Some Problems of Soil Pressure

Quelques Problèmes de la Pression du Sol

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Summary

This paper is devoted to two plane problems of limiting equilibrium of a soil, taking into account the weight. The main system of equations is of a hyperbolic type and has two families of real characteristics.

The first problem deals with the construction of the free curvilinear contour for a stable semi-arch the lower part of which is a slope, while the second problem deals with the determination of contact stresses acting on the curvilinear contour of a rigid wall.

In general, the foregoing problems are solved by the numerical integration of corresponding differential equations of the characteristics, and for some particular cases, even taking into account the weight, the closed form solutions have been developed.

Sommaire

La présente étude est consacrée à deux problèmes plans de l'équilibre limite des sols compte tenu du poids. Le système principal des équations, appartient au type hyperbolique et a deux familles de caractéristiques réelles.

Le premier problème vise à construire le contour libre curviligne d'une semi-arche stable, dont la partie inférieure passe en pente et le deuxième problème trouve les tensions de contact sur le contour curviligne de la paroi rigide.

En principe, la solution de ces problèmes a été effectuée à l'aide de l'intégration numérique des équations différentielles correspondantes des caractéristiques, et pour les cas particuliers, on a déterminé des solutions, approchées tenant compte même du poids.

The plane limiting equilibrium of a soil with the unit weight γ, the angle of internal friction ϕ, and the cohesion coefficient c is known (SOKOLOVSKY, 1942) to be described by the differential equations of equilibrium (with the y-axis directed downwards)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \gamma \qquad \ldots. \ (1)$$

and by the yield condition

$$\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau^2 = \frac{\sin^2 \phi}{4}(\sigma_x + \sigma_y + 2H)^2, \qquad H = c \cot \phi \qquad \ldots. \ (2)$$

It is convenient to transform this system of equations by changing the stress components for the new variables σ and θ according to the formulae

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = \sigma(1 \pm \sin \phi \cos 2\theta) - H, \qquad \tau = \sigma \sin \phi \sin 2\theta \ \ldots. \ (3)$$

By introducing expressions 3 into differential equations of equilibrium 1 we get a system of two equations for the variables σ and θ. It belongs to the hyperbolic type and possesses two families of real characteristics which are defined by the following differential equations:

$$\mathrm{d}y = \mathrm{d}x \tan(\theta \mp \mu)$$
$$\mathrm{d}\sigma \mp 2\sigma \tan \phi \, \mathrm{d}\theta = \gamma(\mathrm{d}y \mp \tan \phi \, \mathrm{d}x) \qquad \ldots. \ (4)$$

In the x, y plane, the foregoing characteristic lines form two isogonal families, coincide with the slip lines and intersect each other at the angles $2\mu = \pi/2 - \phi$.

Let us determine the limiting contour of a stable semi-arch which in its lower part is changing into a slope, assuming that along the horizontal boundary, i.e. along the positive x semi-axis, a uniformly distributed normal pressure p is given (Fig. 1).

Consider small values of the pressure p when the semi-arch limiting equilibrium gives the curve of stress discontinuity OA.

In the domain xOA there is the simplest stress field which is defined by the values

$$\sigma = \frac{\gamma y + p + H}{1 + \sin \phi}, \qquad \theta = \frac{\pi}{2} \qquad \ldots. \ (5)$$

and the characteristic lines are the parallel straight lines inclined to the x-axis at the angles $\pi/2 \mp \mu$.

Consider the boundary conditions along the curves OA and OB taking into account that the normal and tangential stress components N and T, according to equation 3, are expressed by the formulae

$$N = \sigma[1 - \sin \phi \cos 2(\theta - \alpha)] - H$$
$$T = \sigma \sin \phi \sin 2(\theta - \alpha) \qquad \ldots. \ (6)$$

or by the similar formulae in which, instead of α, we have β.

First of all it is obvious that along the curve OA the exterior stress components N and T are continuous, but the interior component N' may be discontinuous, consequently $N'_+ \neq N'_-$.

