$$

Numerator brackets

$$\begin{aligned} N_{1-1}(x) &= [A](1 + bx)e^{-bx} + [E](1 - bx)e^{-3bx} \\ &= -\tfrac{1}{2}\{[B] + [C](1 + zbx)\}e^{-bx} \\ &= -\tfrac{1}{2}\{[B] + [C](1 - zbx)\}e^{-3bx} \\ &= -[D]\{(1 + bx)e^{-bx} - (1 - bx)e^{-3bx}\} \end{aligned} \quad \ldots\ldots (4a)$$

$$\begin{aligned} N_{1-2}(x) &= -[A]bxe^{-bx} - [E]bxe^{-3bx} \\ &= -\tfrac{1}{2}\{[B] - [C](1 + zbx)\}e^{-bx} \\ &= +\tfrac{1}{2}\{[B] - [C](1 - zbx)\}e^{-3bx} \\ &= -[D]bx(e^{-bx} - e^{-3bx}) \end{aligned} \quad \ldots\ldots (4b)$$

$$N_{1-3}(x) = -2(1 - \mu_1)\{[A] + 4[C]bxe^{-2bx} + 2[D]e^{-2bx} - [E]e^{-4bx}\} \quad \ldots\ldots (4c)$$

Stress and displacement equations

$$\sigma_z = -p\frac{a}{h}\int_0^\infty \frac{N_{1-1}(x)}{\Delta_3(x)} J_1\left(\frac{a}{h}x\right)dx \quad \ldots\ldots (5a)$$

$$\tau r_z = \frac{p}{2\pi h^2}\int_0^\infty x\frac{N_{1-2}(x)}{\Delta_3(x)} J_1\left(\frac{t}{h}x\right) dx \quad \ldots\ldots (5b)$$

$$W_0 = -pa\frac{1 + \mu_1}{E_1}\int_0^\infty \frac{1}{X}\frac{N_{1-3}(x)}{\Delta_3(x)} J_1\left(\frac{a}{h} - x\right)dx \quad \ldots\ldots (5c)$$

The solution for the actual stresses and displacements was performed at the Watson Scientific Computing Laboratory, using IBM computing machine models, 602–A, 604 and CPC.

Numerical Analysis

The computation of the integrals of equations 5a, 5b and 5c was accomplished by numerical integration using polynomial approximation.

The first step of the solution was to evaluate the functions involved at various mesh points. The mesh size was based on the machine and card volume required, and the overall accuracy desired. The basis of these decisions was influenced by previous trial computations.

Since it is obviously impossible to evaluate numerically an infinite integral, a finite upper limit was chosen. This limit was established by the decay of the N/Δ function, or by the extent of the available tables of Bessel functions. Allowing this practical upper limit to be (X_1), the integrals of equation 5 become:

$$I_\sigma\left(\frac{a}{h}\right) = \int_0^{X_1} x\frac{N_{1-1}(x)}{\Delta_3(x)}J_1\left(\frac{a}{h}x\right)dx + \int_{X_1}^\infty \frac{N_{1-1}(x)}{\Delta_3(x)}J_1\left(\frac{a}{h}x\right)dx \quad \ldots\ldots (6a)$$

$$I_\tau\left(\frac{r}{h}\right) = \int_0^{X_1} x\frac{N_{1-2}(x)}{\Delta_3(x)}J_1\left(\frac{r}{h}x\right)dx + \int_{X_1}^\infty x\frac{N_{1-2}(x)}{\Delta_3(x)}J_1\left(\frac{r}{h}x\right)dx \quad \ldots\ldots (6b)$$

$$I_w\left(\frac{a}{h}\right) = \int_0^{X_1} \frac{N_{1-3}(x)}{x\Delta_3(x)}J_1\left(\frac{a}{h}x\right)dx + \int_{X_1}^\infty \frac{N_{1-3}(x)}{x\Delta_3(x)}J_1\left(\frac{a}{h}x\right)dx \quad \ldots\ldots (6c)$$

The remainder integrals of equations 6a and 6b can be treated as a single analytical problem. In each case the N/Δ function has a zero limit at infinity. Thus, if the value of (X_1) is taken as large enough the remainder will be small. In addition, since N/Δ is a finite series of exponentials with different frequencies, as (x) becomes large, the higher order frequences vanish, thus simplifying the N/Δ function to manageable proportions.

