Saturated Steady Flow in Non-Homogeneous Media and its Applications to Earth Embankments, Wells, Drains

Ecoulement permanent en milieux saturés hétérogènes avec application aux barrages en terre, puits et drains

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Summary

The *Darcy* law, when applied to saturated isotropic non-homogeneous media, combined with the equation of continuity, leads towards a partial differential potential equation, of the *Poisson* type. Neither conformal mapping nor the usual flownet method is applicable. The introduction of hydraulic resistivity (impermeability), the inverse of the hydraulic conductivity (permeability), simplifies the computations.

In one-dimensional confined flow, the pressure decreases horizontally or upwards. In descending flow, it may increase or present maxima and minima, where resistivity attains its average value along the flowline. The maximum is pronounced where resistivity increases rapidly and the flowline flattens out. The minimum is pronounced where conductivity increases rapidly and the flowline steepens. This is confirmed by pore-pressure measurements, in earthfill zoned embankments, and by the author's observations on asphalt lined ponds (formation of huge asphalt bubbles), on brine pans on the Dead Sea, and on sand dunes after rainfall. A cavitation phenomenon is foreseen and observed in the soil whenever a thin impervious layer overlies a thick permeable one.

Average resistivity, an important factor, is computed in several cases.

A formula is developed for a horizontal drainage gallery or ditch and found to be identical with that for homogeneous soils with average resistivity. When two galleries drain a uniform rainfall, the water table is no longer elliptic. In radial unconfined flow to a well the discharge obeys a formula different from *Dupuit*'s formula which is used by *Thiem*, and the correction is determined.

Sommaire

La loi de *Darcy*, appliquée aux milieux isotropes hétérogènes saturés, et la continuité, conduisent à une équation aux dérivées partielles du type *Poisson*. Ni la représentation conforme ni la méthode des petits carrés ne sont applicables. Les calculs sont simplifiés par l'introduction de la résistivité hydraulique (imperméabilité) qui est l'inverse de la conductivité hydraulique (perméabilité).

Pour un écoulement confiné à une dimension, la pression décroît horizontalement ou vers le haut. Pour un écoulement descendant elle peut croître ou présenter des maxima et minima, là où la résistivité atteint sa valeur moyenne le long d'une trajectoire. Le maximum est prononcé là où la résistivité croît brusquement et la trajectoire s'aplatit. Le minimum est prononcé là où la conductivité croît brusquement et la trajectoire se raidit. Ceci est confirmé par des mesures de pression sur des barrages en terre, et par les observations recueillies par l'auteur sur des étangs avec perré asphaltique (formation de larges bulles d'asphalte), sur les marais salants de la Mer Morte, et sur des dunes après la pluie. Un phénomène de cavitation est prévu et observé dans le sol chaque fois qu'une mince couche imperméable se trouve au-dessus d'une couche perméable épaisse.

La résistivité moyenne, facteur important, est calculée pour quelques cas.

Une formule est présentée pour une galerie drainante ou fossé horizontal; elle est identique à celle des sols homogènes de résistivité moyenne. Lorsque deux galeries drainent une pluie uniforme, le niveau d'eau n'est plus elliptique. En écoulement non-confiné vers un puits, le débit obéit à une formule qui diffère de celle de *Dupuit*, utilisée par *Thiem*, et la correction en est calculée.

Introduction

The flow of liquids through saturated homogeneous isotropic inert porous media at low *Reynold*'s numbers and constant temperature is governed by the *Darcy* formula (*Irmay*, 1947):

$$q = -k\nabla E = -k \operatorname{grad} E \quad \ldots \quad (1)$$

where: q = vector of specific discharge (dimensions LT^{-1}); and E = scalar total energy of flow (dimensions L).

$$E = h + v^2/2g = (z + p/\gamma) + v^2/2g \approx h \quad \ldots \quad (1')$$

h = piezometric head; p/γ = pressure head, referred to atmospheric pressure head as datum; γ = unit fluid weight; $v^2/2g$ = kinetic energy, usually neglected; k = scalar (in non-isotropic media k becomes the second order hydraulic conductivity tensor, which degenerates in isotropic media into an isotropic tensor, whose single component is the hydraulic conductivity) hydraulic conductivity, constant for given porous medium and fluid (*Richards*, 1952), often called coefficient of permeability (dimensions LT^{-1}). Then:

$$q = -\nabla(kE) = -\nabla\varphi;\quad \varphi = kE \qquad (2)$$

φ, which is proportional to E, is the flow potential (Fig. 1).