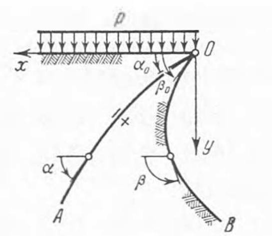

Fig. 1 The semi-arch and discontinuous stress fields
La semi-arche et les champs de tension discontinus

Therefore, from equations 5 and 6, after simple transformations we obtain:

$$\sin(2\alpha - \theta) = \sin \phi \sin \theta, \qquad 0 \leqslant 2\alpha - \theta \leqslant \frac{\pi}{2}$$

and in addition

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan \alpha, \qquad \sigma = \frac{\gamma y + p + H}{1 + \sin \phi} \frac{\sin 2\alpha}{\sin 2(\theta - \alpha)} \qquad \ldots. \ (7)$$

Then it is evident that along the curve OB, which is a semi-arch contour, the stress components $N = T = 0$. Therefore, from 6 it may be found

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan \beta, \qquad \theta = \beta, \qquad \sigma = \frac{H}{1 - \sin \phi} \qquad \ldots. \ (8)$$

In the domain AOB the stress field is not the simplest and can be determined by the numerical integration of differential equations 4 of characteristics using the above-mentioned boundary conditions 7 and 8 along the curves OA and OB.

The values of the angles $\alpha = \alpha_0$ and $\beta = \beta_0$ at the upper point O are expressed by the given pressure p. Indeed, assuming that $\alpha = \alpha_0$ and $\theta = \beta_0$ we obtain

$$\sin(2\alpha_0 - \beta_0) = \sin\phi \sin\beta_0, \qquad 0 \leqslant 2\alpha_0 - \beta_0 \leqslant \frac{\pi}{2} \quad \ldots . \ (9)$$

Equating the values of σ in conditions 7 and 8 when $y = 0$ we shall find:

$$p = H\left[\frac{1+\sin\phi}{1-\sin\phi}\,\frac{\sin 2(\beta_0 - \alpha_0)}{\sin 2\alpha_0} - 1\right] \quad \ldots . \ (10)$$

It is not difficult to show that for small values of the angles α_0 and β_0 instead of 9 and 10 we have the approximate formulae

$$\alpha_0 = (1 + \sin\phi)\frac{\beta_0}{2}, \qquad p = H\sin\phi\beta_0^2 = c\cos\phi\beta_0^2$$

Hence, when p/c is fixed, the angles α_0 and β_0 increase as the angle of internal friction ϕ grows.

It must be noted that when $\gamma = 0$ the desired curves OA and OB become the straight lines, and the semi-arch changes into a weightless acute wedge of a soil which has already been considered by Shapiro (1952) in somewhat different form.

The inclination angles $\alpha = \alpha_0$ and $\beta = \beta_0$ which the above-mentioned straight lines OA and OB make with the x axis are determined by the same formulae 9 and 10; for the small values these angles are determined by the suitable approximated formulae originally obtained by Shield (1954).

The proper weight acting completely changes the limiting equilibrium of a soil; then OA and OB become curved so that the angles α and β are increasing from $\alpha = \alpha_0$ and $\beta = \beta_0$ at the point O up to $\alpha = \pi/2 - \mu$ and $\beta = \pi - \phi$ in the limit.

This proper weight influence is especially evident when the normal pressure $p = 0$; then the weightless wedge turns into a horizontal semi-straight line, i.e. into the positive x semi-axis, and a weightful semi-arch has only a horizontal tangent at the point O.

As an example the semi-arch form for $\phi = \pi/6$ and $\beta_0 = \pi/4$ was established. Usual methods of the numerical integration of differential equations 4 were applied. As a result, the dimensionless co-ordinates $\gamma x/c$ and $\gamma y/c$ of the curve OA:

$\gamma x/c$:	0·08	0·18	0·30	0·43	0·60	0·79	1·01	1·25
$\gamma y/c$:	0·05	0·12	0·21	0·30	0·44	0·59	0·79	1·01

and those of the curve OB:

$\gamma x/c$:	0·00	0·09	0·14	0·20	0·27	0·33	0·40	0·46
$\gamma y/c$:	0·00	0·09	0·15	0·22	0·29	0·38	0·47	0·56
$\gamma x/c$:	0·54	0·62	0·70	0·78	0·86	0·93	1·00	1·05
$\gamma y/c$:	0·70	0·83	0·99	1·16	1·36	1·57	1·82	2·09

were obtained. The characteristic lines, i.e. the slip lines, of both families were also constructed (Fig. 2).

It is interesting to note that when $p = (2H\sin\phi)/(1 - \sin\phi)$ the angles

$$\alpha = \alpha_0 = \frac{\pi}{2} - \mu, \qquad \beta_0 = \frac{\pi}{2}, \qquad N'_+ = N'_-$$

and the discontinuity curve turns into the characteristic straight line, which makes the inclination angle $\pi/2 - \mu$ with the x axis. Then there is no overhanging part, only a slope with a vertical tangent at the point O.

