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VALIDITY OF COULOMB' LAW OF STABILITY

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To characterize the state of failure in Soil Mechanics Coulomb's equation of stability: $t = n\,tg\phi + C$ is made use of. This law has been proved experimentally partly only, because in the range of tension, the envelope curve characterising failure differs greatly from the straight line determined by the above mentioned equation (Fig. 1.a.) In the following an attempt is made to deduce the exact law of failure. It is assumed that Mohr's stress circles have an envelope, but this is generally a curve, its coordinates t and n are functions of angle ε (Fig. 1.b.).

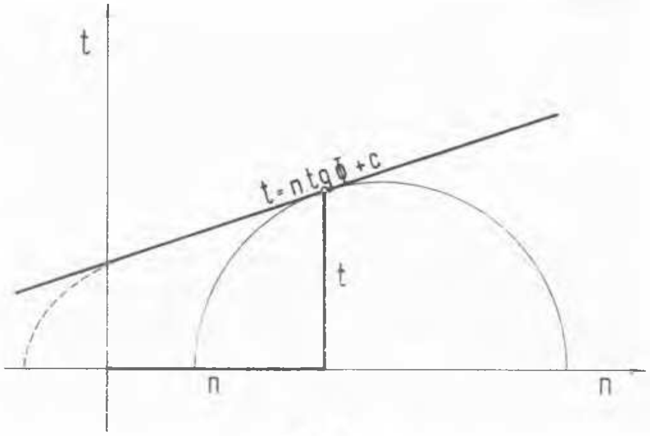

FIG.1 a

I. AUXILIARY THESIS.

From stresses t and n acting on the sliding surface inclined to the horizontal at angle α and from conditions of the envelope curve, i.e. $tg(2\alpha-\varepsilon) = \frac{\sigma_y - \sigma_x}{2\tau}$ equations of the conjugate stresses can be deduced, i.e.

$$\left.\begin{aligned} \sigma_y &= n + 2t\,\frac{\sin\alpha\cos(\alpha-\varepsilon)}{\cos\varepsilon} \\ \sigma_x &= n - 2t\,\frac{\sin(\alpha-\varepsilon)\cos\alpha}{\cos\varepsilon} \\ \tau &= t\,\frac{\cos(2\alpha-\varepsilon)}{\cos\varepsilon} \end{aligned}\right\} \quad (1)$$

That is, all stresses are combined functions of angles ε and α.

In a shearing test, according to arrangement on Fig. 2. after compression of the soil, angle α is equal to zero, the sliding surface becomes horizontal, therefore, substituting in equation (1) $\alpha = 0$, we obtain

$$\left.\begin{aligned} \sigma_y &= n \\ \sigma_x &= n + 2t\,tg\,\varepsilon \\ \tau &= t \end{aligned}\right\} \quad (2)$$

all stresses are functions of ε only.

II. AUXILIARY THESIS.

The soil sample is under arbitrary stress and its weight per unit volume varies either under an external load or under its own weight, therefore γ is not constant but generally: $\alpha = \alpha(x,y)$. Since γ is function of stresses σ_y, σ_x and τ and the latter are functions of ε only, $\gamma = f(\varepsilon)$ (3)

because either α is constant or $\alpha = f(\varepsilon)$, hence all stresses are functions of ε only.

The conditions of equilibrium of the elementary prism can be expressed by Cauchy's equations:

$$\left.\begin{aligned} \frac{d\sigma_y}{d\varepsilon}\,\frac{\partial\varepsilon}{\partial y} + \frac{d\tau}{d\varepsilon}\,\frac{\partial\varepsilon}{\partial x} &= \gamma(\varepsilon) \\ \frac{d\sigma_x}{d\varepsilon}\cdot\frac{\partial\varepsilon}{\partial x} + \frac{d\tau}{d\varepsilon}\cdot\frac{\partial\varepsilon}{\partial y} &= 0 \end{aligned}\right\} \quad (4)$$