Saturated steady or unsteady flow of a chemically inactive incompressible fluid (water) in the absence of compressible free gases (air, vapour) and in a geometrically stable porous medium obeys the continuity law:

$$\operatorname{div} q = -\operatorname{div}(k\nabla E) = -k\nabla^2 E = 0 \qquad (3)$$

$$\nabla^2 E = \partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 + \partial^2 E/\partial z^2 = 0 \qquad (3')$$

E (or φ) is a potential obeying *Laplace* equation (3'). The solutions for steady flow and given boundary conditions may be computed directly, or in two-dimensional flow, by the method of conformal mapping, known also as the method of flownets or the little squares.

Darcy Law for Non-Homogeneous Media

Most soils are not homogeneous, but vary in conductivity continuously, as in the neighbourhood of a well after development or after clogging by prolonged use; or discontinuously, as in a zoned earthfill embankment or when the formation changes suddenly. Here k is a function of x, y, z.

Darcy law (1) was applied to (2) by *Boussinesq* (1904):

$$\operatorname{div} q = -\operatorname{div}(k\nabla E) = -[k\nabla^2 E + (\nabla k\,\nabla E)] = 0 \qquad (4)$$

$$\nabla^2 E + (\nabla \ln k\,\nabla E) = 0 \qquad (4')$$

$$\partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 + \partial^2 E/\partial z^2 + (\partial \ln k/\partial x)\cdot \partial E/\partial x + (\partial \ln k/\partial y)\cdot \partial E/\partial y + (\partial \ln k/\partial z)\cdot \partial E/\partial z = 0 \qquad (4'')$$

Here: ∇k = grad k; ln = natural logarithm; (AB) = scalar product of the vectors A, B.

E is a potential function of the *Poisson* type. As it does not obey *Laplace* equation (3'), the method of conformal mapping is not applicable in two-dimensional flow; (2) is no more true and φ loses any physical meaning.

(4'') is a second order partial differential equation, linear and homogeneous in E, with additive properties: if E_1, E_2 are any two solutions, then $C_1E_1 + C_2E_2$ is a solution with C_1, C_2 any two constant numbers.

The component of the specific discharge in the direction of flow (path s) q_s is

$$q_s = -k\,\partial E/\partial s = kJ \qquad (5)$$

$J = -\partial E/\partial s$ is the energy or hydraulic slope, practically identical with the piezometric slope $-\partial h/\partial s$. Thus (4') becomes

$$\nabla^2 E = J\cdot \partial \ln k/\partial s \qquad (6)$$

The vector q is normal to any equipotential surface E = const.

