Approximate Solution of the Progress of Consolidation in a Sediment

Solution approximative du processus de consolidation dans un sédiment

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Summary

In a sediment the vertical pressure depends on the depth below ground level and the time which has elapsed since the sedimentation started. In this paper the vertical effective grain pressure is expressed as a product of a function of depth and time, respectively.

Using the two-dimensional continuity equation for a compressible soil and *Darcy*'s law for seepage through porous materials, one arrives at a differential equation of the first order for the time function, $\zeta(t)$, the solution of which contains the incomplete Gamma-function.

Assuming the pressure distribution, with respect to depth, to be given by an exponential function, a complete numerical calculation is carried out and plotted on arithmetic and semi-logarithmic papers. An approximate solution, given by the exponential integral, is added on both plots. It is also demonstrated how one can obtain the solution for linear pressure distribution as a special case, in which the more general solution is reduced to contain the error integral as the main part.

In this paper it is also shown how one can calculate the pressure distribution during the consolidation which will take place after sedimentation is completed. In this case the time function is obtained by integration of *Darcy*'s equation, leading to exponential functions. From the origin $\zeta(t_1)$, located on the ζ-curve, the corresponding curves are drawn tangential to the asymptote $\zeta(0)f(1) = 1$. By means of this set of curves the effective grain pressure distribution is determined both during and after the sedimentation process.

Sommaire

La pression verticale dans un sédiment dépend de la profondeur au-dessous du niveau du sol et du temps écoulé depuis le début de la sédimentation.

Cet article introduit une équation exprimant la pression verticale réelle entre les particules en fonction de la profondeur et du temps. Partant de la supposition que la distribution de la pression est une fonction exponentielle, le résultat est donné par un graphique. Une solution de la distribution linéaire de la pression est également exposée.

L'article comprend en plus une méthode pour le calcul de la distribution de la pression pendant la consolidation, une fois la sédimentation achevée.

Introduction

K. Terzaghi (1924), in one of the earlier editions of his book on soil mechanics, has given a treatment of the progress of consolidation in a sediment which is being deposited at a constant rate per unit of time. For the mathematical treatment of this problem, *Terzaghi* established a linear differential equation of the first order, the solution of which was given in integral form. In a paper published some years ago the writer presented a solution given by the error integral and elementary functions, satisfying the boundary conditions at any time (*Gran Olsson*, 1949).

K. Terzaghi made the assumption that the pressure transmitted from grain to grain in the sediment is at any given time constant, is similar and modified only by the increasing height of the sediment and a time factor ζ which is a function of time only. For this condition it will be shown that the differential equation of the problem may be solved by the incomplete

Gamma-function being tabulated in an extension satisfying all practical purposes. The proof of this important property of the equation of *Terzaghi* will be given in this paper.

Symbols and Notations

In the following investigation the symbols commonly employed in soil mechanics are used. Let

c = time constant
C = constant of integration
e = base of natural logarithm
h = head of sediment
i = hydraulic gradient
k = the coefficient of permeability (*Darcy*'s coefficient)
m_v = the coefficient of volume decrease
n = ratio between total volume of voids and total volume of soil
p = effective normal pressure per unit of area
q = quantity of sedimentation per unit of area and time
t = time
u = excess hydrostatic pressure
v = discharge velocity
x = depth from level of sedimentation
α, β = coefficients
γ_1 = unit weight of the sediment
γ = submerged unit weigth ($\gamma = \gamma_1 - \gamma_w$)
γ_w = unit weight of water
Γ = gamma-function
ζ = time function
ξ = x/h = ratio between depth x and head of sediment, h.

