The Consolidation of a Layer, which Modulus of Elasticity is proportional to the Depth

Le tassement d'une couche, dont la compressibilité diminue linéairement avec la profondeur

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Summary

This paper deals with the hydro-dynamical consolidation of a soil layer, supposing, that the compressibility of the soil decreases in a linear way with the depth. The formulae of the time-settlement-curve is derivated for this case.

Sommaire

Dans cette communication le tassement d'une couche est étudié comme phénomène variable dans le temps, et dépendant de l'écoulement des eaux souterraines. En admettant que la compressibilité du sol diminue linéairement avec la profondeur, l'auteur établit une formule donnant le tassement en fonction du temps.

Physical Problem

A soil-layer, with thickness D, a coefficient of permeability K (cm/sec) and a modulus of elasticity E (kg/cm²), confined above by the free air and at the bottom by an impermeable layer, is loaded at the time $T = 0$ by a load p (kg/cm²).

At the first moment the water within the voids bears this load, and at once this void-water becomes an increase of head: φ_0. After a little time the head has decreased to $\varphi < \varphi_0$ by the flow of water, and consequently the pressure σ_k between the particles of the soil has increased. Always

$$\sigma_k + \gamma_w \varphi = p \qquad (\gamma_w = \text{spec. weight of water}).$$

Than: $d\sigma_k = -\gamma_w \cdot d\varphi$.

The increase of σ_k causes a compression of the soil and means a decrease of the voids-ratio with:

$$-\frac{d\sigma_k}{E} F dx = \gamma_w \frac{d\varphi}{E} F dx .$$

Within the time-interval dT the decrease of the volume Fdx is equal to $\frac{\gamma_w}{E} \cdot \frac{\partial \varphi}{\partial T} \cdot Fdx$.

According to *Darcy*'s law the quantity of water, flowing out of the volume Fdx during ΔT equals: $k \frac{\partial^2 \varphi}{\partial x^2} \cdot Fdx$.

Both quantities have to be equal, thus

$$K \frac{\partial^2 \varphi}{\partial x^2} \cdot F dx = \frac{\partial \varphi}{\partial T} \cdot \frac{\gamma_w}{E} F dx$$

or:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\gamma_w}{KE} \cdot \frac{\partial \varphi}{\partial T}$$

Supposing E being a constant, *Terzaghi* and *Fröhlich* (1936) calculated this case by means of a *Fourier*-series. In this paper, however, is supposed, that E is not a constant, but that $E = C\sigma_k + \sigma$, where C = the constant of compressibility and σ = a small initial pressure. We put $\sigma_k = \gamma x$ (σ_k increasing with the depth) and thus:

$$E = C\gamma x + \sigma = C\gamma \left(x + \frac{\sigma}{c\gamma} \right)$$

For brevity we put $\frac{\sigma}{C\gamma} = a$ (cm) and consequently the diff. equation becomes:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\gamma_w}{KC\gamma} \cdot \frac{1}{x + a} \cdot \frac{\partial \varphi}{\partial T}$$

Putting $t = TCK \frac{\gamma}{\gamma_w}$ (t is a length), we obtain:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{x + a} \frac{\partial \varphi}{\partial t} \; .. \qquad (1)$$

Solution of the Partial Differential Equation

We want a solution in the shape: $\varphi = f(x) \cdot F(t)$ where $f(x)$ is a function, independent of t, and $F(t)$ is another function, independent of x. Equation (1) becomes:

$$\frac{x+a}{f} \cdot \frac{d^2 f}{dx^2} = \frac{1}{F} \frac{dF}{dt}$$

Thus on the left-hand side we get a function of the single variable x, and on the right-hand side a function of the other variable t. This can only be true, if both sides of the equation are equal to some constant: $-\frac{\beta^2}{4a}$. Thus we obtain two equations:

$$\frac{x+a}{f} \cdot \frac{d^2 f}{dx^2} = -\frac{\beta^2}{4a} \quad \text{and} \quad \frac{1}{F} \cdot \frac{dF}{dt} = -\frac{\beta^2}{4a}$$