From Eq. (4) we obtain $\frac{\partial\varepsilon}{\partial x}$ and $\frac{\partial\varepsilon}{\partial y}$

$$\left.\begin{aligned} \frac{\partial\varepsilon}{\partial y} &= \gamma\,\frac{\frac{d\sigma_x}{d\varepsilon}}{\frac{d\sigma_x}{d\varepsilon}\cdot\frac{d\sigma_y}{d\varepsilon} - \left(\frac{d\tau}{d\varepsilon}\right)^2} = \gamma\,f_2(\varepsilon) \\ \frac{\partial\varepsilon}{\partial x} &= -\gamma\,\frac{\frac{d\tau}{d\varepsilon}}{\frac{d\sigma_x}{d\varepsilon}\,\frac{d\sigma_y}{d\varepsilon} - \left(\frac{d\tau}{d\varepsilon}\right)^2} = -\gamma\,f_1(\varepsilon) \end{aligned}\right\} \quad (5)$$

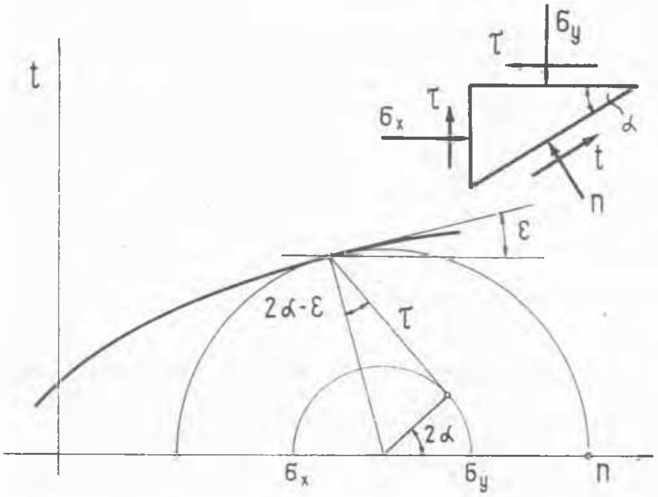

FIG. 1 b

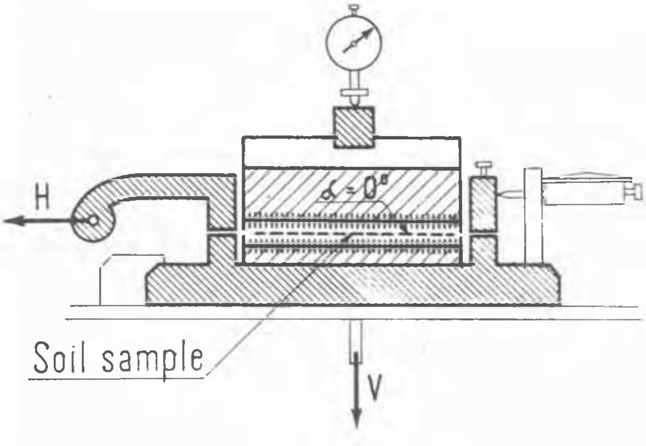

FIG. 2

Generally, the denominator cannot be zero or it would follow that $\gamma = 0$, which is possible only in the case of the weightless mass.

Differentiating the first equation under (4) with respect to x, the second with respect to y and equating $\frac{\partial^2\varepsilon}{\partial x\partial y}$ to $\frac{\partial^2\varepsilon}{\partial y\partial x}$ we obtain:

$$\frac{d\gamma}{d\varepsilon}\frac{\partial\varepsilon}{\partial x} f_2 + \gamma f_2' \frac{\partial\varepsilon}{\partial x} = -\frac{d\gamma}{d\varepsilon}\frac{\partial\varepsilon}{\partial y} f_1 - \gamma f_1' \frac{\partial\varepsilon}{\partial y}$$

substituting $\frac{\partial\varepsilon}{\partial y}$ and $\frac{\partial\varepsilon}{\partial x}$ from Eq.(5)

$$-\frac{d\gamma}{d\varepsilon} f_2\gamma f_1 - \gamma^2 f_2' f_1' = -\frac{d\gamma}{d\varepsilon}\gamma f_2 f_1 - \gamma^2 f_1' f_2'$$

hence $\frac{f_2'}{f_2} = \frac{f_1'}{f_1}$

that is $f_1 = K f_2$ (6)

substituting from Eq. (4)

hence
$$\left.\begin{aligned} \frac{d\tau}{d\varepsilon} &= k_1\frac{d\sigma_x}{d\varepsilon} \\ \tau &= k_1\sigma_x + K_2 \end{aligned}\right\} \qquad (6_a)$$

that is, the shearing stress (τ) at an arbitrary point in the interior of the body is a linear function of the normal stress σ_x. This law, as the basic law of the state of stress of a body possessing weight is satisfied as long as stresses are functions of ε only.