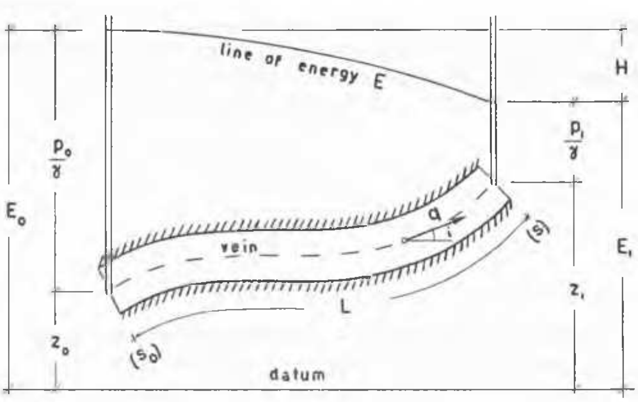

Fig. 1 Energy Line
Ligne d'énergie

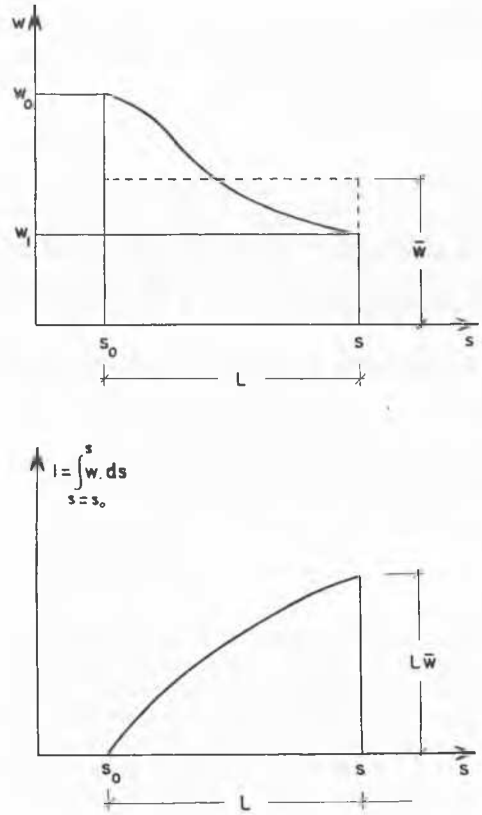

Fig. 2 Average Hydraulic Resistivity
Résistance hydraulique moyenne

A very useful concept is that of *hydraulic resistivity* w

$$w = 1/k \qquad (7)$$

of dimensions $L^{-1}T$.—Just as the intrinsic permeability $K = k\nu/g$ (where ν = kinematic viscosity of fluid, g = gravity acceleration) is independent of the fluid, so also the *intrinsic impermeability* may be defined by $W = 1/K$, and depends on the porous medium alone. In non-isotropic media both w and W become second order tensors bearing the same names.

The above equations become

$$wq = -\nabla E \qquad (8)$$

$$wq_s = J \qquad (8')$$

$$\nabla^2 E = (\nabla \ln w\,\nabla E) = -J\cdot \partial \ln w/\partial s \qquad (9)$$

$$\partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 + \partial^2 E/\partial z^2 = (\partial \ln w/\partial x)\,\partial E/\partial x + (\partial \ln w/\partial y)\cdot \partial E/\partial y + (\partial \ln w/\partial z)\cdot \partial E/\partial z \qquad (9')$$

One-Dimensional Steady Confined Flow

Many problems in steady flow can be reduced to one-dimensional ones. Here:

$$q = -k\,dE/ds = -w^{-1}dE/ds = \text{const.} \qquad (10)$$

Integrating between any two points s_0 and s along the streamline, we get the loss of energy (or loss of head) H:

$$H = E_0 - E = q\int_{s=s_0}^{s} w\,ds = qI;\quad I = \int_{s=s_0}^{s} w\,ds \qquad (11)$$

The integral may be computed by graphical integration (Fig. 2). The upper curve $w(s)$ is plotted, and the area subtended by it is the integral I represented by the lower curve. The loss of energy is $H = qI$.

The average resistivity $\bar{w}$ over the length $L = s - s_0$ is

$$\bar{w} = I/L = L^{-1} \int_{s=s_0}^{s} w ds \quad (12)$$

The average hydraulic slope

$$\bar{J} = H/L = \bar{w}q \quad (12')$$

The total discharge Q of a fluid vein of constant cross-section A

$$Q = qA = A\bar{J}/\bar{w} \quad (12'')$$

The pressure head from (1′) is

$$p/\gamma \approx E - z = E_0 - (H + z) = (E_0 - z_0) - [qI + (z - z_0)] \quad (13)$$

The slope i of the vein axis is

$$i = \sin \alpha = dz/ds \quad (14)$$

so

$$p/\gamma = (E_0 - z_0) - \int_{s=s_0}^{s} F ds; \quad F = wq + i; \quad dp/ds = -\gamma F;$$

$$d^2p/ds^2 = -\gamma dF/ds = -\gamma [q dw/ds + di/ds] \quad (13')$$

(*a*) In steady *horizontal flow*,

$$i = 0; \quad z = z_0; \quad \int_{s=s_0}^{s} F ds = qI = H = \bar{J}L; \quad F = wq > 0;$$

$$dp/ds < 0 \quad (15)$$

The pressure *decreases* in the *direction* of flow, until at the outlet $p = 0$ (atmospheric pressure).

(*b*) In *ascending flow*, whether vertical or inclined,

$$i > 0; \quad z > z_0; \quad \int_{s=s_0}^{s} F ds = H + (z - z_0) = \bar{J}L + (z - z_0);$$

$$F = wq + i > 0; \quad dp/ds < 0 \quad (15)$$

The pressure *decreases* rapidly in the direction of flow. If the upper outlet has contact with the atmosphere, the vein may reach the point $p = 0$ ($z = z_p$; $s = s_p$) called *phreatic water table*. Above this point the pressure becomes subatmospheric, sometimes called negative pressure, suction, or tension. The vein may break up, air or vapour collecting in pockets, and the whole system does not obey *Darcy* law except in a completely modified form (*Irmay*, 1953). When there is no atmosphere above the nappe, the vein breaks up only at considerably lower pressures.