Analyse a particular case when the normal pressure $p = O$, and the semi-arch contour, therefore, has a horizontal tangent at the upper point O (Fig. 3). Here is the way to construct approximated solution near the point O, assuming that $\sigma - \sigma_0$, θ and x, y are small and the relation y/x is also small.

By introducing expressions 3 into equilibrium equations 1 and by estimating the order of their various terms, we neglect those which are comparatively small. Approximately

$$\frac{\partial\theta}{\partial y} = 0, \qquad (1 - \sin\phi)\frac{\partial\sigma}{\partial y} = \gamma - 2\sigma_0\sin\phi\frac{\partial\theta}{\partial x} \quad \ldots . \ (11)$$

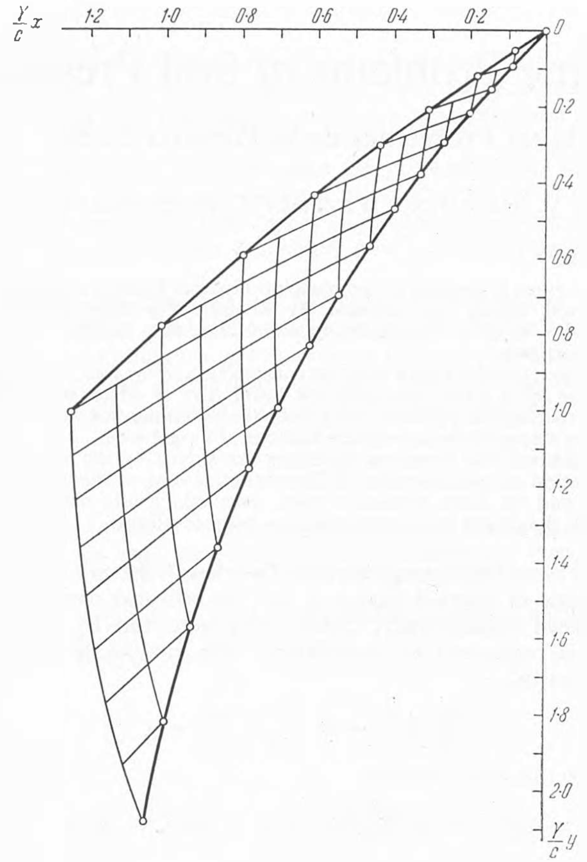

Fig. 2 An example of the slip lines for a semi-arch
Un exemple des lignes de glissement pour une semi-arche

These equations have the integrals:

$$\theta = \theta(x), \qquad (1 - \sin\phi)\sigma = [\gamma - 2\sigma_0\sin\phi\theta'(x)]y + f(x) \quad \ldots . \ (12)$$

which contain two arbitrary functions of x.

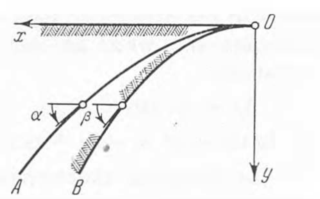

Fig. 3 The semi-arch in the case when $p = 0$
La semi-arche dans le cas où $p = 0$

The conditions 7 along the curve OA may be reduced to

$$\frac{dy}{dx} = \alpha = (1 + \sin\phi)\frac{\theta}{2}, \qquad \sigma = \frac{\gamma y}{1 - \sin\phi} + \sigma_0(1 - \sin\phi\theta^2) \quad \ldots . \ (13)$$

and the conditions 8 along the curve OB may be written as

$$\frac{dy}{dx} = \beta = \theta, \qquad \sigma = \sigma_0 \quad \ldots . \ (14)$$

The integrals 12, after determining the arbitrary functions

$\theta(x)$ and $f(x)$ from conditions 13 and 14, make it possible to find the equations of the curves OA and OB:

$$y = (1 + \sin\phi)\frac{x^2}{4R_0}, \qquad y = \frac{x^2}{2R_0} \qquad \ldots\ldots (15)$$

and as well to get the desired functions

$$\sigma - \sigma_0 = \frac{\sigma_0 \sin\phi(3\sin\phi - 1)(x^2 - 2R_0 y)}{2R_0^2(1 - \sin\phi)} \qquad \theta = \frac{x}{R_0} \qquad \ldots\ldots (16)$$

where $R_0 = (3H\sin\phi)/\gamma$ is the value of the OB curvature radius at the point O, and

$$\sigma_0 = \frac{H}{1 - \sin\phi}$$

The approximated inclination angles of the curves OA and OB to the x axis near the upper point O will be:

$$\alpha = (1 + \sin\phi)\frac{\beta}{2}, \qquad \beta = \frac{\gamma x}{3H\sin\phi} = \frac{\gamma x}{3c\cos\phi}$$

Hence, when $\gamma x/c$ is fixed, the angles α and β are increasing as the angle of internal friction ϕ grows.

