SUB-SECTION I e

SHEARING STRENGTH AND EQUILIBRIUM OF SOILS

I e 1

ON THE LAW OF FRICTION OF SAND

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1. The law of friction of sand tells us that the shearing stress on the sliding surface is proportional to the normal stress on it, when the failure of the sand mass is going on. On the other hand, Mr. Reynolds 1) has pointed out that the characteristic property of sand concerning the motion of it is what he called "dilatancy", namely that the volumetric change in sand occurs when the shearing motion begins. Mr. Jenkin paid his serious attention to this property 2), and he performed careful experiments. He pointed out that the friction angle between sand layers varies as the compactness changes. He concluded that these facts show what is called "dilatancy". Mr. A. Casagrande 3) has also shown that the "critical density" is observed when the sand mass begins to fail. Mr. Casagrande's "critical density" may be closely related to Mr. Reynolds's dilatancy.

We intend to investigate in detail the nature of failure of sand mass along these lines of idea.

2. EXPERIMENTAL FACTS.

a) We performed the so-called shearing tests with various kinds of sand. The apparatus is that described by Mr. N. Yamaguchi in the Proceedings of the international conference on soil mechanics and foundation engineering, held in June 1936, vol. II p. 42.

Our experiments were performed for the following kinds of sand; two kinds of sand used for the standard tests of cement mortar, in our country; the one finer, (0.3 mm in diameter), the other coarser (1.2 mm in diameter); a kind of river sand (larger than 2.38 mm 15.6%, 2.38 mm to 1.19 mm 30.15%, 1.19 mm to 0.59 mm 28.05%, 0.59 mm to 0.297 mm 16.70%, smaller than 0.297 mm 9.10%).

All kinds of sand are tested in dry and wet conditions.

b) Tests performed.

The shearing test of soil usually performed in our country is executed as follows. At first, the soil sample is placed in a shearing box of our apparatus and then applying a constant load vertically on the upper surface of the sample, then the middle box is pushed laterally, the soil sample is sheared off at the boundary planes which separate the middle box from neighbouring upper and lower boxes. There are many defaults about this method of test in case of sand; then the new method which is effected by the same equipment is proposed. This method is verified to be very satisfactory by experiments. But the most important point is that the mechanism of motion of sand may be investigated closely by this method of experiment. A new method, we proposed, is as follows. No load is applied vertically on the upper surface of the soil sample at first, shearing off the soil sample as above stated. At the same time, the heaving up motion of the upper cover plate is measured. As the shearing motion advances the upper cover plate is heaved up by sand, this shows the property "dilatancy". Some results obtained are illustrated in the figures 1 to fig. 3 and fig. 5.

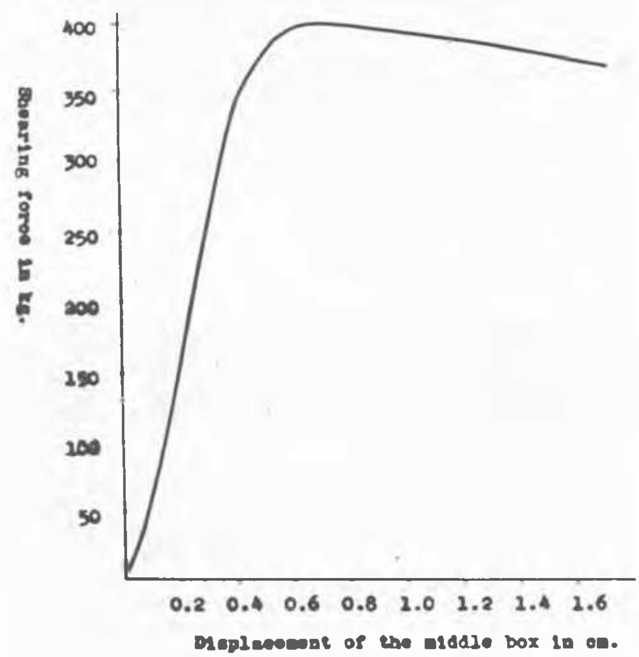

FIG. 1

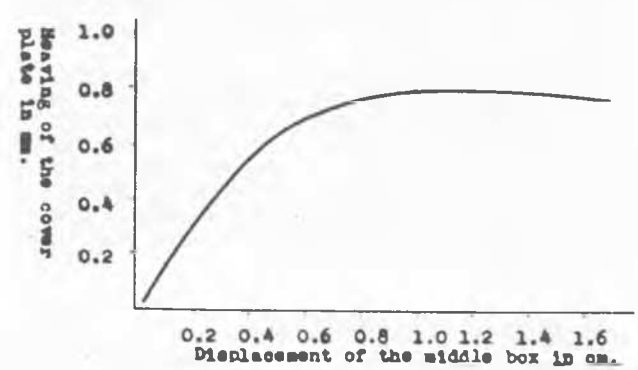

FIG. 2

c) The motion of sand.

