Determination by the Theory of Plasticity of the Bearing Capacity of Continuous Footings on Sand

Détermination de la force portante d'empattements sur sable par la théorie de plasticité

by H. Lundgren, Professor, Dr. techn., and Knud Mortensen, M.Sc., Soil Mechanics Laboratory, Technical University of Denmark, Copenhagen, Denmark

Summary

By means of the theory of plasticity, methods are developed for the exact determination of the rupture lines as well as the bearing capacity of continuous footings on a horizontal sand surface for any value of the surface load. The methods are applicable to rough as well as to smooth foundations.

For a rough base and $\varphi = 30°$, N_γ is found to be 14.8, which is only 70% of the value generally applied in the *Terzaghi* formula. This formula is based upon a linear combination of two terms containing γ (unit weight) and q (surface load), respectively. This paper shows that the linear combination is up to 17% conservative.

Sommaire

Cette communication étudie l'emploi de la théorie de la plasticité pour la détermination de la force portante des fondations de longueur infinie reposant sur du sable. En appliquant la théorie de la plasticité les auteurs ont developpé des méthodes pour la détermination rigoureuse des lignes de glissement et de la force portante des empattements reposant sur une surface de sable horizontale avec une surcharge arbitraire. Les méthodes sont applicables à des fondations rugueuses aussi bien qu'à des fondations lisses.

Pour des fondations rugueuses et $\varphi = 30°$, N_γ est calculé à 14,8; cette valeur représente seulement 70% de la valeur généralement employée dans la formule de *Terzaghi*. Cette formule est basée sur une combinaison linéaire de deux termes contenant respectivement γ (poids spécifique) et q (surcharge). Dans cet exposé, il est démontré que l'emploi de la combinaison linéaire donne une marge montant jusqu'à 17%.

General Principles

Fig. 1 shows an infinitesimal element of a sand mass, which is assumed to be in the state of two-dimensional plastic flow with the intermediate principal stress σ_2 perpendicular to the paper plane. The major and minor principal stresses at a point satisfy the relation

$$\frac{\sigma_1 - \sigma_3}{\sigma_1 + \sigma_3} = \sin \varphi, \tag{1}$$

where φ is the angle of internal friction. The plane contains two systems of rupture lines, the α-lines and the β-lines, which intersect at an angle of $\frac{\pi}{2} + \varphi$. The element in Fig. 1 is enclosed by two sets of consecutive rupture lines.

From the equations of equilibrium the following relations may be derived after some calculation:

$$\frac{\partial}{\partial s_1} (\ln t + 2\theta \tan \varphi) = \frac{\gamma}{t} \sin (\theta + \varphi) \tag{2}$$

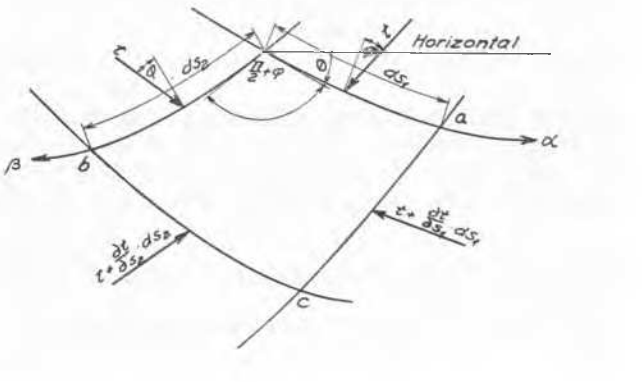

Fig. 1 Stresses Acting on an Infinitesimal Element between Rupture Lines

Contraintes d'un élément infinitésimal entre des lignes de glissement

$$\frac{\partial}{\partial s_2}(\ln t - 2\theta \tan \varphi) = \frac{\gamma}{t}\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Here ∂s_1 and ∂s_2 are the length elements along the rupture lines; t is the total stress on the faces of the element (forming the angle φ with the normal); θ is the clockwise angle from the horizontal to the positive α-direction; and γ is the unit weight of the sand. Eqs. (2–3) may be considered as a transcription of the well-known *Kötter*'s equation, *Kötter* (1888).

If the points a and b (Fig. 1) are given, as well as the values of θ and t at these points, the point c can be found by intersecting the β-line through a and the α-line through b, and Eqs. (2–3) can be used to determine the values of θ and t at point c. By repeated application of this procedure it will be seen that the whole system of rupture lines between a given α-line and a given β-line can be found, provided that the values of t along these lines are known. This method of constructing the lines of rupture is a special example of "the general method of characteristics", which is commonly used for solving partial differential equations of the hyperbolic type. The method of characteristics has been applied to plastic flow problems in purely cohesive materials by several authors, for example *Ischlinsky* (1944).

