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I 94

THE INFLUENCE OF THE HETEROGENEITY OF THE SOIL MASS ON ITS DEFORMATION

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SUMMARY.

In some cases, it may be appropriate to use the theory of a heterogeneous semi-infinite solid for the purpose of deriving settlement estimates from observations on full-size structures. Solutions for different load distributions are given.

In general, the compressibility of soil masses below foundations is variable from spot to spot. In most cases, it decreases in vertical direction downwards. The phenomena observed under loads distributed over a limited portion of the surface of sand deposits load to the conclusion that in such cases variations of the compressibility in horizontal direction are also bound to occur. For the purpose of stress computations, we simplify the actual state of things and represent the foundation soil by a semi-infinite elastic body. On two conditions, this working assumption may lead to suitable results:

i/ The configuration of the foundation soil is such that there are no important irregularities in the degree of consolidation of individual strata and that the rock surface is not, or but sligthly, inclined.

ii/ The intensity of loading does not exceed a certain limit so that it is reasonably possible to admit the law of proportionality between unit deformations and stresses produced in single elements of the mass.

In many cases of foundation practice both these conditions are satisfied with a sufficient approximation. They definitely are not fulfilled, however, with laboratory experiments either on sand fillings confined in rigid vessels or on earth accumulations placed on a concrete floor. The type of heterogeneity of such artificial models is obviously essentially different from that of an unlimited soil mass which has been consolidated by the pressures of its upper layers, the compressibility of its elements being dependent upon the intensity of the principal stresses acting on them.

If the foundation soil can be considered as corresponding to both the conditions mentioned above, it will be possible to derive the constants characterising its behaviour from the settlement measurements on full-size structures and to forecast, consequently, the expected settlement of a proposed structure from the data gained by the observation of the existing ones. In such investigations, however, it has proved inappropriate to neglect the influence of the variations of the compressibility of the soil mass. Thus, in the following, we propose to analyse the deformation of an elastic semi-infinite solid with a modulus of elasticity which depends upon the position of the mass element in the body and which can be represented by a continuous function of the coordinates. We shall confide our considerations to those cases only in which it is possible to express the resulting deformation of the solid by simple closed relations suitable for use in foundation practice.

1. FORCE APPLIED AT A POINT.

Let a force P acting along the Z-axis of the system of spherical coordinates R,φ,ψ, (Fig.1) be applied to the point O of the boundary plane $Z = 0$.

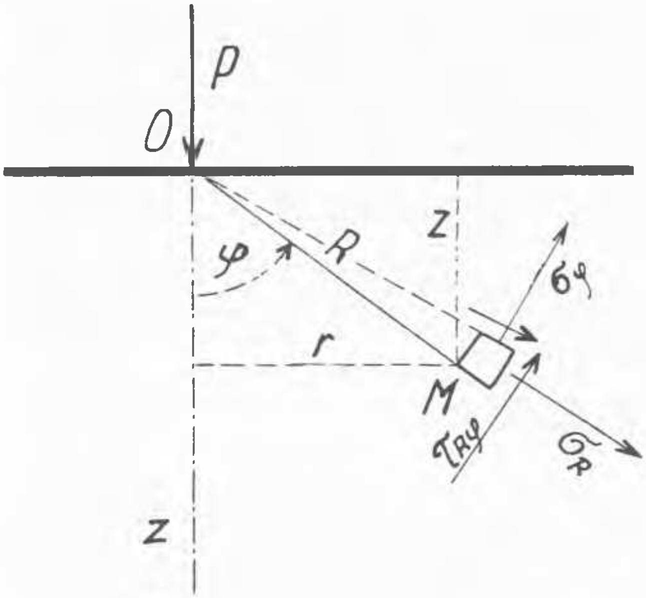

FIG. 1

We shall use the following notations:

u , v , w	components of displacements,
ε_R , ε_φ , ε_ψ	unit elongations,
$\gamma_{R\varphi}$, $\gamma_{R\psi}$, $\gamma_{\varphi\psi}$	shearing strain components,
σ_R , σ_φ , σ_ψ	normal components of stress,
$\tau_{R\varphi}$, $\tau_{\varphi\psi}$, $\tau_{R\psi}$	shearing-stress components.