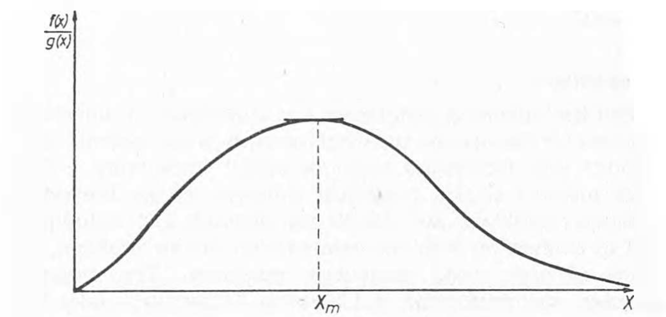

Fig. 2

Considering the general form of the remainder term:

$$R = \int_{X_1}^\infty \frac{f(x)}{g(x)}J_1(rx)dx \quad \ldots\ldots (7)$$

where the worst case of $f(x)/g(x)$ is graphed as in Fig. 2.

If the restriction on (X_1) is that $X_1 \geqslant X_m$ and considering that the Bessel function is oscillatory and decreasing with an increasing argument, the upper bound of R may be written as:

$$|R| \leqslant |J_1(rx)|_{\max}\left|\int_{X_1}^\infty \frac{f(x)}{g(x)}dx\right| \quad \ldots\ldots (8)$$

However, since $x \geqslant X_1$ then from equation 3, $g(x) = 1 + h(x)$ where $h(x)$ is positive.

$$\left|\int_{X_1}^\infty \frac{f(x)}{1 + h(x)}dx\right| \leqslant \left|\int_{X_1}^\infty f(x)\,dx\right| \quad \ldots\ldots (9)$$

Thus the maximum absolute upper bound of R is:

$$|R| \leqslant |J_1(rx)|_{\max}\left|\int_{X_1}^\infty f(x)\,dx\right| \quad \ldots\ldots (10)$$

Using the value of $|R|$ from equation 10, the value of (x_1) can be pre-determined to result in a remainder with a minimumal error of approximation in the limit. The absolute value of the remainder can also be determined for the integrals of equations 6a and 6b.

The remainder of equation 6c is similar to 6a and 6b, but contains an additional term of the form:

$$\int_{X_1}^\infty \frac{J_1(rx)}{x}dx \quad \ldots\ldots (11a)$$

This integral is referred to by its standard notation:

$$Ji_1(rX_1) = \int_{X_1}^{\infty} \frac{J_1(rx)}{x}\,dx \qquad \ldots\ldots (11b)$$

This function was evaluated for the useful range of (X_1) by means of asymptotic expansion 4.

The evaluation of the integrals 6a, 6b and 6c from zero to (X_1) was accomplished by polynomial approximation, similar to Simpson's rule.

It was found in computing the integrands that some of these functions had an essential singularity on the negative real axis, in the vicinity of the origin. Although this would not produce an improper integral in the range considered, the ordinary polynomial approximation would have a descent so steep as to require a very close mesh spacing. In order to increase the mesh size, a special polynomial was developed that used 'outrigger' points on either side of the integrated region.

$$\begin{aligned}\int_{X_0}^{X_n} f(z)dz = \frac{\delta}{7560}[&10(f_{-2} + f_{n+2}) - 144(f_{-1} + f_{n+1}) \\ &+ 3016(f_0 + f_n) + 9232(f_1 + f_{n-1}) \\ &+ 6022(f_2 + f_{n-2}) + 9088(f_3 + f_{n-3}) \ldots \qquad (12a) \\ &+ 6032f_4 + 9088f_5 + \ldots + 6032f_{n-4}] \\ &- \epsilon_I\end{aligned}$$

$$\epsilon_I = \frac{x_n - x_0}{9600} f^{(8)}(\bar{x}_m)\delta^8 \qquad \ldots\ldots (12b)$$

Where δ is the mesh size.