Derivation of the Differential Equation for the Time Function ζ

The requirement for continuous flow of water leads to the following equation between the discharge velocity v, and the pore water pressure u (*Terzaghi*, 1946)

$$\frac{\partial v}{\partial x}dx = dv = -m_v\frac{\partial u}{\partial t}dx \qquad (1)$$

The sum of the change in pore water pressure and the change in grain pressure p must be zero, hence

$$\frac{\partial p}{\partial t} = -\frac{\partial u}{\partial t} \qquad (2)$$

The combination of Equations (1) and (2) yields:

$$dv = m_v\frac{dp}{dt}dx \qquad (3)$$

from which

$$v = \int_x^h m_v\frac{dp}{dt}dx \qquad (3\text{a})$$

In this equation h designates the thickness of the entire layer of sediment, while x represents an arbitrary depth below the ground surface at a certain time t, see Fig. 1.

According to *Darcy*'s law the discharge velocity v may be expressed as follows:

$$v = \frac{k}{\gamma_w}\frac{d(\gamma x - p)}{dx} \qquad (4)$$

where $(\gamma x - p)$ is the pore water pressure, since γx represents the total pressure at depth x while p is the effective grain pressure. Integration of Equation (4) leads to:

$$\gamma x - p = \frac{\gamma_w}{k}\int_0^x v\,dx \qquad (4\text{a})$$

in which the discharge velocity formula, Equation (3a), must be introduced.

For the effective grain pressure p, *Terzaghi* (1924) assumes an exponential function, where the power exponent contains the depth x as well as a time function ζ. The latter is found by solving a differential equation for ζ.

The expression for the grain pressure p may, with some modification, be written as follows:

$$p = \gamma h(1 - e^{-\zeta x/h}) \quad \ldots\ldots\ldots\ldots \qquad (5)$$

in which e is the base of the Napierian logarithm.

Particularly, when taking only the two first terms into account in a series for $e^{-\zeta x/h}$, Equation (5) yields

$$p = \zeta\gamma x \quad \ldots\ldots\ldots\ldots\ldots \qquad (5)$$

For this grain pressure variation, which is linear with respect to depth, solutions are given by *Terzaghi* (1924) and *Gran Olsson* (1949).

The following derivations will be based on the assumption that the grain pressure profiles are similar with respect to depth and time. In terms of mathematics:

$$p = \gamma h\,\zeta(t)\,f(\xi) \quad \ldots\ldots\ldots\ldots\ldots \qquad (5\text{b})$$

in which equation the depth x is represented by the dimensionless free variable $\xi = x/h$.

The change in grain pressure is obtained from Equation (5b)

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx = \gamma h\left[\frac{\partial \zeta}{\partial t}f(\xi)dt + \frac{\partial f}{\partial \xi}\zeta(t)\,d\zeta\right] \ldots \qquad (6)$$

and hence, since $\frac{dx}{dt} = \frac{q}{\gamma}$,

$$\frac{dp}{dt}dx = \gamma h\left[\frac{\partial \zeta}{\partial t}f(\xi)\,dx + \frac{\partial f}{\partial \xi}\zeta(t)\frac{q}{\gamma}d\xi\right] \qquad (6\text{a})$$

Since $f(\xi)$ and $\zeta(t)$ are functions of ξ and t, respectively, the partial derivatives ∂ may be replaced by the usual symbols d, leading to

$$\frac{dp}{dt}dx = \gamma h\left[\frac{d\zeta}{dt}f(\xi)\,dx + df(\xi)\zeta(t)\frac{q}{\gamma}\right] \qquad (6\text{b})$$

The combination of Equations (3a) and (6b) yields

$$v = m_v\gamma h\int_\xi^1\left[\frac{d\zeta}{dt}f(\xi)dx + df(\xi)\zeta(t)\frac{q}{\gamma}\right] \qquad (3\text{b})$$