The axis of z being that of symmetry, it may be written

$$\left.\begin{aligned} & w=0\,,\quad \gamma_{R\psi}=\gamma_{\varphi\psi}=0\,,\\ & \varepsilon_R=\frac{\partial u}{\partial R}\,,\quad \varepsilon_\varphi=\frac{1}{R}\left(u+\frac{\partial v}{\partial\varphi}\right),\quad \varepsilon_\psi=\frac{1}{R}\left(u+v\,\mathrm{cotg}\,\varphi\right),\\ & \gamma_{R\varphi}=\frac{\partial v}{\partial R}+\frac{1}{R}\left(\frac{\partial u}{\partial\varphi}-v\right), \end{aligned}\right\} \quad (1)$$

$$\varepsilon_R=\frac{1}{E}\left(\sigma_R-\frac{\sigma_\varphi+\sigma_\psi}{m}\right),\quad \varepsilon_\psi=\frac{1}{E}\left(\sigma_\psi-\frac{\sigma_R+\sigma_\varphi}{m}\right),\quad \tau_{R\varphi}=\frac{mE}{2(m+1)}\,\gamma_{R\varphi} \quad (2)$$

Herein m is the reciprocal of Poisson's ratio, which is assumed to be constant throughout the body, and $E=f(R,\varphi)$ is the variable modulus of elasticity.

Adding the differential equations of equilibrium to the expressions (1) and (2), the conditions of compatibility may be derived in terms of strain or stress components.

If the compressibility of the material decreases with the depth z, the modulus of elasticity may be expressed by the equation $E=Cz^n$ where C is constant and $n \geq 0$ is a positive number.

It may be shown 1) that the displacements are in this case

$$u = \frac{n+3}{n+1}\,\frac{P\cos\varphi}{2\pi C R^{n+1}}, \quad v = -\frac{n+3}{(n+1)(n+2)}\,\frac{P\sin\varphi}{2\pi C R^{n+2}}, \qquad (3)$$

under the condition that $m = n+2$. (4)
The stress distribution is a simple radial one, the principal stress amounting to

$$\sigma_R = -\frac{n+3}{2\pi}\,\frac{P\cos^{n+1}\varphi}{R^2} \qquad (5)$$

and all other stress components being nil.

Possible value of n.

Tests and observations have shown, that the settlement of circular areas under equal load per square unit increases with their radius, but not in direct proportion to it. Excluding very small areas, the relation between the settlement and the radius of the loaded circle can be represented by a curve of parabolic type. Accordingly, the settlement of an individual footing or of a single pile is always smaller than the settlement of an entire group of such units.

If our semi-infinite body shall be an appropriate representation of the foundation soil, we have to choose for n such a number that the deformation of the solid corresponds with these observations.

From the second of Eqs. (3) we conclude that the deflections ξ of the points of the boundary plane are $\xi = \frac{P}{K r^{n+1}}$

r being the distance of the point in consideration from the point of the application of the force P, and K constant.

Let us compute the deflection ξ_c of the centre of a loaded circular area with the radius a , if the intensity of a uniform loading is p. We obtain

$$dP = 2\pi r\,dr\,p\ ,$$

$$\xi_c = \int_0^a \frac{p\,2\pi r\,dr}{K r^{n+1}} = \frac{2\pi p}{K}\int_0^a \frac{dr}{r^n}\ ,$$

As the deflection is in no case infinitely large, the exponent n must be smaller than 1.

For n = 0, we should have $\xi_c = \frac{2\pi p}{K} a$; the deflection would increase in direct proportion to the radius. This is, however, not in accordance with observations.
Therefore, it must be $0 < n < 1$ (6)
In this case

$$\xi_c = \frac{2\pi p}{K(1-n)}\,a^{1-n}$$

This is actually a relation of a parabolic type. The number n being greater than 0 and smaller than 1, the reciprocal of Poisson's ratio lies, according to Eq. (4), between 2 and 3, which also is applicable to soils.

Let us consider more closely the case when n has the middle value, i.e. $n = \frac{1}{2}$
That means $E = C\sqrt{z}$ the modulus of elasticity increasing in direct proportion to the square root of the depth below the surface of the semi-infinite solid and m = 2.5.
Eqs. (3) yield the displacements

$$u = \frac{7P\cos\varphi}{6\pi C\sqrt{R^3}}\ , \quad v = -\frac{7P\sin\varphi}{15\pi C\sqrt{R^3}}\ . \qquad (7)$$

It may easily be verified that this is the exact solution of the problem. Substituting into Eqs. (1) and (2), we find that there is only one component of stress different from zero, i.e. $\sigma_R = -\frac{7P}{4\pi R^2}\sqrt{\cos^3\varphi}\ ,$ (8)

which satisfies the only remaining condition of equilibrium

$$2\sigma_R + R\frac{\partial\sigma_R}{\partial R} = 0$$

It is also, in fact,

$$P = -2\pi R^2\int_0^{\pi/2}\sigma_R \sin\varphi\cos\varphi\,d\varphi$$

which is necessary because of statical equivalency.