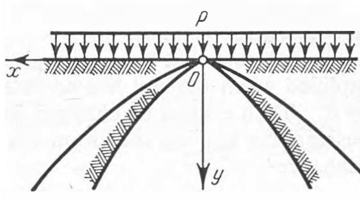

Fig. 4 The arch consisting of two identical semi-arches
L'arche qui consiste en deux semi-arches identiques

The considerations given above may be taken to determine completely the limiting contour of a stable arch, assuming that along the horizontal boundary, i.e. along the x axis a uniformly distributed normal pressure p is given, because such an arch consists of two identical semi-arches symmetrical with respect to the y axis (Fig. 4).

Now let us determine the contact stresses along the wall contour, assuming that along the horizontal boundary a uniformly distributed normal pressure p_0 is given (Fig. 5).

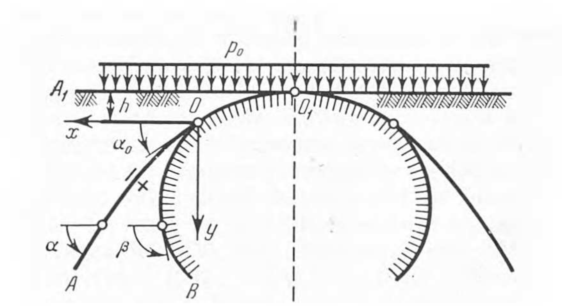

Fig. 5 The rigid wall and a discontinuous stress field
La paroi rigide et le champ de tension discontinu

In the domains A_1O_1Ox and xOA the simplest stress fields come into being, which are determined by the values 5, where $p = p_0 + \gamma h$. Therefore, along the curve O_1O from 6 we shall have

$$\left.\begin{aligned} N + H &= (\gamma y + p + H)\frac{1 + \sin\phi\cos 2\beta}{1 + \sin\phi} \\ T &= (\gamma y + p + H)\frac{\sin\phi\sin 2\beta}{1 + \sin\phi} \end{aligned}\right\} \ldots\ldots (17)$$

Assume that along the curve OB which is a wall contour between the components N and T the following relations hold

$$T = (N + H)\tan\delta, \qquad \delta \leqslant \phi$$

Therefore, owing to 6 it is found that

$$\sin[2(\theta - \beta) + \delta] = \frac{\sin\delta}{\sin\phi}, \qquad 0 \leqslant 2(\theta - \beta) + \delta \leqslant \frac{\pi}{2} \ldots\ldots (18)$$

Hence, in the most interesting particular cases, when $\delta = 0$ and $\delta = \phi$, respectively, we shall have

$$\theta = \beta \qquad \text{and} \qquad \theta = \beta + \mu$$

The point O must be so chosen that the components N and T be continuous. From equations 17 and 18 it can be found

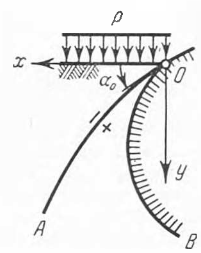

Fig. 6 Two domains of a stress field
Deux domaines du champ de tension

that the curves OA and OB have the common tangent inclined to the x axis at the angle $\alpha_0 = \beta_0$ while

$$\sin(2\alpha_0 - \delta) = \frac{\sin\delta}{\sin\phi}, \qquad 0 \leqslant 2\alpha_0 - \delta \leqslant \frac{\pi}{2}$$

In the domain AOB the stress field will not be the simplest already and must be determined by the numerical integration of differential equations 4 of characteristics according to the above-mentioned boundary condition 7 and 18 along the curves OA and OB (Fig. 6).