or:

$$\frac{d^2 f}{dv^2} - \frac{1}{v}\frac{df}{dv} + f = 0 \quad \text{and} \quad \frac{dF}{F} = -\frac{\beta^2}{4a} dt$$

where $v = \beta \sqrt{\frac{x}{a} + 1}$

The solutions are:

$$f = v Z_1(v) \quad \text{and} \quad F = e^{-\frac{\beta^2}{4a} t}$$

$$Z_1(v) = C_1 J_1(v) + C_2 N_1(v)$$

$J_1(v)$ is the *Bessel* function of the first kind and first order. $N_1(v)$ is the *Bessel* function of the second kind (*Neumann*-function) and the first order.

Thus the solution of equation (1) becomes:

$$\varphi = v e^{-\frac{\beta^2}{4a} t} Z_1(v) \quad .. \tag{2}$$

Boundary Conditions

(1e) When $x = 0$, always $\varphi = 0$; or when $v = \beta$, $\varphi = 0$. This condition is satisfied when $Z_1(\beta) = 0$; or:

$$C_1 J_1(\beta) + C_2 N_1(\beta) = 0 \quad .. \quad .. \tag{3}$$

(2e) When $x = D$, always $\frac{\partial \varphi}{\partial x} = 0$, or:

when $v = v_D = \beta \sqrt{\frac{D}{a} + 1}$ is $\frac{\partial \varphi}{\partial x} = 0$.

Since:

$$\frac{\partial \varphi}{\partial x} = \frac{\beta^2}{2a} e^{-\frac{\beta^2}{4a} t} \{C_1 J_0(v) + C_2 N_0(v)\},$$

the second boundary-condition is satisfied when:

$$C_1 J_0(v_D) + C_2 N_0(v_D) = 0 \tag{4}$$

(3e) When $t = \infty$, everywhere $\varphi = 0$. This boundary-condition is always satisfied by the equation (2).

(4e) When $t = 0$, everywhere $\varphi = \varphi_0$. This leads to:

$$\varphi_0 = v \{C_1 J_1(v) + C_2 N_1(v)\},$$

and this is impossible. Thus the solution

$$\varphi = v e^{-\frac{\beta^2}{4a} t} Z_1(v)$$

does *not* satisfy the fourth boundary condition.

From the equations (3) and (4) we derive

$$\frac{J_1(\beta)}{N_1(\beta)} = -\frac{C_2}{C_1} \quad \text{and} \quad \frac{J_0(v_D)}{N_0(v_D)} = -\frac{C_2}{C_1}$$

Consequently:

$$\frac{J_1(\beta)}{N_1(\beta)} = \frac{J_0(v_D)}{N_0(v_D)}$$

Putting $\varrho = \frac{1}{\sqrt{\frac{D}{a} + 1}}$ we obtain $\beta = \varrho v_D$ and thus:

$$\frac{J_0(v_D)}{N_0(v_D)} = \frac{J_1(\varrho v_D)}{N_1(\varrho v_D)} \tag{5}$$

For each value of v_D, satisfying equation (5), we obtain a solution of the partial diff.-equation which satisfies the first three boundary-conditions. These values of v_D can be found in tables, f.i. in *Jahnke* and *Emde* (1945). (Here ϱ is named K.) Every value of v_D gives one solution, which can be written:

$$\varphi = A e^{-\frac{v_D^2 \varrho^2}{4a} t} \cdot v \cdot \left\{ J_1(v) - \frac{J_0(v_D)}{N_0(v_D)} N_1(v) \right\}$$

By addition of these solutions (every solution with its own A) we obtain a series, and we will choose such values for A, that the series satisfies the fourth boundary-condition.