As shown below the method of characteristics can be used to find the whole system of rupture lines under a centrally loaded footing of width b resting on a horizontal surface of sand which, outside the footing, carries a uniform surface load q. The system of rupture lines depends partly on the roughness of the base and partly on the dimensionless ratio $\gamma b/q$. In the sections below the whole range of this parameter from zero to infinity will be considered.

Weightless Sand with a Surface Load

When $\gamma = 0$ and the base is smooth, the bearing capacity problem is easily solved. The solution shown in Fig. 4*a* was found by *Prandtl* (1920). Each α-line consists of two straight elements and a part of a logarithmic spiral, whereas all the β-lines are straight. The two triangles where all rupture lines are straight present Rankine zones.

For a rough base the same system of rupture lines applies, the only modification being that the Rankine zone under the footing will be in an elastic state (Fig. 4*b*). Since elastic deformations are neglected in comparison with plastic deformations, the triangle under the footing may be considered rigid.

For $\gamma = 0$ the right hand sides of (2) and (3) vanish. Hence t is a constant along straight rupture lines. In the Rankine zone above OA

$$t = q \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (4)$$

In the zone OAC the following relation can be derived from (3), AC being a β-line:

$$t = q \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \cdot \exp\left[2\left(\theta - \frac{\pi}{4} + \frac{\varphi}{2}\right)\tan\varphi\right], \quad (5)$$

where $\theta = \frac{\pi}{4} - \frac{\varphi}{2}$ along the α-line OA will reduce (5) into (4) and thus satisfy the boundary condition along this line.

The Sand Has Weight and Carries a Surface Load

In this case (Figs. 4*c–d*) it will be a question of the plastic flow properties of the sand whether the plastic zones will actually reach the surface outside the footing. Since this discussion is beyond the scope of this paper, it will simply be assumed that the plastic zones do extend to the surface. Then it follows that Rankine zones exist above the lines OA.

The construction of the rupture lines by the method of characteristics must start from OA and in all directions from O. It is evident that in the immediate vicinity of the edge O the weight of the sand has negligible influence, the stresses being determined only by the surface load, for which the *Prandtl* (1920) solution is known. Therefore, the subsequent considerations will deal with the state of stress near the edge; cf. Fig. 2 where the width of the footing is assumed to be very large.

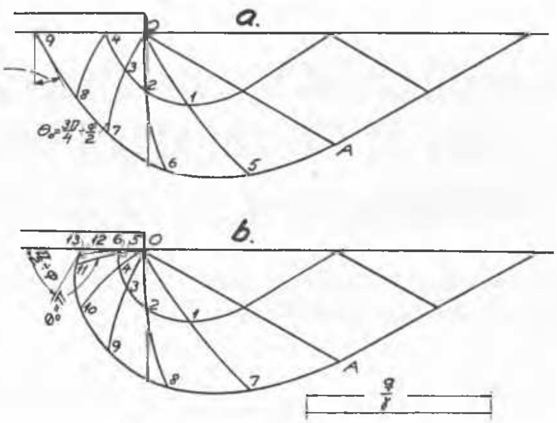

Fig. 2 Rupture Lines near the Edge of the Footing
a = Smooth Base b = Rough Base
Lignes de glissement près du bord de la fondation
a = fondation lisse b = fondation rugueuse

Along OA the stresses are given by the expression

$$t = \left(q + \frac{\gamma s_1}{2}\right)\tan\left(\frac{\pi}{4} + \varphi\right), \quad (6)$$

where s_1 is measured from O. At the point of singularity O the values of θ and t will be designated θ_0 and t_0. From this point, for a smooth base (Fig. 2*a*), a family of α-lines radiate covering the right angle between the lines OA and O–3–7, i.e. for these α-lines θ_0 varies from $\frac{\pi}{4} - \frac{\varphi}{2}$ to $\frac{3\pi}{4} - \frac{\varphi}{2}$. The corresponding values t_0 can be found from (5) if θ is substituted by θ_0. The α-lines to the left of O–3–7 start from the base under the same angle as O–3–7 (cf. Fig. 4*a*). A β-line in the vicinity of O consists, as in the *Prandtl* case, of two straight parts and a logarithmic spiral. With these remarks sufficient boundary conditions are given for the construction of the whole system of rupture lines (cf. the sequence 1, 2, ... 9 of points). When moving away from O, the α-lines curve because of the influence of γ.