2. PLANE DEFORMATION.

Using the notations indicated in Fig. 2, the expression for the unit strain components are

$$\varepsilon_r = \frac{\partial u}{\partial r}\ , \quad \varepsilon_s = \frac{1}{r}\left(u + \frac{\partial v}{\partial\varphi}\right), \quad \gamma_{rs} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial\varphi} - v\right). \qquad (9)$$

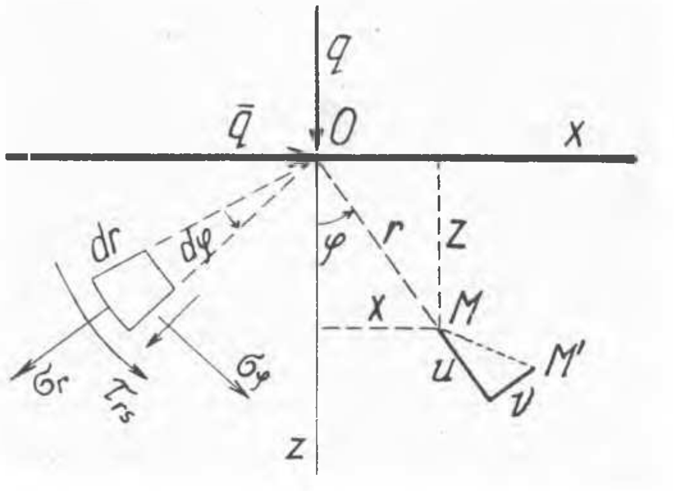

FIG. 2

The relations connecting stress and strain components may be written down as follows:

$$\sigma_r = \frac{mE}{m+1}\left(\frac{m-1}{m-2}\varepsilon_r + \frac{\varepsilon_s}{m-2}\right), \quad \sigma_s = \frac{mE}{m+1}\left(\frac{m-1}{m-2}\varepsilon_s + \frac{\varepsilon_r}{m-2}\right), \qquad (10)$$

$$\tau_{rs} = \frac{mE}{2(m+1)}\gamma_{rs}\ .$$

Let us consider the case of $E = C\sqrt{z}$, $m = 2.5$

i) If pressures of intensity q per unit length, acting perpendiculary to the surface plane, are distributed uniformly over the y-axis as shown in Fig. 2, the exact solution is given by the following equations:

$$u = 1{,}1685\,\frac{q\cos\varphi}{C\sqrt{r}}\ , \quad v = -0{,}7790\,\frac{q\sin\varphi}{C\sqrt{r}}\ , \quad \sigma_r = 0{,}6955\,\frac{q\sqrt{\cos^3\varphi}}{r} \qquad (11)$$

which may be easily verified.
Herein, the coefficients are expressed by ratios of Gamma-functions which have been calculated to four decimal places, so that

$$\frac{42\,\Gamma\left(\frac{9}{4}\right)}{25\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{4}\right)} = 1.1685\ , \quad \text{etc.}$$

ii) If loads of intensity $\bar{q}$ per unit length, distributed uniformly over the y-axis, are applied in the surface plane in the direction of x (Fig. 2), we find the following solution:

$$u = 1.7527\,\frac{\bar{q}\sin\varphi}{C\sqrt{r}}\ , \quad v = 1.1685\,\frac{\bar{q}\cos\varphi}{C\sqrt{r}}\ ,$$

$$\sigma_r = -1.0433\,\frac{\bar{q}}{r}\sin\varphi\sqrt{\cos\varphi}\ , \qquad (12)$$

where it was taken

$$\frac{42}{25}\,\frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{4})}=\frac{21\Gamma^4(\frac{1}{4})}{20\pi\sqrt{2\pi}}=1.7527 \quad \text{etc.}$$

The normal stress components in rectangular coordinates x, z,

$$\cdot\ \sigma_z=-1.0433\frac{\bar{q}}{z}\sin\varphi\sqrt{\cos^7\varphi}\ , \quad \sigma_x=-1.0433\frac{\bar{q}}{x}\sin^4\varphi\sqrt{\cos\varphi}$$

may be represented by curves of the form shown in Fig. 3.

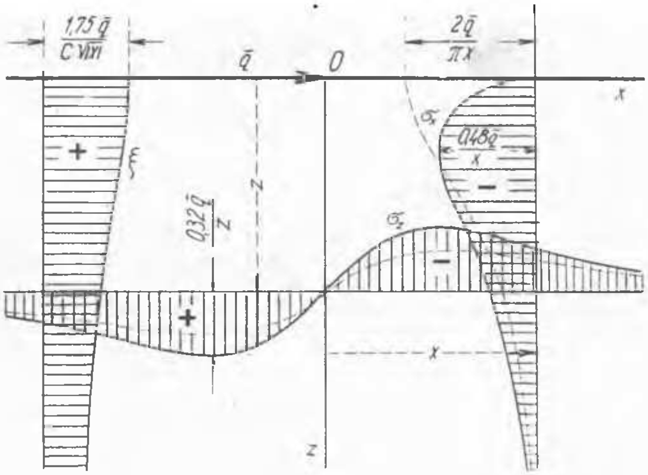

FIG.3

The horizontal displacements

$$\xi=u\sin\varphi+v\cos\varphi=\frac{1.7527}{C\sqrt{|x|}}\left(\sin^2\varphi+\frac{2}{3}\cos^2\varphi\right)\sqrt{|\sin\varphi|}$$