(*c*) In *descending flow*, $i < 0$; $z < z_0$.

When $wq > -i$; $F > 0$: the pressure decreases in the direction of flow at high discharges, impervious medium, flat slopes.

When $wq < -i$; $F < 0$: the pressure increases in the direction of flow at low discharges, permeable medium, steep slopes.

When $wq = -i$; $F = 0$: the pressure remains constant along the flow line, or it passes through a maximum ($dF/ds > 0$) or minimum ($dF/ds < 0$). This point is reached for

$$w/\bar{w} = -i/\bar{J}, \text{ practically} \approx 1, \text{ or } w \approx \bar{w} \quad (17)$$

The maximum or minimum pressure is reached where resistivity attains its average value.

All the above cases may be reproduced in a vertical permeameter (Fig. 3) the water flowing downwards under $\bar{J} = H/L = 1$. The pressure diagrams correspond to (*a*) w = constant; (*b*) decreasing downwards slowly; and (*c*) rapidly; (*d*) increasing; (*e*) decreasing, then increasing; (*f*) increasing, then decreasing.

Any minimum or maximum of w corresponds to an inflexion point of p.

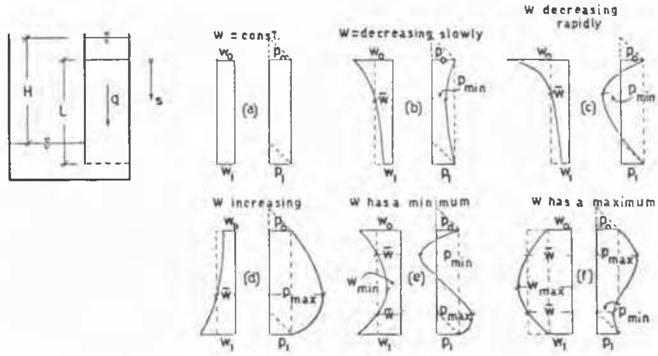

Fig. 3 Pressure Diagrams in Vertical Descending Flow
Diagrammes de pression en écoulement vertical descendant

Experimental Verification

The main new feature is the formation of zones of maximum and minimum pressures in descending flow. This has been observed in the so-called pore-pressure diagrams in many *earth-fill zoned embankments* with an impervious core (*Walker*, 1948). The maxima are more pronounced as $-d^2p/ds^2$ is larger or where the impermeability increases rapidly, and the slope of flowlines flattens out. The minima are more pronounced as d^2p/ds^2 is larger or where the permeability increases rapidly, and the flowline slope steepens [case (*f*)].

At Ebron (Israel) the author inspected a 10 foot deep *pond*, with a gravelly earthfill embankment lined with a 10 cm thick asphalt-sand mixture. After filling with water, some seepage occurred. Due to the sudden increase of permeability from asphalt to gravelly fill, a pronounced pressure minimum was produced [case (*c*)]. In this tension zone air and vapour bubbles accumulated at the asphalt-lining, tore part of it and heaved it up in form of huge bubbles, 2 cm wall thickness, 30–40 cm in diameter. When the pond was emptied, the gases escaped through cracks and the bubbles collapsed. We may call this phenomenon *cavitation* in the soil.

After rainfall the upper layers of coastal *sand dunes* in Israel become slightly cemented and a zone of tension is formed. Due to evaporation some water menisci disappear and air is sucked in producing a hissing sound. (Oral communication by Dr. *J. Vroman*, Haifa.)

The author inspected open *brine pans* of the Palestine Potash Co. works at the Dead Sea. The subsoil is a permeable marly soil, but a heavy deposit of impervious salt overlies it. When occasionally a pole penetrating the subsoil is extracted, numerous gas bubbles accumulated below escape into the brine.