The motion of sand near the shearing plane is observed closely and the conclusions as described below are obtained. In the first stage of the shearing motion, the sand layer near the shearing plane is made to be loose, in this stage the motion is similar to that of viscous liquid and this layer of motion has a finite breadth and the cover plate on the upper surface is heaved up by sand continuously. But when the second stage which we call it as steady state of motion is reached, the motion becomes stationary and the

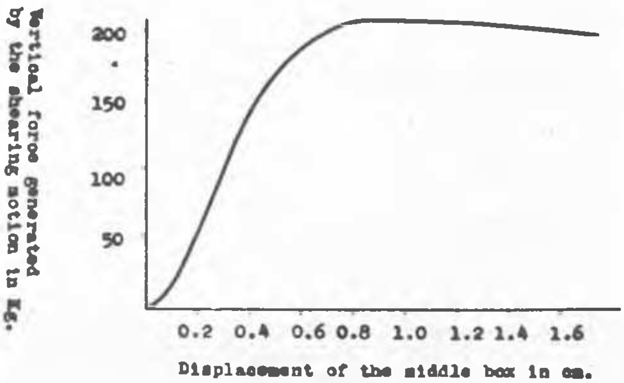

FIG. 3

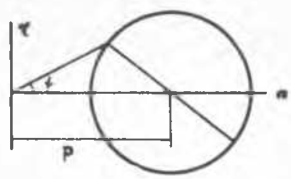

FIG. 4

heaving up motion of the cover plate and the generation of the vertical force will stop. And the shearing force becomes constant. In the first stage the angle of internal friction calculated by the following formula, namely

$\tan\theta = \tau/n$

, where

$\tau = S/2A$: The shearing stress,
S : The shearing force,
A : The sectional area of the shearing box,
$n = N/A$: The vertical normal stress,
N : The developed vertical force due to shearing motion,

is not constant, and in the steady state it attains a constant value, approximately equal to the angle of repose of the sand tested or in some cases, after reaching a minimum, it increases a little.

This shows that the Coulomb's law of friction does not hold when the state of motion of sand is not steady. But it holds in the steady state of motion of sand. When the steady state of motion is reached the region of motion is confined in a very thin layer and then it may be concluded that the sliding surface is found in the steady state in place of sliding layer.

3. THEORETICAL CONSIDERATIONS.

a) As the relations between stresses and strain velocities of the sand in motion, we assumed the following formula.

$$\left.\begin{aligned}
&\sigma_x = -p + 2mp\frac{\delta u}{\delta x}, \quad \sigma_y = -p + 2mp\frac{\delta v}{\delta y},\\
&\sigma_z = -p + 2mp\frac{\delta w}{\delta z}, \quad \tau_{yz} = mp\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right),\\
&\tau_{zx} = mp\left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right), \quad \tau_{xy} = mp\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right).
\end{aligned}\right\} \quad (2)$$

, where u, v, w are the velocity components and p is the mean pressure, namely

$$-p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z). \qquad (3)$$

and m is a constant.

In the above relations, if we consider u, v, w as the displacement components, these formulae are the same as those due to Boussinesq in his theory of the pulverulent masses. When the motion is not steady, the above mentioned formula are considered as to be a rough approximation, because the rate of the volume dilatation is not zero.

The validity of these relations would be inferred from the results which are stated in the followings.

b) The principle of minimum rate of energy loss.

When the motion of sand is steady it may be verified that the possible manner of motion is determined from the principle above stated. The time rate of the loss of energy in the sand in motion is formulated from the relations (2) as follows.

$$\left.\begin{aligned}
\frac{dI}{dt} &= -\int\Big[\sigma_x\frac{\delta u}{\delta x} + \sigma_y\frac{\delta v}{\delta y} + \sigma_z\frac{\delta w}{\delta z} + \tau_{yz}\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right) +\\
&\quad + \tau_{zx}\left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right) + \tau_{xy}\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)\Big]\,dV\\
\frac{dI}{dt} &= -\int\Big[\left(-p + 2mp\frac{\delta u}{\delta x}\right)\frac{\delta u}{\delta x} + \left(-p + 2mp\frac{\delta v}{\delta y}\right)\frac{\delta v}{\delta y} +\\
&\quad + \left(-p + 2mp\frac{\delta w}{\delta z}\right)\frac{\delta w}{\delta z} + mp\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right)^2 +\\
&\quad + mp\left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right)^2 + mp\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)^2\Big]\,dV.
\end{aligned}\right\} \quad (4)$$