Thus

$$\varphi = \sum_{n=1}^{\infty} A_n e^{-v_{Dn}^2 \varrho^2 \frac{t}{4a}} v \left\{ J_1(v) - \frac{J_0(v_{Dn})}{N_0(v_{Dn})} N_1(v) \right\}$$

Let be $v = v_D \varepsilon$ then $\varepsilon = \varrho \sqrt{\frac{x}{a} + 1}$

(when $x = 0$ is $\varepsilon = \varrho$; when $x = D$ is $\varepsilon = 1$)

Thus:

$$\varphi = \sum_{n=1}^{\infty} A_n e^{-v_{Dn}^2 \cdot \varrho^2 \frac{t}{4a}} v_{Dn} \varepsilon Z_1(v_{Dn} \varepsilon)$$

in which:

$$Z_1(v_{Dn}\varepsilon) = J_1(v_{Dn}\varepsilon) - \frac{J_0(v_{Dn})}{N_0(v_{Dn})} N_1(v_{Dn}\varepsilon)$$

The fourth boundary-condition is satisfied if the constant φ_0 can be written in the form

$$\varphi_0 = \sum_{n=1}^{\infty} A_n v_{Dn} \varepsilon Z_1(v_{Dn}\varepsilon) \quad \text{for} \quad \varrho < \varepsilon < 1$$

The theory of the *Bessel* functions shows that this is possible when the constants A_n are probably chosen. The determination of these constants is similar to the determination of the constants in an ordinary *Fourier*-series, using in our case the property of orthogonality of the *Bessel* functions. The nth coefficient A_n is given by the formula:

$$A_n = \varphi_0 \frac{2 Z_0(\varrho v_{Dn})}{\{v_{Dn} Z_1(v_{Dn})\}^2 - \{\varrho v_{Dn} Z_0(\varrho v_{Dn})\}^2} \tag{6}$$

Thus the solution of the equation, which satisfies the *four* boundary-conditions is:

$$\varphi = \sum_{n=1}^{\infty} A_n e^{-v_{Dn}^2 \varrho \cdot \frac{t}{4a}} v Z_1(v)$$

wherein A_n has to be calculated from the equation (6)

$$v = \varrho v_{Dn}\sqrt{\frac{x}{a}+1}$$

$$\varrho = \frac{1}{\sqrt{\frac{D}{a}+1}}$$

$$Z_1(v) = J_1(v) - \frac{J_0(v_{Dn})}{N_0(v_{Dn})} N_1(v)$$

and the values of v_{Dn} have to be calculated from the equation:

$$\frac{J_0(v_{Dn})}{N_0(v_{Dn})} = \frac{J_1(\varrho v_{Dn})}{N_1(\varrho v_{Dn})}$$

Settlement

The settlement z_D of the soil layer (thickness D), is equal to

$$\int_{x=0}^{x=D} \frac{\sigma_k}{E}\, dx\,,$$

wherein:

$$\sigma_k = \gamma_w(\varphi_0 - \varphi) \quad \text{and} \quad E = C\gamma\,(x+a)$$

Thus:

$$z_D = \int_{x=0}^{x=D} \frac{\gamma_w(\varphi_0-\varphi)}{C\gamma(x+a)}\,dx = \int_{x=0}^{x=D} \frac{\gamma_w \varphi_0}{C\gamma}\cdot\frac{dx}{x+a} - \int_{x=0}^{x=D} \frac{\gamma_w}{C\gamma}\cdot\frac{\varphi}{x+a}\,dx\,.$$

Since

$$v = \varrho v_D\sqrt{\frac{x}{a}+1}\,, \quad \frac{dx}{x+a} = \frac{2}{v}\,dv\,;$$

thus

$$z_D = \frac{2\gamma_w\varphi_0}{\gamma C}\int_{v=\varrho v_D}^{v=v_D}\frac{dv}{v} - \frac{2\gamma_w}{\gamma C}\int_{v=\varrho v_D}^{v=v_D}\frac{\varphi}{v}\,dv\,,$$

thus

$$z_D = \frac{2\gamma_w\varphi_0}{\gamma C}\lg\frac{1}{\varrho} - \frac{2\gamma_w\varphi_0}{\gamma C}\int_{v=\varrho v_D}^{v=v_D}\sum e^{-v^2_{Dn}\varrho^2\frac{t}{4a}}\,\frac{A_n}{\varphi_0}\,Z_1(v)\,dv$$