Substituting τ and its differential-quotient with respect to ε in the second equation under (4) we obtain:

$$\frac{d\sigma_x}{d\varepsilon}\left(\frac{d\varepsilon}{dx} + k_1\frac{\partial\varepsilon}{\partial y}\right) = 0 \qquad (7)$$

either 1.) $\frac{d\sigma_x}{d\varepsilon} = 0$ thus $\sigma_x = C_1$ (8)

or 2.) $\frac{\partial\varepsilon}{\partial x} + k_1\frac{\partial\varepsilon}{\partial y} = 0$

This equation is satisfied, if

$$\varepsilon = f(k_1 x - y) \qquad (9)$$

ad 1.) If $\sigma_x = C_1$ it follows after Eq.(6.a):

$$\tau = C_2$$

In this case Mohr's stress circles have a point in common and have no envelope curve (Fig. 3.). Therefore, this case lies outside of the conception of the envelope and is thus impossible.

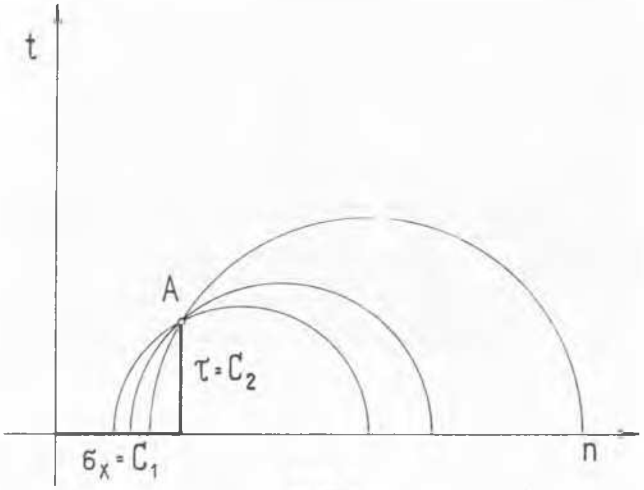

FIG. 3

ad 2.) Eq.(9) yields the required function $\varepsilon = f(x,y)$. There is an infinite number of possible solutions, for instance

$$\varepsilon = C_1(k_1x - y)$$
$$\varepsilon = \frac{C_1}{k_1x - y} \qquad \text{etc.}$$

a.) Stress conditions in the shearing test.

In the shearing test, the sample has such small dimensions that its own weight can be neglected in comparison with the external load, thus $\gamma = 0$. The system of equations (4) holds, if:

1) $\frac{\partial\varepsilon}{\partial x} = 0$. $\frac{\partial\varepsilon}{\partial y} = 0$ that is

$$\underline{\varepsilon = \text{constant}}$$

it means, that the envelope is a straight line. This is Coulomb's solution: $t = n\,\mathrm{tg}\,\phi + C$. or 2.) if the main determinant of the equation system:

$$\begin{vmatrix} \frac{d\sigma_y}{d\varepsilon} & \frac{d\tau}{d\varepsilon} \\ \frac{d\tau}{d\varepsilon} & \frac{d\sigma_x}{d\varepsilon} \end{vmatrix} = 0$$

that is
$$\frac{d\sigma_x}{d\varepsilon}\,\frac{d\sigma_y}{d\varepsilon} = \left(\frac{d\tau}{d\varepsilon}\right)^2 \qquad (11)$$

Substituting the values of Eq.(2) in the preceding equations and considering that along the envelope curve: $\frac{dt}{d\varepsilon} = \frac{dn}{d\varepsilon}\,\mathrm{tg}\,\varepsilon$ we get after reductions:

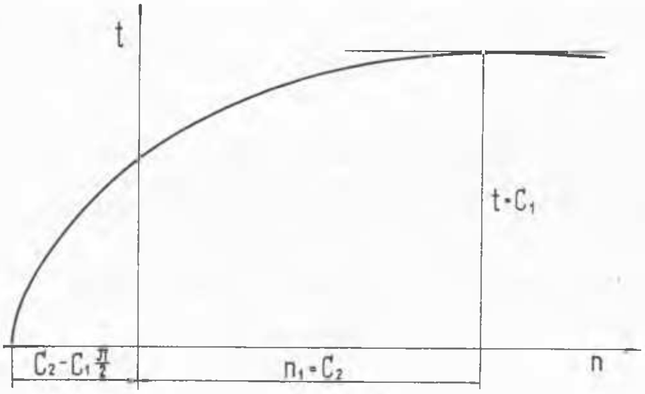

FIG. 4 a

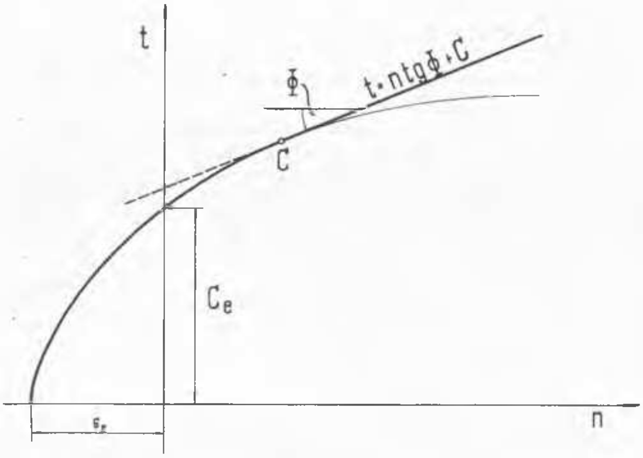

FIG. 4 b

$$\frac{dt}{d\varepsilon} + 2t\,tg\,\varepsilon = 0 \qquad (12)$$

Solving this differential equation we obtain

$$\left.\begin{aligned} t &= C_1 \cos^2\varepsilon \\ n &= C_2 - C_1(\varepsilon + \sin\varepsilon\cos\varepsilon) \end{aligned}\right\} \qquad (13)$$

respectively.

These are the equations of the envelope curve with parameter ε. We plotted same in Fig. 4.a.

This is a novel and as yet unknown law of failure, valid mainly in the range of tension, the curve of which is joined tangentially at angle ϕ by Coulomb's boundary straight line. (Fig. 4.b.)

Computing angle ϕ from the compression-shearing test and the tensile strength (σ_z) from the tension test, we obtain the characteristics of the combined envelope curve (Fig.4.a.): $C_2 - C_1\frac{\pi}{2} = -\ \sigma_z$

In the point of contact $\underline{C}$: $\varepsilon = \phi$ substituting (13) in Eq. (10)

$$C_1\cos^2\phi = \left[C_2 - C_1(\phi + \sin\phi\cos\phi)\right] tg\,\phi + C$$

The constants C_1 and C_2 may be determined and the envelope curve will be known in the range of tension too.

Indeed, the envelope curve intersects the t-axis below the Coulomb line, thus, the effective cohesion (C_e) will be smaller than obtained from the straight line (C_c). This fact was suggested long ago by researchers, because the Coulomb line gave cohesion values unsuitable to explain stability conditions and slope failures. Fig. 5. shows a combined envelope curve obtained by tests on clay.

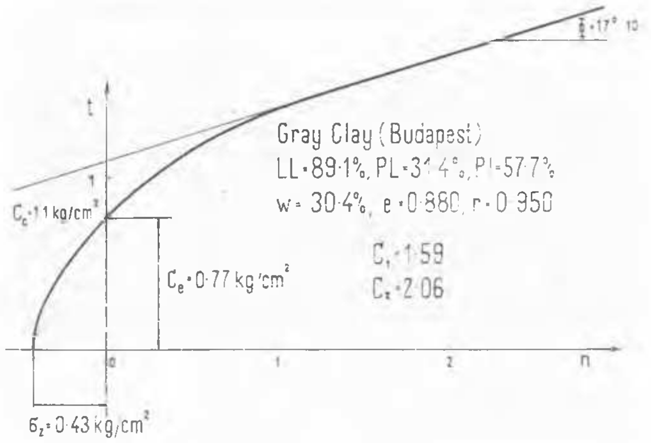

FIG. 5

b.) <u>Law of shearing in mass possessing weight.</u>

Let the law of shearing in a heavy earth mass be investigated. There exists in this case the relationship (6.a), substituting values of σ_x and τ from (1) and equating $k_1 = tg\,\psi$ we obtain

$$t\,\frac{\cos(2\alpha-\varepsilon)}{\cos\varepsilon} = tg\,\psi\left(n - 2t\,\frac{\sin(\alpha-\varepsilon)\cos\alpha}{\cos\varepsilon}\right) + k_2$$