Although this formulation appears complex, in terms of a computing machine, where the coefficients are stored internally, this polynomial is quite as easy to use as the more simple forms such as Simpson's rule. Its advantages in terms of larger mesh sizes and increased accuracy make it more efficient in the final analysis.

Computing Procedures

Although the swift advances of machine technology have made much of the equipment actually used in the solution of this problem obsolete, the methods presented here are applicable to the most modern machines.

The integrands being composed of two basic functions were computed separately. The N/Δ functions were computed for a fixed range of values, that being set by the previously determined upper limit (X_1). The steps in the mesh size of the N/Δ function were also pre-determined and held fixed so that for the full range of (r/h) and (a/h) evaluated, a single set of N/Δ functions was necessary for each stress or displacement, per set of parameters. The Bessel functions $J_1\left(\frac{r}{h}x\right)$ were key punched from the Harvard Annuals (Harvard University Computation Laboratory, 1947), and the argument values $\left(\frac{r}{h}x\right)$ were matched to the N/Δ function by sorting and collating externally. Using the more recent types of stored address computers with large memory capacity, this operation can be performed internally.

The integrations themselves were performed on the CPC. The CPC was programmed in such a way that the coefficients of the integrating polynomial were stored internally. With this set-up, a table of N/Δ functions, matched with a separate table of Bessel functions, were fed into the machine and by the time the last entry was read the results were printed out. Timewise a 3000 entry integrand table was completely integrated in 20 minutes.

The most important feature of any computing procedure is the checking process. A machine, or for that matter a human being, is subject to random error. In any computational procedure checks must be set up beforehand to account for this error. The faster the computation the more extensive and detailed these checks become.

The simplest and crudest type of checking procedure is to do everything twice, using different equipment. In terms of hand computations this would mean two computers doing identical computations, with frequent comparisons. In terms of machines, this means using two machines, or different sections of a single machine.

The more sophisticated methods of checking are many and varied, and usually are unique to the problem and computing equipment at hand. For this problem, many different procedures were followed. For example: to check the assembly of computed functions to be operated upon by values from library tables, it was necessary to determine if the matching of arguments was correct. This was done by building into the computing routine a counting routine. Wherever possible, after each operation, its inverse was performed, and checking accomplished by comparing the result of the inverse with the operands. In many cases it was necessary to double the operation and compare separate results. As a final check; at several key stages in the computation, the results to that point were differenced, and the differences were observed for divergence.

It is a well established principle in numerical mathematics that for smooth functions the differences must decrease as the order of the difference increases. Thus if the converse happens, the source must be an error. In general, the checking procedures were carried along in parallel with the computations, so that when an error cropped up the corrective measures would be minimal, and would not entail going back to the beginning of the problem.

The total computational process for all the stresses and displacements that are discussed in this paper took 643 machine hours. This is approximately equivalent to 300 man years of effort.

Error Analysis

All numerical operations have a tacit implication of error. In many cases, where the precision is slight and the number of arithmetical operations small, an analysis of the errors in computation can be neglected. In the problem under consideration, however, just the reverse was true. The precision desired was great and the number of operations considerable. Thus, it was a necessity to perform a detailed error analysis, so that one might know the extent to which the final results were accurate.

The total numerical error of this computation can be broken into three types:

(1) Round-off errors due to the large number of arithmetical operations performed on numbers that have been arbitrarily truncated after a finite number of decimal places. Since the computations were performed on a so-called 'fixed decimal' basis, i.e. the number of decimal places was held fixed at all times to eleven places, the errors follow the following formulation:

$$\begin{aligned}(A + \epsilon_A) \pm (B + \epsilon_B) &= A \pm B + (\epsilon_A \pm \epsilon_B) \\ (A + \epsilon_A)(B + \epsilon_B) &= AB + (A\epsilon_B + B\epsilon_A) \qquad \ldots\ldots (13) \\ \frac{(A + \epsilon_A)}{(B + \epsilon_B)} &= \frac{A}{B} + \frac{B\epsilon_A - A\epsilon_B}{B^2}\end{aligned}$$

where A and B are the truncated number, and ϵ_A and ϵ_B are the errors of truncation.