$Z_0(v) = 0$, when $v = v_D$, thus

$$z_D = \frac{2\gamma_w\varphi_0}{\gamma C}\left[\lg\frac{1}{\varrho} - \sum\frac{2\{Z_0(\varrho v_{Dn})\}^2}{\{v_{Dn}Z_1(v_{Dn})\}^2 - \{\varrho v_{Dn}Z_0(\varrho v_{Dn})\}}\,e^{-v_{Dn}^2\varrho^2\frac{t}{4a}}\right]$$

wherein $\gamma_w\varphi_0 = p$ = the increase of loading.

The final settlement z_∞ is reached after an infinite time period ($t = \infty$). Thus:

$$z_\infty = \frac{2\gamma_w\varphi_0}{\gamma C}\lg\frac{1}{\varrho} = \frac{p}{\gamma C}\lg\left(\frac{D}{a}+1\right)$$

$z = 0$ when $t = 0$, thus:

$$\frac{1}{2}\lg\frac{1}{\varrho} = \sum\frac{\{Z_0(\varrho v_{Dn})\}^2}{\{v_{Dn}Z_1(v_{Dn})\}^2 - \{\varrho v_{Dn}Z_0(\varrho v_{Dn})\}^2}$$

This equation allows to draw some conclusions concerning the convergence of the series. We try: $\varrho = 0.25$, therefore

$$\frac{1}{2}\lg\frac{1}{\varrho} = 0.693\,.$$

Calculating the first four terms of the series we obtain:

n	$Z_0(\varrho v_{Dn})$	$\{Z_0(\varrho v_{Dn})\}^2$	$\{v_{Dn}Z_1(v_{Dn})\}^2$	$\{\varrho v_{Dn}Z_0(\varrho v_{Dn})\}^2$	nth term
1	0.83	0.69	1.90	0.31	0.434
2	1.22	1.48	17.60	4.10	0.110
3	1.15	1.33	40.00	9.40	0.044
4	0.41	0.17	9.70	2.30	0.024
The first four terms give:					0.612

Using only the first four terms we obtain an error of *12%*. However, when t is not equal to zero, but reaches greater values, the convergence is better. Since the e-coefficient contains the form $(v_D)^2$, this coefficient will increase very soon in the further terms of the series, thus $e^{-v_{Dn}^2\varrho^2\frac{t}{4a}}$ will decrease very soon. For not to small values of t it may be sufficient to calculate only a few terms of the series.

Thus it may be possible to calculate the time-settlement-curve with sufficient precision, except for small values of t. The course of the curve for these small values of t, however, is not very interesting, this part of the curve being very steep; errors of 10% or 20% will not be perceptible here.

Example

Let be: $\sigma = 5$ kg/cm²; $C = 50$; $\gamma = \gamma_w = 1$ g/cm³

$D = 15$ m; $K = 1.48 \times 10^{-6}$ cm/sec (clay).

Thus $a = \dfrac{\sigma}{C\gamma} = 1$ m and $\varrho = 0.25$.

We put $\lambda = v_{Dn}^2\,\varrho^2\,\dfrac{t}{4a} = v_{Dn}^2\,\dfrac{\varrho^2}{4a}\cdot C\cdot K\dfrac{\gamma}{\gamma_w}\,T$

The table gives values of λ for different values of n and T.

$(v_{Dn})^2$	n	*T = 1 days*	*T = 7 days*	*T = 30 days*	*T = 150 days*	*T = 365 days*	*T = 1000 days*
7.20	1	0.007	0.05	0.22	1.08	2.70	7.20
43.—	2	0.043	0.30	1.30	6.45	15.70	43.—
113.—	3	0.113	0.80	3.40	17.—	41.50	113.—
218.—	4	0.218	1.53	6.60	33.—	80.—	218.—

When $\lambda = 1{,}0$ we obtain $e^{-\lambda} = 0.0005$. This is very small and consequently we may stop when $\lambda < 5$. When $T = 30$ days the first three terms are sufficient already.