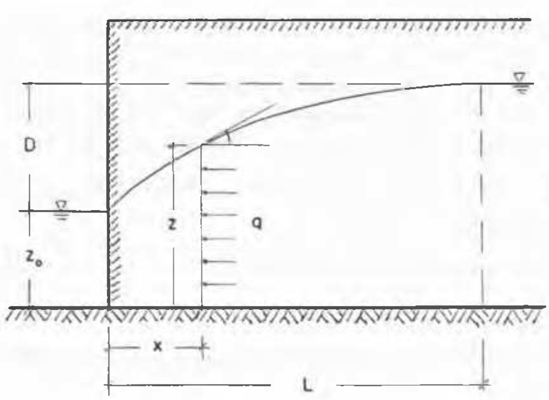

Fig. 4 Horizontal Gallery or Ditch
Galerie ou fossé horizontaux

Mean Resistivity $\bar{w}$

We have seen that the average resistivity $\bar{w}$ (or impermeability) is much more important than the average conductivity $\bar{k}$ (or permeability). Its value along a path L, where w varies between w_0 and w_1, is given by

$$\bar{w} = L^{-1} \int_{s=0}^{L} w\,ds \tag{12}$$

The following table gives some values of $\bar{w}$ for different cases of $w(s)$; a is a soil constant, as defined below:

Table 1 Values of Average Resistivity $\bar{w}$

No.	w/w_0	$\bar{w}/w_0$	a
1	1	1	—
2	$1+a\cdot s/L$	$1+0\cdot 5a$	$(w_1-w_0)/w_0$
3	$1-a\cdot s/L$	$1-0\cdot 5a$	$(w_0-w_1)/w_0$
4	$(1+a\cdot s/L)^{-1}$	$a^{-1}\ln(1+a)$	$(w_0-w_1)/w_1$
5	$(1-a\cdot s/L)^{-1}$	$a^{-1}\ln(1-a)$	$(w_1-w_0)/w_1$
6	$e^{-a\cdot s/L}$	$a^{-1}(1-w_1/w_0)$	$\ln(w_0/w_1)$
7	$1+a\sin(f\cdot s/L)$	$1+af^{-1}(1-\cos f)$ $(\to 1$ for $f\to\infty)$	
8	$1+a(s/L)^n$	$1+a/(n+1)$	$(w_1-w_0)/w_0$
9	$w=w_1$ for $0\leqslant s\leqslant a_1L$; $w=w_2$ for $a_1L\leqslant s\leqslant(a_1+a_2)L$; $w=w_3$ for $(a_1+a_2)L\leqslant s\leqslant$ $\leqslant(a_1+a_2+a_3)L$	$\bar{w}=\sum_{i=1}^{3} w_i a_i$	

No. 7 shows that in alternatively permeable and impervious layers w is practically constant and independent of the amplitudes of variation.

No. 9 shows that permeable layers (e.g. if $w_2 \ll w_1$) may be replaced by voids ($w_2 = 0$) without sensibly modifying the flow. This explains why the resistivity of a lined canal is directly proportional to the thickness and resistivity of the lining, and almost independent of the permeable soil.

Horizontal Gallery or Drainage Ditch—Unconfined Flow

An infinite permeable water-bearing stratum contains groundwater (depth H above an impervious bottom, Fig. 4), and is drained by a horizontal gallery or ditch of great length b at a rate Q, the water table dropping at the gallery ($x = 0$) to level z_0, elsewhere (at x) to z. The drawdown $D = H - z_0$ is felt up to a distance $x = L$.

Let the soil layers be vertical. As the slope of the water table at equilibrium is very flat, we may assume with *Dupuit* (1863) that the flow is two-dimensional, horizontal up to the water table, and that the velocity along any vertical depends only on the slope dz/dx of the water table at that vertical. Hence

$$Q_x = bzq = bzk\,dz/dx = (bz/w)\,dz/dx = \text{const} = Q \tag{18}$$

and by integration

$$H^2 - z_0^2 = 2Q/b \int_{x=0}^{L} w\,dx = 2Q\bar{w}L/b \;.. \tag{19}$$

$$(Q\bar{w}/H^2)(L/b) = D/H\,[1 - 0.5\,D/H] \quad .. \tag{19'}$$

which are the usual formulae for homogeneous media, with $\bar{w}$ instead of w.

We cannot apply *Dupuit*'s hypothesis to horizontal layers, as it is quite improbable that the flow along a vertical depends only on the slope of the water table, which may occur in a quite different layer.