Taking the variation of this equation we have

$$\left.\begin{aligned}
\delta\frac{dI}{dt} &= -\int\delta p\Big[\left(-1 + 2mp\frac{\delta u}{\delta x}\right)\frac{\delta u}{\delta x} + \cdots +\\
&\quad + m\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right)^2 + \cdots\Big]\,dV -\\
&\quad -\int\Big[\Big\{\frac{\delta p}{\delta x} - 4m\frac{\delta}{\delta x}\left(p\frac{\delta u}{\delta x}\right) - 2m\frac{\delta}{\delta y}\left\{p\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)\right\} -\\
&\qquad - 2m\frac{\delta}{\delta z}\left\{p\left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right)\right\}\Big\}\delta u +\\
&\quad + \Big\{\frac{\delta p}{\delta y} - 4m\frac{\delta}{\delta y}\left(p\frac{\delta v}{\delta y}\right) - 2m\frac{\delta}{\delta z}\left\{p\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right)\right\} -\\
&\quad - 2m\frac{\delta}{\delta x}\left\{p\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)\right\}\Big\}\delta v + \Big\{\frac{\delta p}{\delta z} - 4m\frac{\delta}{\delta z}\left(p\frac{\delta w}{\delta z}\right) -\\
&\quad - 2m\frac{\delta}{\delta x}\left\{p\left(\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\right)\right\} - 2m\frac{\delta}{\delta y}\left\{p\left(\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\right)\right\}\Big\}\delta w\Big]\,dV
\end{aligned}\right\} \quad (5)$$

Form the equation (2) and (3) we can obtain

$$\delta\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}\right) = 0$$

Then we get

$$\left.\begin{aligned}
&\int p\,\delta\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}\right)dV =\\
&= -\int\left(\frac{\delta p}{\delta x}\delta u + \frac{\delta p}{\delta y}\delta v + \frac{\delta p}{\delta z}\delta w\right)dV.
\end{aligned}\right\} \quad (6)$$

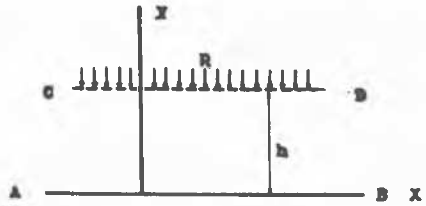

FIG. 5

If the body force has the potential Ω (The components of the body force are X, Y, Z)

$$\left.\begin{aligned} &-\int \varrho\left(X\delta u + Y\delta v + Z\delta w\right) dV = \\ &= \int \varrho\left(\frac{\delta\Omega}{\delta x}\delta u + \frac{\delta\Omega}{\delta y}\delta v + \frac{\delta\Omega}{\delta z}\delta w\right) dV = \\ &= -\int \varrho\,\Omega\,\delta\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}\right) dV = 0 \end{aligned}\right\}(7)$$

If we combine the formula (5), (6) and (7) we get

$$\begin{aligned} \delta\frac{dI}{dt} = &-\int \delta p\left[\left(-1+2mp\frac{\delta u}{\delta x}\right)\frac{\delta u}{\delta x}+\cdots+ \right. \\ &\left. +m\left(\frac{\delta w}{\delta y}+\frac{\delta v}{\delta z}\right)^2+\cdots\right]dV + 2\int\left[\left\{-\frac{\delta p}{\delta x}+2m\frac{\delta}{\delta x}\left(p\frac{\delta u}{\delta x}\right)+\right.\right. \\ &+m\frac{\delta}{\delta y}\left\{p\left(\frac{\delta v}{\delta x}+\frac{\delta u}{\delta y}\right)\right\}+m\frac{\delta}{\delta z}\left\{p\left(\frac{\delta u}{\delta z}+\frac{\delta w}{\delta x}\right)\right\}+\varrho X\Big\}\delta u+ \\ &+\left\{-\frac{\delta p}{\delta y}+2m\frac{\delta}{\delta y}\left(p\frac{\delta v}{\delta y}\right)+m\frac{\delta}{\delta z}\left\{p\left(\frac{\delta w}{\delta y}+\frac{\delta v}{\delta z}\right)\right\}+\right. \\ &+m\frac{\delta}{\delta x}\left\{p\left(\frac{\delta v}{\delta x}+\frac{\delta u}{\delta y}\right)\right\}+\varrho Y\Big\}\delta v+ \\ &+\left\{-\frac{\delta p}{\delta z}+2m\frac{\delta}{\delta z}\left(p\frac{\delta w}{\delta z}\right)+m\frac{\delta}{\delta x}\left\{p\left(\frac{\delta u}{\delta z}+\frac{\delta w}{\delta x}\right)\right\}+\right. \\ &\left.+m\frac{\delta}{\delta y}\left\{p\left(\frac{\delta w}{\delta y}+\frac{\delta v}{\delta z}\right)\right\}+\varrho Z\Big\}\delta w\right]dV. \qquad (8) \end{aligned}$$