We calculate:

when T =	1	7	30	150	365	1000 days
$\dfrac{z_D}{p}$ =	2.85	13.—	25.—	43.60	53.20	55.35

When $T = \infty$ is $\dfrac{z_D}{p} = \dfrac{z_\infty}{p} = 55.4$ (z_D is measured in cm and p in kg/cm²).

The figure shows this time-settlement curve; within 5 weeks the layer of clay consolidated to 50%, within 5 months to 75%, within 8 months to 90% and within one year to more than 95% of the final settlement.

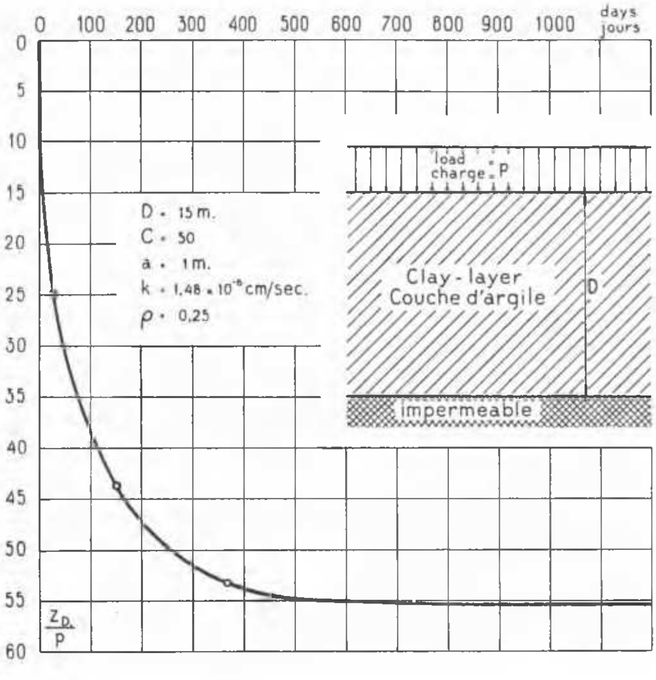

Fig. 1 Hydro-dynamic Time-Settlement Curve
Courbe de tassement hydro-dynamique

Final Form of the Formulae

We know:

$$\varrho = \frac{1}{\sqrt{\frac{D}{a}+1}}, \text{ so } \frac{\varrho^2}{a} = \frac{1-\varrho^2}{D}$$

$$t = TCK\frac{\gamma}{\gamma_w}, \text{ so } e^{-\nu_{Dn}^2 \varrho^2 \frac{t}{4a}} = e^{-\nu_{Dn}^2 (1-\varrho^2)\frac{C}{4}\frac{K}{D}\cdot\frac{\gamma}{\gamma_w}T}$$

Also:

$$z_\infty = \frac{2\gamma_w \varphi_0}{\gamma C}\lg\frac{1}{\varrho} \quad \text{and} \quad \gamma_w \varphi_0 = p.$$

For brevity we put:

$$F_n(\varrho) = \frac{4\{Z_0(\varrho \nu_{Dn})\}^2}{\{\nu_{Dn} Z_1(\nu_{Dn})\}^2 - \{\varrho \nu_{Dn} Z_0(\varrho \nu_{Dn})\}^2}$$

Thus:

$$(z_\infty - z_D)\frac{\gamma}{p} = \frac{1}{C}\sum e^{-\nu_{Dn}^2(1-\varrho^2)\frac{C}{4}\cdot\frac{K}{D}\frac{\gamma}{\gamma_w}T} F_n(\varrho).$$

References

Jahnke, E. and *Emde, F.* (1945): Tables of Functions. 4th edition, Dover Publications, New York.

Terzaghi, K. and *Fröhlich, O. K.* (1936): Theorie der Setzung von Tonschichten. Leipzig und Wien.