The value $\sigma = \sigma_0$ at the point O may be expressed by the pressure p. In fact, from 7 and 18 when $y = 0$ we obtain:

$$\sigma_0 = \frac{p + H}{1 + \sin\phi}\,\frac{\sin 2\alpha_0}{\sin 2(\alpha_0 - \delta)} \qquad \ldots\ldots (19)$$

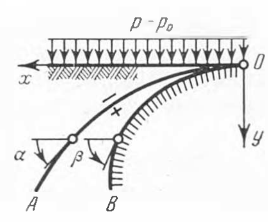

Fig. 7 The rigid wall in the case when $\delta = 0$
La paroi rigide dans le cas où $\delta = 0$

It must be noted that when $\delta = \phi$, the angles

$$\alpha_0 = \beta_0 = \frac{\pi}{2} - \mu, \qquad N'_+ = N'_-$$

and the discontinuity curve turns into the characteristic straight line which forms the angle $\pi/2 - \mu$ with the x axis.

Consider the particular case, when $\delta = 0$, $p = p_0$ and, consequently the wall contour has a horizontal tangent at the upper point O (Fig. 7).

The integrals 12 after determining the arbitrary functions $\theta(x)$ and $f(x)$ from conditions 13 along the curve OA, and the condition $\theta = \beta = x/R_0$ along the curve OB, give the former equation 15 and the desired functions:

$$\sigma - \sigma_0 = \frac{\sigma_0 \sin\phi(3\sin\phi - 1)x^2 - 2R_0(2\sigma_0 \sin\phi - \gamma R_0)y}{2R_0^2(1 - \sin\phi)},$$

$$\theta = \frac{x}{R_0} \qquad \ldots. \ (20)$$

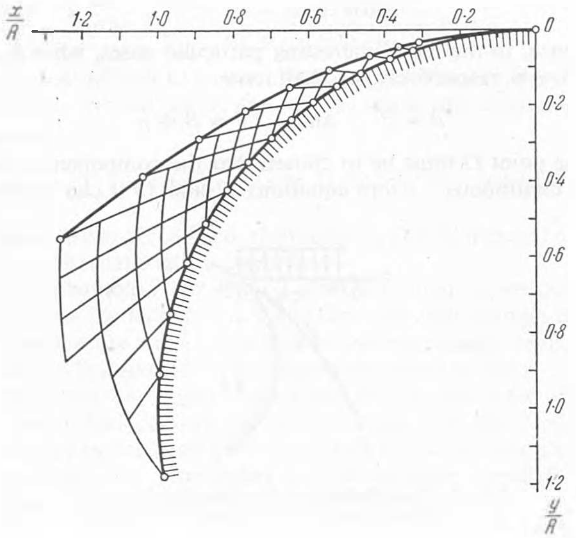

Fig. 8 An example of the slip lines for a circular wall
Un exemple des lignes de glissement pour la paroi circulaire

where R_0 is the value of the OB curvature radius at the point O and

$$\sigma_0 = \frac{p + H}{1 - \sin\phi}$$

It is not difficult to see that, when $p = 0$ and $R_0 = (3H\sin\phi)/\gamma$, formulae 16 and 20 give the same results.

After this, let us give the normal component expression N on the wall contour near the point O as the function of the arc length s. We have:

$$N = p + \left[\gamma - \frac{3}{R_0}(p + H)\sin\phi\right]\frac{s^2}{2R_0}$$

As an example, a circular wall of radius R for $\phi = \pi/6$, $\delta = 0$ and $(p + H)/(\gamma R) = 5$ is considered. Near the point O the approximated formulae are taken, while at a distance from it the methods of numerical integration of differential equations 4 are used. And so the dimensionless co-ordinates x/R and y/R of the curve OA:

x/R:	0·00	0·20	0·31	0·43	0·55	0·65	0·76	0·89	1·04	1·26
y/R:	0·00	0·02	0·04	0·07	0·11	0·15	0·21	0·29	0·38	0·55

and the dimensionless values $(N + H)/(\gamma R)$ for various values of dimensionless length s/R of the curve OB:

s/R:	0·00	0·21	0·33	0·39	0·48	0·55	0·63	0·70
$(N + H)/(\gamma R)$:	5·00	4·86	4·67	4·55	4·35	4·16	3·97	3·78
s/R:	0·77	0·86	0·95	1·06	1·17	1·32	1·47	1·74
$(N + H)/(\gamma R)$:	3·59	3·35	3·12	2·85	2·60	2·29	1·99	1·54

were obtained. The characteristic lines, i.e. the slip lines, of both families are constructed (Fig. 8).

In conclusion we shall note that the above given results are considerably simplified when the soil has no internal friction, that is when $\phi = 0$. Then σ must be changed for $\sigma + H$ and H for $c \cot\phi$; besides, the limiting transition when $\phi \to 0$ and $\delta \to 0$ must be realized.

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