If we put this variation to be zero, remembering that

$$p = \lim_{\substack{k\to 0\\ \Delta\to 0}} k\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta w}{\delta z}\right) \qquad (9)$$

, where $\Delta = \dfrac{\delta u}{\delta x}+\dfrac{\delta v}{\delta y}+\dfrac{\delta w}{\delta z}$,k = a constant,

we can obtain the equations of motion of sand in the steady state, namely when the acceleration terms in equations, are zero.

In these eqpations, if we consider the formulae (2) we have the well known equations of motion in case of the steady state, i. e.,

$$\left.\begin{aligned} &\frac{\delta\sigma_x}{\delta x}+\frac{\delta\tau_{xy}}{\delta y}+\frac{\delta\tau_{xz}}{\delta z}+\varrho X = 0 \\ &\frac{\delta\tau_{yx}}{\delta x}+\frac{\delta\sigma_y}{\delta y}+\frac{\delta\tau_{yz}}{\delta z}+\varrho Y = 0 \\ &\frac{\delta\tau_{zx}}{\delta x}+\frac{\delta\tau_{zy}}{\delta y}+\frac{\delta\sigma_z}{\delta z}+\varrho Z = 0 \end{aligned}\right\}(10)$$

c) The law of friction and the above theorem.

When the motion is considered as 2-dimensional and we can assume that,

$$\tau_{yz} = 0,\quad \tau_{zx} = 0,\quad \partial w/\partial z = 0$$

the variation of the time rate of the energy loss in a moving layer of sand is written as follows.

$$\left.\begin{aligned} \frac{dI}{dt} = -m\int p\left\{2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\right. \\ \left.+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}dV. \end{aligned}\right\}(10)$$

In case of 2-dimension, the formulae (2) becomes very simple and we get

$$\left.\begin{aligned} &p = \frac{\sigma_x+\sigma_y}{2} \\ &2mp\frac{\partial u}{\partial x} = -2mp\frac{\partial v}{\partial y} = \frac{\sigma_y-\sigma_x}{2} \\ &mp\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) = \tau_{xy} \end{aligned}\right\}(11)$$

Then we can transform the equation of the time rate of energy loss as follows (reference may be made to Mohr's circle, figure 4)

$$\frac{dI}{dt} = -\frac{1}{m}\int p\sin^2\psi\, dV. \qquad (12)$$

From the formula (9) we can say $\delta p = 0$ when applying this formula to the elementary volume of sand mass, we can conclude, that if the motion of sand is steady the angle between stress and the normal to the surface which the stress acts is minimum, say, ϕ.

If the real angle between stress and the normal is larger than ϕ the motion is not steady and if it is less than ϕ, the motion of sand cannot occur.

This conclusion is not other than the Coulomb's law of friction.

d) Formation of the sliding surface in sand.

We assume the motion as 2-dimensional and we take the axes X and Y as shown in the fig. 5.

And we consider that the sand layer between y = 0 and y = h is moving from left to right and the sand masses which lies in the region y < 0 and y > h, are in motion as a whole.