After reductions:

$$t\,\frac{\cos(2\alpha-\varepsilon-\psi) - \sin\varepsilon\sin\psi}{\cos\varepsilon\sin\psi} = n + k_2\cot\psi \qquad (14)$$

If $\alpha = f(\varepsilon)$ is given, Eq. (14) may be integrated and the stresses $\underline{t}$ and $\underline{n}$ will be known, thus the state characterised by stresses $\sigma_x\,\sigma_y$ and τ and also the sliding surfaces will be cleared up.

I. E.g. let the simple case of $\alpha = \varepsilon$ be considered .

It means, that the vertical lines are sliding surfaces, too, because - according to Mohr's theorem - the angle between the sliding surfaces is $(90° - \varepsilon)$ thus, if $\alpha = \varepsilon$ the angle of inclination of the other sliding surface

$$\alpha_1 = 90° - \varepsilon + \varepsilon = 90° \qquad (15)$$

In this case from Eq. (14)

$$t\cot\psi = n + k_2\cot\psi$$

thus

$$t = n\,tg\,\psi + k_2$$

We obtain Coulomb's law, that is $\varepsilon = \alpha = \psi$, whence, the sliding surfaces are planes inclined at $\psi\measuredangle$ on the one hand and vertical planes on the other hand. That is, Coulomb's law is valid for the heavy earth mass, too, but in this case, the sliding surfaces are planes inclined at $\alpha = \psi$

Moreover, Coulomb's linear law of failure holds also in the case of other (plane of curved) sliding surfaces. Since $\varepsilon = \phi$, this time Eq. (4) cannot be used, stresses σ_x, σ_y and τ being functions of the angle of the sliding surface and not of angle ε

II. Let now $\alpha = 45° + \frac{\phi}{2}$, then after Eq.(1)

$$\tau = 0 \qquad (16)$$

and from Eq.(4)

$$t\,\frac{1-\sin\varepsilon}{\cos\varepsilon} = n + k_2\cot\psi$$

Differentiating with respect to ε and substituting the relationship

$$\frac{dn}{d\varepsilon} = \frac{dt}{d\varepsilon}\cot\varepsilon$$

we obtain

$$\frac{dt}{d\varepsilon} + t\,tg\,\varepsilon = 0$$

whence $t = C_1\cos\varepsilon$

and $n = C_2 - C_1\sin\varepsilon$ $\qquad (17)$

That means, the body is in a homogeneous state of stress (Fig.6.) i.e. $\sigma_y = C_1 + C_2$

$\sigma_x = C_2 - C_1$

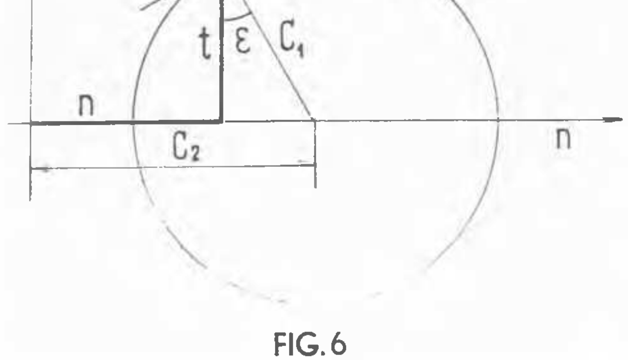

FIG. 6

If all stresses are constant, it follows from Cauchy's law of equilibrium: $\gamma = 0$, therefore the case of $\alpha = 45° + \frac{\varepsilon}{2}$ does not refer to the heavy earth mass, and must be excluded.

It would lead far to deal in detail with a given problem, therefore only the way, how to solve it is shown now.