(2) The second type of error to consider is the error in the infinite integration, when the integration is carried only to a finite upper limit. This was previously covered by the evaluation of the maximum absolute upper bound $|R|$ in equation 10.

(3) The third error is the error inherent in a numerical integration, by the polynomial approximation; equation 12b. This

was computed by means of finite difference expansions. Having computed the error in the N/Δ functions, the general form of the solution then becomes:

$$I(r) = \int_0^{X_1} f(x)J_1(rx)\mathrm{d}x + \int_0^{X_1} \epsilon_f(x)J_1(rx)\mathrm{d}x + \int_0^{X_1} f(x)\epsilon_J(rx)\mathrm{d}x + \int_{X_1}^{\infty} f(x)J_1(rx)\mathrm{d}x + \epsilon_I \quad \ldots. \quad (14)$$

Putting the second and third terms of equation 14 in terms of the absolute maximum upper bound

$$|\, I_f \,| = \left|\, \int_0^{X_1} \epsilon_f(x)J_1(rx)\mathrm{d}x \,\right| \leqslant X_1 \,|\, \epsilon_f(x) \,|_{\max} \,|\, J_1(rx) \,|_{\max}$$

$$|\, I_J \,| = \left|\, \int_0^{X_1} f(x)\epsilon_J(rx)\mathrm{d}x \,\right| \leqslant X_1 \,|\, f(x) \,|_{\max} \,|\, \epsilon_J(rx) \,|_{\max} \quad \ldots. \quad (15)$$

Assuming all errors to be additive, the total possible error was determined. In examining these functions it was found that the errors increased as (r/h) or (a/h) increased. Thus, to determine working limits of the values of the stresses and displacements, the errors were determined only for the lowest and highest value of (r/h) and (a/h).

Stresses and Displacements

The results of the previous analysis are present in form in the table, together with the estimated probable errors involved in these computations. It is evident that a high degree of real accuracy can be achieved by such computational methods and computation checks. In all cases the stresses and displacements were in terms of an influence value (I).

$$\sigma_z = -\, p[I(\sigma_z)] \quad \ldots. \quad (16a)$$

$$\tau_{rz} = -\, \frac{p}{H^2}[I(\tau_{rz})] \quad \ldots. \quad (16b)$$

$$W_0 = \frac{z(1-\mu_3{}^2)}{E_3} pa[I(W_0)] \quad \ldots. \quad (16c)$$

Other Procedures

The results given above are limited to the lower range of the radial parameter (r/h) and (a/h). In dealing with a fixed table of Bessel functions, as (r/h) becomes larger, the mesh size of the Bessel function becomes greater. At a certain point the table would have to be expanded in order to follow the same procedure and obtain nearly the same accuracy. Since it is generally easier to compute the N/Δ functions than the Bessel functions, by a transformation of variables the integral

$$I(r) = \int_0^{X_1} \frac{f(x)}{g(x)} J_1(rx)\mathrm{d}x \quad \ldots. \quad (17)$$

can be transformed to

$$I(r) = \frac{1}{r}\int_0^{(rX_1)} \frac{f(z/r)}{g(z/r)} J_1(z)\mathrm{d}z \quad \ldots. \quad (18)$$

Using the above, the range of (r) can be considerably extended.

Considering the exact form of equation 17; i.e. the infinite integral, it is actually the Hankel transform of $f(x)/g(x)$, of order one. All one need do, therefore, is to find the inverse of this transform, and the solution is in closed form. Since a Hankel inverse involving a denominator as complex as $g(x)$ does not exist, other procedures must be used. If $1/g(x)$ could be approximated to an equivalent numerator form, then the transform would be immediately subject to inversion.