In the case of two galleries or ditches (at $x = \pm L$, Fig. 5) of great length b, draining a rainfall, the rate of which

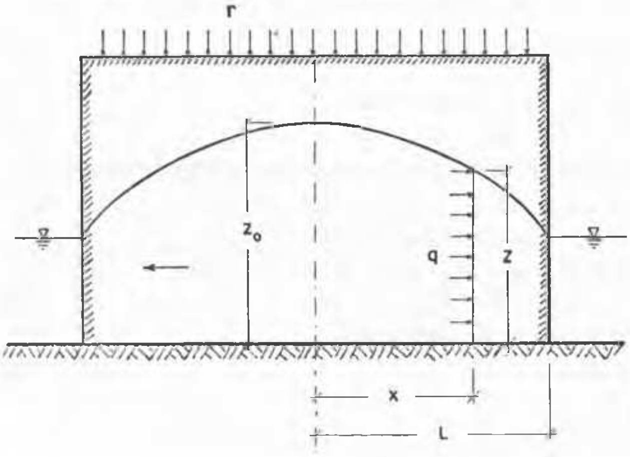

Fig. 5 Two Ditches and Rainfall
Deux fossés et précipitations

r (m/sec) is uniform, and assuming vertical layers, symmetrical with respect to the median plane ($x = 0$), we may write:

$$Q_x = -k\,(dz/dx)\,bz = bxr; \quad w = 1/k \tag{20}$$

Integrating

$$z^2 = 2r \int_{x'=x}^{L} wx'\,dx' \tag{20'}$$

Only when $w = \text{const}$, do we obtain an elliptic water table, as usual.

Radial Flow to a Well—Unconfined Flow

Here the *Dupuit* approximation may again be used (Fig. 6) if the layers have vertical rotational symmetry and the well penetrates completely to the impervious subsoil. At the radial distance x:

$$Q = 2\pi xzq = 2\pi xz\cdot w^{-1}dz/dx; \quad w = w(x) \quad .. \tag{21}$$

Integrating between $x = r$ (well radius) and $x = L$ (radius of drawdown cone) we get

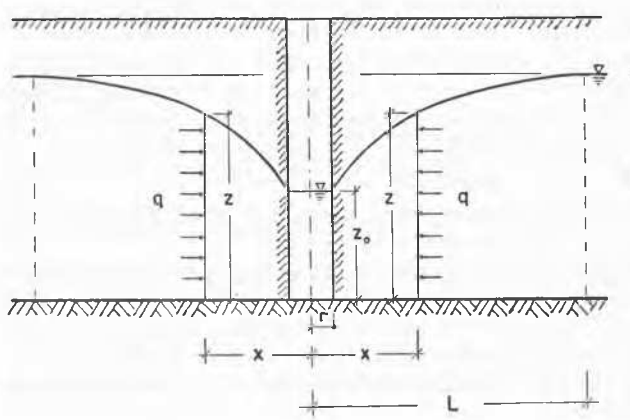

Fig. 6 Well in Unconfined Flow
Puits en écoulement sans pression

$$(H^2 - z_0^2)\,\pi / Q = \int_{x=r}^{L} w x^{-1} dx = M w_0 \quad .. \qquad (22)$$

Referring to the table of section 5, we find for case:

$$\left.\begin{array}{lll} \text{No. 1;} & w/w_0 = 1; & M = \ln(L/r) \\ \text{No. 2, 3;} & w/w_0 = 1 \pm ax/L; & M = \ln(L/r) \pm a \\ \text{No. 4, 5;} & w/w_0 = (1 \pm ax/L)^{-1}; & M = \ln(L/r) - \\ & & \quad \ln(1 \pm a) \pm ar/L \\ \text{No. 8;} & w/w_0 = 1 + a(x/L)^n; & M = \ln(L/r) + a/n \end{array}\right\} \qquad (23)$$

In homogeneous soils ($a = 0$) we get the usual *Dupuit* formula

$$(H^2 - z_0^2)\,\pi / Q = w \ln(L/r) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (24)$$

which serves for the determination of k (or w) by the *Thiem* method. In heterogeneous soils a correction is required.

Example: $H = 20$ m; $z_0 = 18$ m; $k_0 = 0.001$ m/s (at the well); $k_1 = 0.0005$ m/s at a distance $L = 50$ m; $r = 0.25$ m; $L/r = 200$; $w_0 = 1000$ s/m; $w_1 = 2000$ s/m. In the case of No. 2, $a = 1$; $M = 6.3$; $Q = 0.038$ m³/s.

If we apply (24) with $\bar{w} = 0.5\,(w_0 + w_1) = 1500$ s/m, we get $Q = 0.030$ m³/s, or 21 % less.

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