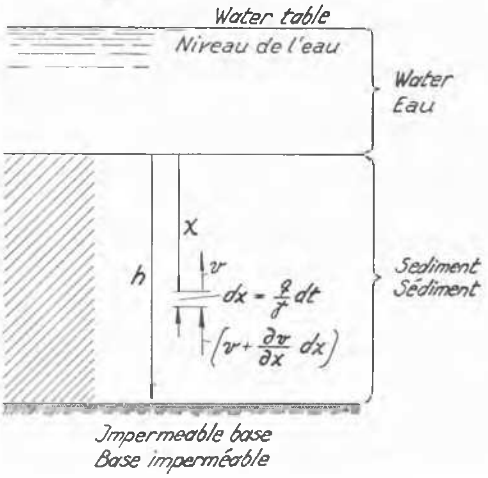

Fig. 1 One-Dimensional Flow in a Sediment
Filtration uniaxiale dans un sédiment

or, when integrated

$$v = m_v \gamma h \left[\frac{d\zeta}{dt} \{F(1) - F(\xi)\} h + \frac{q}{\gamma} \zeta(t) \{f(1) - f(\xi)\} \right] \quad (3c)$$

where $F(1) - F(\zeta) = \int_{\xi}^{1} f(\zeta)\, d\xi$.

The combination of Equations (4a), (5a) and (3c) leads to the following

$$\gamma h [\xi - \zeta(t) f(\xi)] = \frac{\gamma_w}{k} m_v \gamma h \left\{ \frac{d\zeta(t)}{dt} [F(1)\xi - G(\xi) + G(0)]\, h^2 + \right.$$

$$\left. + \frac{q}{\gamma} \zeta(t) [f(1)\xi - F(\xi) + F(0)]\, h \right\} \quad (4b)$$

where $G(\xi) - G(0) = \int_{0}^{\xi} F(\xi)\, d\zeta$.

For $x = h = \frac{tq}{\gamma}$, that is $\zeta = 1$, Equation (4b) is reduced to

$$[1 - \zeta(t) f(1)] = \frac{\gamma_w m_v q^2 t}{k\gamma} \left[\beta \frac{\partial \zeta}{dt} + \frac{1}{t} \alpha\beta\zeta \right] \quad (4c)$$

where

$$\left. \begin{aligned} \alpha\beta &= f(1) - F(1) + F(0) \\ \beta &= F(1) - G(1) + G(0) \end{aligned} \right\}$$

The following abbreviation for the "time constant" is introduced

$$c = \frac{f(1)}{\beta} \frac{k\gamma^2}{\gamma_w m_v q^2} \quad (7)$$

by means of which Equation (4c) becomes

$$\frac{d\zeta}{dt} + \left(\frac{a}{t} + \frac{c}{t^2} \right) \zeta = \frac{c}{f(1)} \frac{1}{t^2} \quad (8)$$

This is the general differential equation in ζ, provided that the effective grain pressure distribution is given by Equation (5b).

The boundary condition requires that the pore water pressure

$$\gamma x - p = \gamma h [\xi - \zeta(t) f(\xi)]$$

be equal to zero for $t = 0$ and $x = h$, i.e. $\xi = 1$. This requirement is seen to be satisfied when $\zeta(0) f(1) = 1$, from which relationship one obtains the only constant of integration which will appear in the solution of the differential Equation (8).

Solution of the Differential Equation (8)

A general solution of Equation (8) may be established for any values of the parameters a, c and $f(1)$, by substituting

$$z = \frac{c}{t} \text{ i.e., } dz = -ct^{-2} dt \quad (9)$$

after which Equation (8) reads

$$\frac{d\zeta}{dz} - \left(\frac{a}{z} + 1 \right) \zeta = -f(1)^{-1} \; .. \quad (8a)$$

Neglecting the term on the right hand side of this equation one arrives at the homogeneous solution

$$\zeta_0(z) = c_0 z^a e^z \quad .. \;.. \;.. \;.. \;.. \;.. \quad (10)$$

The complete solution is then obtained by varying the constant of integration

$$\zeta(z) = [C_0 + C_p(z)]\, z^a e^z \; .. \;.. \;.. \qquad .. \;.. \;.. \quad (10a)$$

where $C_p(z)$ is found by introducing Equation (10a) into Equation (8a), yielding

$$C_p(z) = -f(1)^{-1} \int z^{-a} e^{-z} dz \; .. \quad (11)$$

The combination of Equations (10a) and (11) offers the complete solution

$$\zeta(z) = [C_0 - f(1)^{-1} \int_0^z z^{-a} e^{-z} dz]\, z^a e^z \quad .. \quad (10b)$$

The integral is known as the incomplete Gamma-function, the numerical values of which may be found in *Pearson* (1946).