Consider an approximation

$$h(x) \doteq \frac{1}{g(x)} \qquad 0 \leqslant x \leqslant \infty \quad \ldots. \quad (19)$$

The accuracy of the results will depend on the closeness of this approximation. If the error of fit is kept to a known value of ϵ by letting:

$$\sqrt{\int_0^{\infty} \left[\frac{1}{(g(x)} - h(x)\right]^2 \mathrm{d}x} < \epsilon \quad \ldots. \quad (20)$$

Table

Stress and displacements in a three-layer soil system

Tension et déplacements dans un sol composé de trois couches distinctes

Burmister problem

A. *Vertical stress at first interface*

	$H/h = 1{\cdot}0$		$H/h = 0{\cdot}2$	
a/h	$I(\sigma_z)$	*Maximum error*	$I(\sigma_z)$	*Maximum error*
0·1	0·01376	$1{\cdot}88 \times 10^{-9}$	0·28580	$8{\cdot}85 \times 10^{-9}$
0·2	0·05290		0·63986	
0·3	0·11164		0·80980	
0·4	0·18233		0·88190	
0·5	0·25738		0·91560	
0·6	0·33058		0·93332	
0·8	0·45662		0·94924	
1·0	0·54755		0·95502	
1·5	0·65869		0·96186	
2·0	0·70122		0·97057	
3·0	0·77217		0·98771	$4{\cdot}38 \times 10^{-7}$
4·0	0·84602			
5·0	0·90524			
6·0	0·94748	$1{\cdot}01 \times 10^{-5}$		

B. *Shearing stress at first interface*

	$H/h = 1{\cdot}0$		$H/h = 0{\cdot}2$	
r/h	$I(\tau_{rz})$	*Maximum error*	$I(\tau_{rz})$	*Maximum error*
0·1	0·05384	$1{\cdot}94 \times 10^{-9}$	0·15706	$1{\cdot}12 \times 10^{-9}$
0·2	0·10051		0·09495	
0·3	0·13500		0·04407	
0·4	0·15564		0·02165	
0·5	0·16361		0·01212	
0·6	0·16181		0·00772	
0·8	0·14173		0·00417	
1·0	0·11532		0·00285	
1·5	0·06572		0·00154	
2·0	0·04073		0·00089	
3·0	0·01907		0·00026	$7{\cdot}46 \times 10^{-8}$
4·0	0·00967			
5·0	0·00489			
6·0	0·00236	$3{\cdot}24 \times 10^{-7}$		

C. *Settlement at surface*

	$H/h = 1{\cdot}0$		$H/h = 0{\cdot}2$	
a/h	$I(W_0)$	*Maximum error*	$I(W_0)$	*Maximum error*
0·1	0·11918	$2{\cdot}42 \times 10^{-6}$	0·13119	$2{\cdot}84 \times 10^{-7}$
0·2	0·13834		0·16311	
0·3	0·15746		0·19568	
0·4	0·17588		0·22841	
0·5	0·19550		0·26082	
0·6	0·21437		0·29262	
0·8	0·25170		0·35381	
1·0	0·28834		0·41128	
1·5	0·37578		0·53721	
2·0	0·45575		0·63829	
3·0	0·59017	$7{\cdot}16 \times 10^{-5}$	0·77857	$1{\cdot}93 \times 10^{-5}$
4·0	0·69235	$9{\cdot}55 \times 10^{-5}$		
5·0	0·76799	$1{\cdot}19 \times 10^{-4}$		
6·0	0·82313	$1{\cdot}43 \times 10^{-4}$		

Then, by the Schwartz inequality, the total error of approximation can be found.

$$\left| \int_0^{\infty} \frac{f(x)}{g(x)} J_1(rx)\mathrm{d}x - \int_0^{\infty} h(x)f(x)J_1(rx)\mathrm{d}x \right| \leqslant \epsilon \sqrt{\int_0^{\infty} f^2(x)J_1{}^2(rx)\mathrm{d}x} \quad \ldots. \ (21)$$

Since (ϵ) can be arbitrarily set, the order of approximation can be as precise as desired.

Another possible procedure is based on the possibility that the decay of the Bessel function can be faster in the complex plane than in the real. If this is so, the evaluation of the integral might be hastened by integrating through the complex plane and evaluating results by means of the Residue theorem.

Conclusion

The presentation of the results of the series of computations are a further addition to information on this stress problem in Soil Mechanics and Foundations. It is the writer's hope that this paper will, in some way, contribute a stimulus to the use of new techniques and methods in the analysis of our increasingly complex problems.

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