For a rough (and very wide) footing, Fig. 2*b*, the logarithmic spiral near O continues to the base. The α-lines radiating from O cover the whole angle between OA and the base with θ_0-values varying from $\frac{\pi}{4} - \frac{\varphi}{2}$ to π. On the whole, the construction of rupture lines proceeds in the same manner as for the smooth base (cf. the sequence 1, 2, ... 13). The curve O–5–11 is the last α-line radiating from O. The subsequent α-lines start from the base to the left of O, and since the base is rough, they are all tangential to the base at their starting points. The curve 6–12 is one of these α-lines.

Figs. $2a$–b show the rupture lines only in the vicinity of the edge (cf. the length q/γ indicated in the figure). The system of lines in a larger region will appear for a rough base from Fig. 3 (for which the scale is about 1/10 of that for Fig. 2).

For a footing of finite width b, the width must be plotted in Fig. 3 to the same scale as q/γ. When this is done, a similar system of rupture lines must be imagined to exist under the left side of the footing. The result of this symmetrization is the failure conditions illustrated by Fig. $4c$ for a smooth base and Fig. $4d$ for a rough base. Fig. $4c$ requires no further explanation. In Fig. $4d$ the "triangle" $O\,O\,C$ can be considered rigid. The point C is found on the axis of symmetry by the condition $\theta = \frac{3\pi}{4} - \frac{\varphi}{2}$ for the α-line $O\,C$ from the right edge, thus giving a smooth transition at point C between the lines from the right and the (symmetrical) lines from the left.

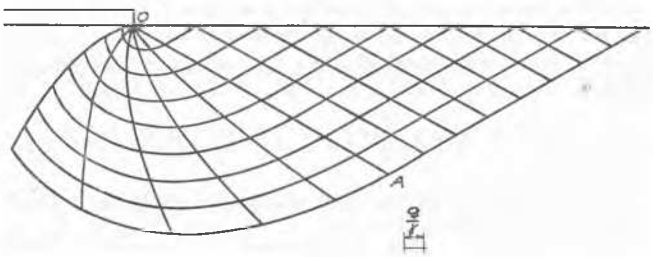

Fig. 3 Rupture Lines under a Rough Base for $\varphi = 30°$
Lignes de glissement sous une fondation rugueuse pour $\varphi = 30°$

It should be mentioned that all figures in this paper are drawn for $\varphi = 30°$. The solution in Fig. 3 has been carried so far that it can be used for footings with values of $\gamma b/q$ less than 11.4. For smaller values of this parameter the α-line in Fig. $4d$ that forms the boundary of the rigid zone will extend from C to the edge O. For values larger than 11.4 the α-line from C will not extend to the very edge, but it will become tangential to the base at some distance from the edge where it will stop (cf. the α-lines in Fig. $2b$).

The wider the footing, the less influence has q on the failure load. Therefore, the deepest β-lines in Fig. 3 must be expected to be very similar to the rupture lines for the case $q = 0$. As a matter of fact, the construction in this figure has been carried so far that the difference in failure load is relatively small. However, it would not be feasible to continue the construction to infinity. Therefore, the case $q = 0$ is discussed separately below.

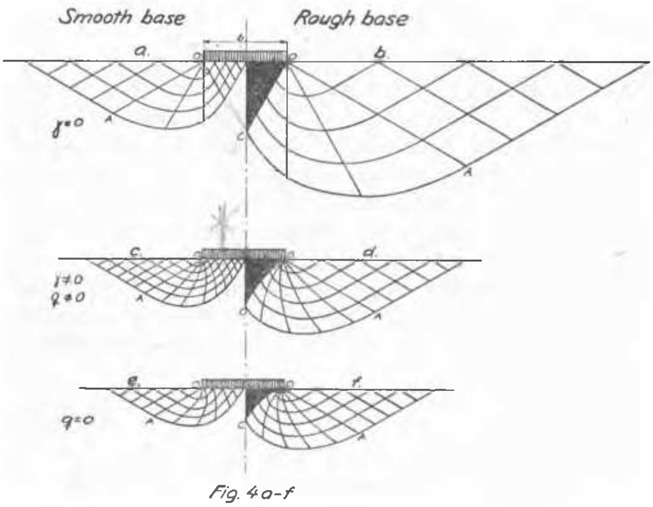

Fig. 4 Rupture Lines under Smooth and Rough Bases for $\varphi = 30°$
Lignes de glissement sous des fondations lisses et rugueuses pour $\varphi = 30°$

The Sand Has Weight, but Carries no Surface Load

This case can be solved by means of the differential equations and the boundary conditions by the method used by *v. Kármán* (1926) for the problem of active earth pressure on a vertical wall without surface load. In our problem the wall is substituted by the footing and active pressure by passive pressure. The edge O of the footing (Figs. $4e$–f) corresponds to the topmost point of the wall.