In order to determine C_0 one must require that ζ is finite for $t = 0$, that is for $z = \infty$, or

$$C_0 = f(1)^{-1} \int_0^\infty z^{-a} e^{-z} dz = f(1)^{-1}\, \Gamma(1 - a) \; .. \qquad .. \quad (11a)$$

where $\Gamma(1 - a)$ denotes the complete Gamma-function, hence

$$\zeta(z) = f(1)^{-1} \{ \Gamma(1 - a) - \int_0^z z^{-a} e^{-z} dz \}\, z^a e^z \; .. \qquad .. \quad (10c)$$

The ordinary symbols for the Γ-function is obtained by introducing $n - 1 = -a$, after which

$$\zeta(z) = f(1)^{-1} [\Gamma(n) - \gamma(n, z)]\, z^{-n+1} e^z \; .. \qquad .. \;.. \quad (10d)$$

where $\gamma(n,z)$ designates the incomplete Γ-function, as defined by the following relationship,

$$\gamma(n, z) = \int_0^z z^{n-1} e^{-z} dz.$$

Equation (10d) represents the general solution of the differential Equation (8a).

Example with Exponential Pressure Distribution

According to the pressure distribution curves, which are given by *Terzaghi* (1924, p. 171, Fig. 30), it appears likely that the distribution can be sufficiently approximated by the following expression,

$$p = \gamma h\, \zeta(t)\, (1 - e^{-\xi}) \quad (12)$$

hence,

$$\left. \begin{array}{lll} f(\xi) = 1 - e^{-\xi}; & \text{i.e. } f(1) = 1 - \frac{1}{e} & \\ F(\xi) = \xi + e^{-\xi}; & \text{i.e. } F(1) = 1 + \frac{1}{e}; & F(0) = 1 \\ G(\xi) = \frac{1}{2}\xi^2 - e^{-\xi}; & \text{i.e. } G(1) = \frac{1}{2} - \frac{1}{e}; & G(0) = -1 \end{array} \right\}$$

whereof

$$\left. \begin{aligned} \beta &= F(1) - G(1) + G(0) = \frac{4 - e}{2e} \\ a\beta &= f(1) - F(1) + F(0) = 1 - \frac{2}{e} \end{aligned} \right\}$$

yielding

$$\left. \begin{aligned} c &= \frac{f(1)}{\beta} \frac{k\gamma^2}{\gamma_w m_v q^2} = \frac{2(e - 1)}{4 - e} \frac{k\gamma^2}{\gamma_w m_v q^2} \\ a &= \frac{2(e - 2)}{4 - e} = 1{,}1208 \\ n &= 1 - a = \frac{8 - 3e}{4 - e} = -0{,}1208 \end{aligned} \right\}$$

Introducing these values of n and $f(1)$ into Equation (10d) one can evaluate ζ for all values of $z = c/t$. Hence the grain pressure p is found for any time, t, and depth, $x = \xi h$, by means of Equation (12).

For the numerical computation, however, it is preferable to substitute the *Whittaker* function (1935) defined as

$$W_{k,m}(z) = \frac{e^{-\frac{z}{2}} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty \varphi^{-k-\frac{1}{2}+m} \left(1 + \frac{\varphi}{z}\right)^{k-\frac{1}{2}+m} e^{-\varphi} d\varphi$$

The relationship between the *Whittaker-* and the Gamma-function is the following

$$\Gamma(n) - \gamma(n, z) = z^{\frac{n-1}{2}} e^{-\frac{z}{2}} W_{\frac{n-1}{2}, \frac{n}{2}}(z)$$