1) If $\alpha = f(\varepsilon)$ is given or assumed relation Eq. (14) leads to a homogeneous linear equation of first degree, from which we obtain $t = F(\varepsilon)$ and by integrating $n = G(\varepsilon)$ respectively and using these relationships, all stresses as functions of ε may be determined from Eq. (1).
2) Assuming now on basis of Eq. (9) function $\varepsilon = f(x,y)$, the stresses, as the functions of coordinates (x,y) will be known as well and determining from the first equation under (4) function $\gamma = \gamma(\varepsilon) = \gamma(x,y)$, we have in every point the value of the weight per unit volume and in connection with it, the value of the void ratio and finally, on basis of $\alpha = f(\varepsilon) = g(x,y)$ the geometry of the sliding surfaces is established.

CONCLUSIONS

In order to examine conditions of equilibrium of earth masses with respect to the failure, Coulomb's linear law has as yet used. This is a very primitive law, since we proved that the form of the function giving the stresses acting on the sliding surfaces, $t = f(n)$ is greatly affected by the shape of the sliding surfaces produced by the movement of the earth masses and by the variation of γ that is of the void ratio, briefly, it is determined by the manner of movement. As soon as we know these, the stress conditions can be computed.

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I e 9

STATE OF STRESS IN GREAT DEPTH

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The author wishes to deal in this paper with two states of stress, namely with the law of compression and shearing due to great pressures. Laboratory tests used to be extended up to the pressure $p \cong 10$ kg/cm² and the results of numerous tests are summarized in the following empirical laws:

1) The compression is well characterized by Prof. Terzaghi's law:

$$e = e_o - c_1 \ln\left(\frac{p + p_c}{p_c}\right) \qquad (1)$$

that is, the void ratio e depends on the logarithm of the p - pressure.

2) The shearing test proves Coulomb's assumption that is on the surface of shear:

$$t = n \operatorname{tg} \phi + c \qquad (2)$$

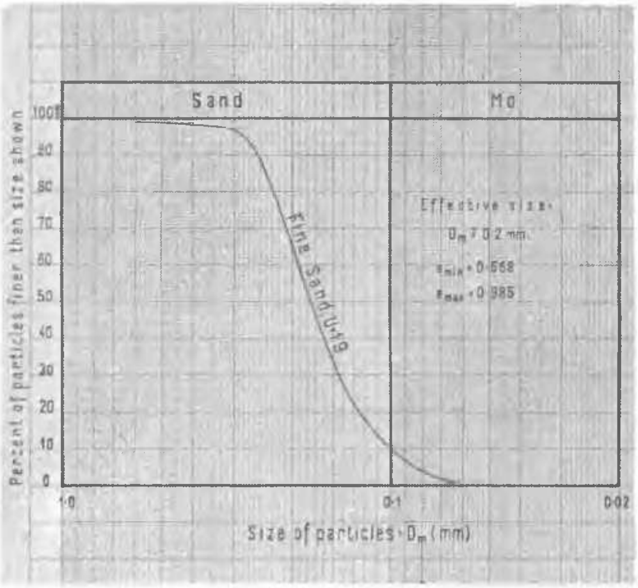

FIG.1

The question is whether the above equations are also valid in the range of great pressures. An effort is made to give an answer partly by experiments partly by theoretical considerations. The tests were made on dry sand only, but the results obtained may be generalized for cohesive soils, too.

The grain distribution curve of the sand used in the experiments is shown on Fig. 1, the effective grain size: D = 0.2 mm. Its void ratio is $e_{min} = 0.568$ in dense state and $e_{max} = 0.985$ in loose state, its specific gravity $s = 2.66$ g/cm³.

a) Compression.

The compression test carried out in an oedometer after Casagrande, dimensions of the soil sample were as follows: diameter ϕ 45 mm, height 18 mm. The water content of the sand was w = 5.8 %. The pressure raised in uniformly graduated steps up to p = 200 kg/cm². The function p = f (e) is plotted on semi and double logarithmic scale (Fig. 2). It may be stated, that the curve follows the logarithmic law to $p_1 = 4.0$ kg/cm² only, from there it obeys a power parabolic law, that is, the more general law of compression:

$$e = e_o - c_1 p^m \qquad (3)$$