The problem is mathematically simple because it depends upon the solution of ordinary differential equations only. This is due to the fact that all rupture lines are similar with O as the point of similarity.

From the equations of equilibrium and the condition of failure the following two differential equations can be derived after a good deal of calculation:

$$(\cos 2w - \cos 2\alpha)\frac{dm}{d\theta} + m \sin 2w = 2\cos^2\alpha \cdot \cos(\theta - 2w),$$

$$\left(\frac{1}{\cos 2\alpha} - \cos 2w\right)\frac{dm}{d\theta} + 2m \sin 2w \frac{dw}{d\theta} - 3m \sin 2w =$$

$$= \frac{2\cos^2\alpha}{\cos 2\alpha}\cos\theta\,.$$

Here $\alpha = \frac{\pi}{4} + \frac{\varphi}{2}$ and $m = \frac{\sigma_3}{\gamma r}$, while r denotes the radius vector from O, w denotes the angle between σ_3 and r, and θ denotes the angle between the horizontal and r.

The boundary conditions for $\theta = \frac{\pi}{4} - \frac{\varphi}{2}$ (at the Rankine zone) are: $w = \pi - \alpha$ and $m = \cos\alpha$; and for $\theta = \pi$ (at the base): $w = \pi$ for a smooth base and $w = \pi - \alpha$ for a rough base.

The numerical integration must start from the base, where only one boundary condition is given. Therefore, the value of m at the base must be estimated and improved by trial and error until the boundary conditions for $\theta = \frac{\pi}{4} - \frac{\varphi}{2}$ are satisfied (*Damgaard*, 1951).

For a smooth base the lines of rupture are sketched in Fig. $4e$. For a rough base the result of the calculation is shown in Fig. $4f$. Through O only one α-line (OA) passes. The α-lines to the left of OA are all tangential to the base at some distances from the edge.

For a footing of given width the symmetrization is done in the same way as in Fig. $4d$. Let Q be the point where the α-line boundary between the plastic and elastic zones touches the base. Then the calculations have given the following values: $OQ = 0.104\frac{b}{2}$; $\theta_C = 140.8°$; and $OA = 2.13\frac{b}{2}$.

Bearing Capacity

Since now the system of rupture lines is known for any value of $\gamma b/q$, Fig. $4d$ giving a completely smooth transition between $4b$ and $4f$, the corresponding bearing capacities can be easily computed.

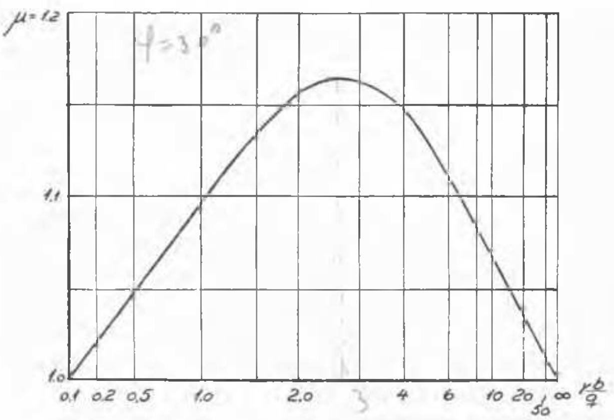

Fig. 5 Variation of μ in (7)
Variation de μ dans la formule (7)

If, in analogy to *Terzaghi*'s formula, the bearing capacity is written as

$$p_{\text{ult.}} = \mu \left(q N_q + \gamma \frac{b}{2} N_\gamma \right) , \tag{7}$$

the coefficient μ will depend upon φ, the ratio $\gamma b/q$, and the roughness, whereas N_q and N_γ are the bearing capacity factors corresponding to $\gamma = 0$ and $q = 0$, respectively.

For $\varphi = 30°$ the values $N_\gamma = 14.8$ and $N_q = 18.3$ have been found. The coefficient μ is plotted as function of $\gamma b/q$ in Fig. 5. It is noteworthy that the value of N_γ amounts to only 70% of the value usually applied (e.g. *Meyerhof*, 1951). In the *Terzaghi* formula μ is neglected; from Fig. 5 it appears that this gives safe values with an error not exceeding 17%.

Finally, it must be emphasized that all the systems of rupture lines in this paper have been derived from pure stress considerations. Therefore, they are statically possible, but they may be or may not be kinematically possible depending on the actual strain properties of the sand during plastic flow.

Therefore, the bearing capacities found may be slightly conservative because the "actual" figure of failure (which would include several rigid bodies) must correspond to a higher load (cf. *Prager* and *Hodge*, 1951).

At the present stage of knowledge, however, we think that the bearing capacities found by the theory of plastic stresses must be considered a satisfactory approximation to the true values.

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