In our case, the *Whittaker* parameters, k and m, are seen to be

$$k = \frac{n-1}{2}, \quad m = \frac{n}{2}$$

after which

$$W_{\frac{n-1}{2}, \frac{n}{2}}(z) = e^{-\frac{z}{2}} z^{\frac{n-1}{2}} \int_0^\infty \left(1 + \frac{\varphi}{z}\right)^{n-1} e^{-\varphi}\, d\varphi$$

since $\Gamma(1) = 1$. Hence, the complete solution for $\zeta(z)$ becomes

$$\zeta(z) = f(1)^{-1} \int_0^\infty \left(1 + \frac{\varphi}{z}\right)^{n-1} e^{-\varphi}\, d\varphi$$

Using this expression $\zeta(z)$ is evaluated for various values of $z = \frac{c}{t}$. The arithmetic and semi-logarithmic plots are found in Figs. 2 and 3, respectively.

(*a*) An approximate solution for $\zeta(z)$ is obtained by introducing $\alpha = 1$ (instead of 1.1208) that is $n = 0$. Hence

$$\Gamma(0) - \gamma(0, z) = z^{-\frac{1}{2}} e^{-\frac{z}{2}} W_{-\frac{1}{2}, 0}(z) = -Ei(-z)$$

where $Ei(-z)$ denotes the exponential integral, defined by *Whittaker* and *Watson* (1935, p. 341) as

$$-Ei(-z) = \int_z^\infty \frac{e^{-\varphi}}{\varphi}\, d\varphi$$

Therefore, the approximate solution for $n = 0$ becomes

$$\zeta(z) = f(1)^{-1} z e^z \int_z^\infty \frac{e^{-\varphi}}{\varphi}\, d\varphi$$

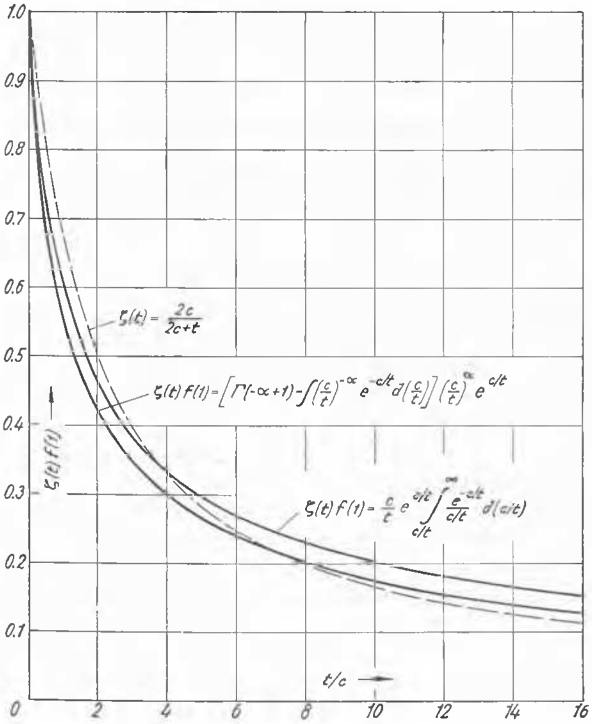

Fig. 2 Time Functions $\zeta(t)$ in Linear Time Scale
Graphique représentant les fonctions du temps $\zeta(t)$ par rapport à une échelle linéaire des temps

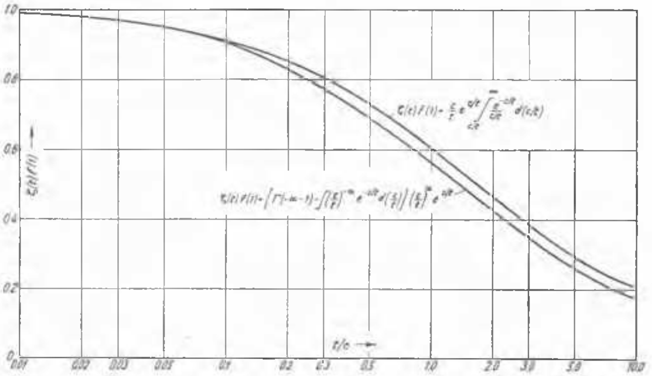

Fig. 3 Time Functions $\zeta(t)$ in Logarithmic Time Scale
Fonctions du temps $\zeta(t)$ en échelle logarithmique