In this case we can put

$$\left.\begin{aligned} &\sigma_x = -p \quad \sigma_y = -p+2mp\frac{\partial v}{\partial y} \quad \sigma_z = -p \\ &\tau_{yz} = 0 \quad \tau_{zx} = 0 \quad \tau_{xy} = mp\frac{\partial u}{\partial y} \end{aligned}\right\}(13)$$

and then the equations of motion of sand mass is as follows,

$$\left.\begin{aligned} &\frac{\delta\tau_{xy}}{\delta y} = 0 \\ &\frac{\delta\sigma_y}{\delta y} - \varrho = 0 \end{aligned}\right\}\text{namely}\quad \left.\begin{aligned} &\frac{d}{dy}\left(p\frac{du}{dy}\right) = 0 \\ &\frac{dp}{dy}+\varrho = 0 \end{aligned}\right\}(14)$$

and $\qquad \dfrac{dv}{dy} = 0 \qquad (15)$

Solving above fundamental equations of steady motion of sand between y = 0 and y = h under the boundary conditions

at $y = 0$ $u = u_0$ $\left| \tau_{xy}/\sigma_y \right| = \tan\phi_0$ (16)

at $y = h$ $u = u_1$ $\left| \tau_{xy}/\sigma_y \right| = \tan\phi_1$ (17)

, where ϕ_0 and ϕ_1 are the angles of friction at $y = 0$ and $y = h$ respectively, we can get the following results,

$$-p = -\varrho(h-y) - p_0 \quad (18) \qquad (18)$$

$$h = \frac{p_0}{\varrho}\left(\frac{\tan\phi_1}{\tan\phi_0} - 1\right) (19) \qquad (19)$$

From the equation (18) we can conclude that the pressure distribution in the moving sand layer is hydrostatic and the equation (19) tells us that if the friction angle at $y = 0$ is less than that at $y = h$ we get $h > 0$ and in the reverse case we have $h < 0$.

And if these angles are equal we have $h = 0$.

Then we can say that if the angle of friction at $y = 0$ is larger than the angle of friction at $y = h$ the steady motion of sand in a layer of finite breadth cannot occur and if both angles are equal with each other the breadth of the layer of the sand in motion reduces to zero.

Of course, the theoretical calculation is for the steady motion of sand, but we may consider that its results are approximately true when the motion is not steady.

Then from these theoretical considerations and the facts obtained from experiments we can conclude the following results.

When the layer is put in motion by the shearing action of the lower sand mass, the layer between AB and CD in the figure 5 expands (dilatancy). The nearer the sand mass lies to AB the more it expands.

Then the effective friction angle of sand near CD is larger than that for the neighbourhood of AB. In this stage the moving portion is approximately confined by CD and AB. As the layer between AB and CD expands the frictional angle in this layer may become more and more uniform and then breadth of the moving layer of sand becomes thinner.

At last the expansion reaches to its maximum then this layer reduces to a very thin sheet, the sliding surface may be formed in this way. From these considerations we expect that the effective angle of friction of sand decreases from a larger value to a smaller constant, namely the angle of repose of the sand tested.

The results of experiments verify this expectation as shown in the figure 6. In this figure the angle θ is calculated

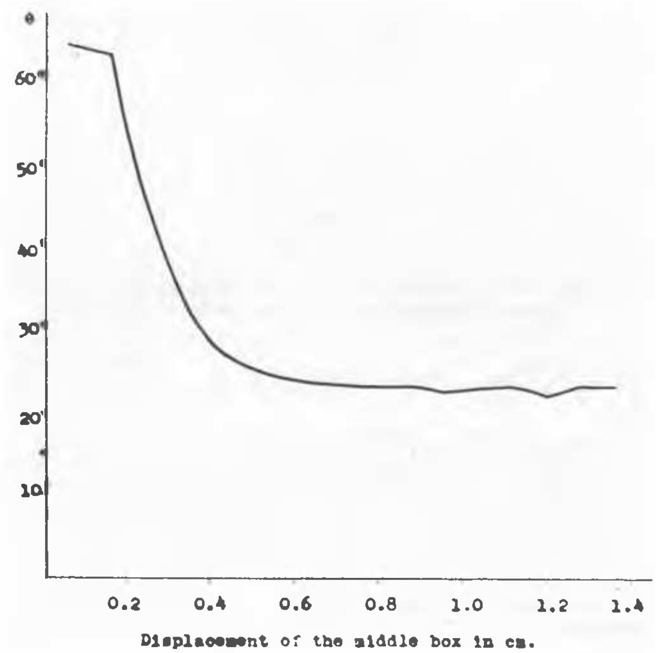

FIG.6

by the following formula,

$$\tan\theta = \tau/\sigma = S/2N$$

, where

S : the shearing force applied,

N : the vertical force generated.

In case of some kind of sand, after decreasing the angle calculated as above from the date of experiments to a minimum then it increases a little. This phenomena may be interpreted as follows.

In this case the layer becomes too loose when the state θ = a minimum is reached, then it must contract a little.

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