Since the integral, $Ei(-z)$, is tabulated, this approximate solution is readily evaluated for various values of $z = \frac{c}{t}$, see Figs. 2 and 3. From these plots it is seen that the approximation is fairly good at the beginning of the process of sedimentation, but the discrepancy between the two curves ($\alpha = 1$ and $\alpha = 1.1208$) increases as the sedimentation progresses.

(*b*) In case the pressure distribution is linear, Equations (5a) and (5b),

$f(\xi) = \xi$; i.e. $f(1) = 1$
$F(\xi) = \frac{1}{2}\xi^2$; i.e. $F(1) = \frac{1}{2}$, $F(0) = 0$
$G(\xi) = \frac{1}{6}\xi^3$; i.e. $G(1) = \frac{1}{6}$, $G(0) = 0$

from which

$$\beta = \tfrac{1}{2} - \tfrac{1}{6} + 0 = \tfrac{1}{3}$$
$$\alpha\beta = 1 - \tfrac{1}{2} + 0 = \tfrac{1}{2}$$
$$\alpha = \tfrac{3}{2} \quad \text{or} \quad n = -\tfrac{1}{2}$$

while

$$c = 3 \frac{k\gamma^2}{\gamma_w m_v q^2}$$

The general solution is obtained from Equation (10b), introducing $\alpha = \frac{3}{2}$, $f(1) = 1$

$$\zeta(z) = \left[C_0 - \int_0^z z^{-\frac{3}{2}} e^{-z}\, dz\right] z^{\frac{3}{2}} e^z$$

Substituting $z = r^2$ the above equation is written as follows

$$\zeta(r) = [C_0 + 2r^{-1}e^{-r^2} + 2\pi^{\frac{1}{2}}\Phi(r)]\, r^3 e^{r^2}$$

where $\Phi(r)$ denotes the error integral. The constant of integration becomes $C_0 = -2\pi^{\frac{1}{2}}$, after which

$$\zeta(r) = [r^{-1}e^{-r^2} + \pi^{\frac{1}{2}}\Phi(r) - \pi^{\frac{1}{2}}]\, 2r^3 e^{r^2}$$

This particular solution is identical with the equation given by *Gran Olsson* (1949, p. 340, last equation), when r^2 is replaced by z.

The Consolidation of the Soil after the Sedimentation is Completed

The sedimentation process is assumed to be completed at a time $t = t_1$. In the following an attempt will be made in order to establish an approximate solution for the consolidation of the soil at a time $t > t_1$.

As soon as the depositing on the surface ceases, i.e. $q = 0$, one can introduce $\frac{dx}{dt} = 0$. Hence, from Equation (6a),

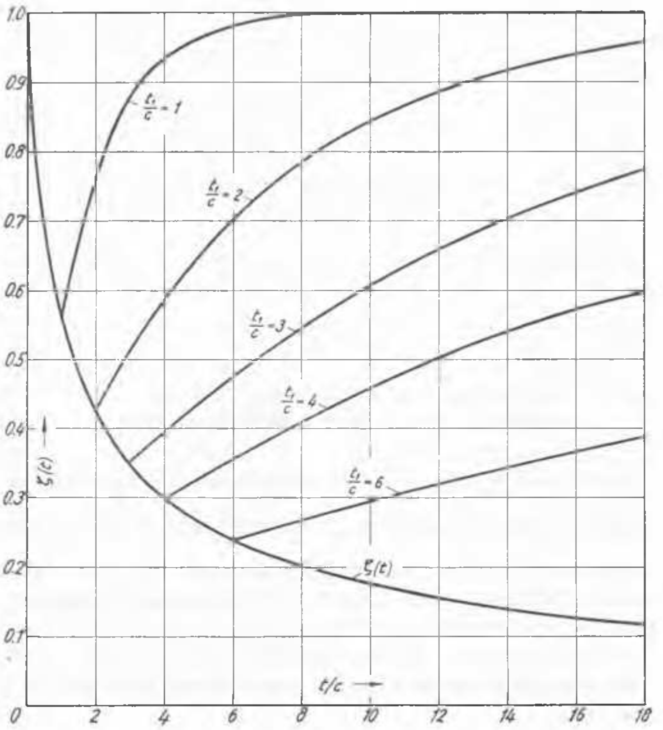

Fig. 4 Time Functions $\zeta(t)$ after the Sedimentation is Completed
Fonctions du temps $\zeta(t)$ une fois la sédimentation achevée

$$\frac{dp}{dt}dx = \gamma h_1 \frac{d\zeta}{dt} f(\xi)\, dx \quad .. \qquad (6c)$$

According to Equation (3b) the discharge velocity of the water percolating through the sedimentated soil becomes

$$v = m_v \gamma h_1^2 \frac{d\zeta}{dt} \int_0^1 f(\xi)\, d\xi = m_v \gamma h_1^2 \frac{d\zeta}{dt} [F(1) - F(0)] \qquad (3d)$$

The decrease in pressure per unit area, causing the water to flow upwards, is

$$\gamma h_1 \{1 - \zeta f(1)\} \frac{1}{h_1} = \gamma \{1 - \zeta f(1)\}$$

According to *Darcy*'s law the following relationship exists between this decrease in pressure and the discharge velocity

$$m_v \gamma h_1^2 \frac{d\zeta}{dt} [F(1) - F(0)] = \frac{k\gamma}{\gamma_w} [1 - \zeta f(1)] \quad .. \qquad (4d)$$

The time $t = t_1$ corresponds to the moment at which the sedimentation process is completed, and the value $\zeta(t_1)$ may be extrapolated from Figs. 2 or 3.

When solving Equation (4d) with respect to dt, and integrating one obtains

$$1 - \zeta(t)f(1) = [1 - \zeta(t_1)f(1)] \exp.\left\{\frac{kf(1)}{\gamma_w m_v h_1^2}(t - t_1)[F(1) - F(0)]\right\}$$

Introducing the "time constant", defined in Equation (7), one obtains

$$1 - \zeta(t)f(1) = [1 - \zeta(t_1)f(1)] \exp.\left\{\frac{c(t - t_1)}{\beta t_1^2}[f(1) - \alpha\beta]^{-1}\right\} \qquad (13)$$

where α and β have the same meaning as before.
Applying the pressure distribution given by Equation (12), Equation (13) becomes

$$1 - \zeta(t)f(1) = [1 - \zeta(t_1)f(1)] \;\exp.\left[\left(2 - \frac{e}{2}\right)\frac{c}{t_1}\left(\frac{t}{t_1} - 1\right)\right] \qquad (13a)$$

or in a more symmetric way,

$$[1 - \zeta(t)f(1)] \exp.\left(\frac{e}{2} - 2\right)\frac{ct}{t_1^2} = [1 - \zeta(t_1)f(1)] \exp.\left(\frac{e}{2} - 2\right)\frac{c}{t_1}$$

Since $2 - e/2 = 0.641$ the above equation may also be written

$$1 - \zeta(t)f(1) = [1 - \zeta(t_1)] \;\exp.\left[0.641\,\frac{c}{t_1}\left(\frac{t}{t_1} - 1\right)\right] \quad .. \qquad (13b)$$

By means of this equation, $\zeta(t)f(1)$ may be evaluated for each chosen point $(t_1 \zeta_1)$ on the curve $\zeta(t)$. Choosing several initial points (t_1, ζ_1) one will obtain a set of curves as shown in Fig. 4, defining the consolidation of the soil after the completion of the sedimentation.

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