at the points of a straight line x = const. are shown in the same figure, where the dotted curves represent corresponding stresses in a homogeneous body. Horizontal displacements in points of a homogeneous solid become logarithmically infinite, if the condition shall be satisfied that they vanish in infinite distances from the boundary plane.

The interesting point brought out by the results shown in Fig. 3 is the remarkable difference in the intensity of horizontal stresses close to the boundary plane. In a homogeneous body, the most intensive tensile stresses are produced in the surface plane. This may be possible only in extremely cohesive materials such as rocks, but not in sandy layers or in clays; the shearing strength of similar types of compressible soils is not sufficient to support the shearing stress originated by the difference between both the extreme principal stresses.

We know, however, that retaining walls and similar structures do transmit considerable horizontal forces into their foundations. Hence, we arrive again at the conclusion that the stress distribution in soil must be different from that occuring in a homogeneous body. It follows, that the theory of an elastic solid with a constant modulus of elasticity, if applied to foundation soils, cannot give satisfactory results.

Some authors suggest, therefore, the law $E=Cz$. This gives, however, infinite deflection even under a distributed load, as we have shown above.

It seems that the results of computations will approach the statistically most probable state of stress and strain in a regularly stratified foundation soil below a normal footing, when they are derived from the theory of a semi-infinite elastic solid the modulus E of which increases in proportion to the square root of the depth beneath the footing.

Eqs. (7),(8),(11) and (12) may serve as the basis for such investigations. Some results are given in the following.

3. INCLINED FORCES.

Solutions may be expressed by superposing the effects due to the horizontal and the vertical component of the inclined load. If those components are q_x and $q_z=3q_x$ respectively, the principal radial stress is

$$\sigma_r=-\frac{0.6955}{r}\left(\frac{3}{2}q_x\sin\varphi+q_z\cos\varphi\right)\sqrt{\cos\varphi}\ .$$

This may be represented by a curve such as shown in Fig. 4.

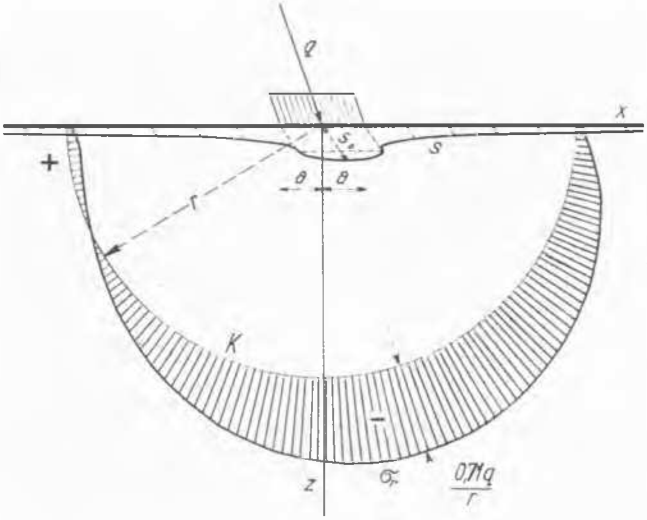

FIG.4

4. STRIP LOADING.

Let the width of strip be 2a (Fig. 5).

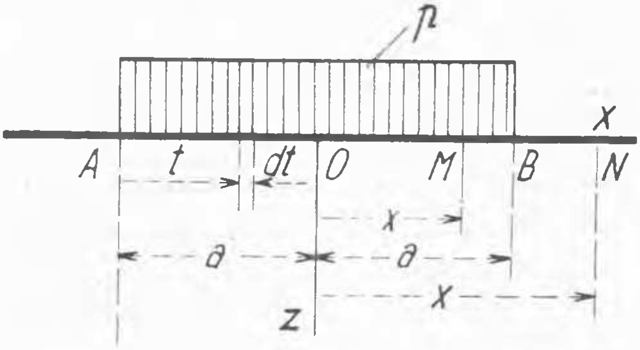

FIG.5

i) Forces of intensity p per unit square acting normally inwards produce at the point A the deflection

$$\zeta_A=\frac{0.779p}{C}\int_0^{2a}\frac{dt}{\sqrt{t}}=\frac{1.558p}{C}\sqrt{2a}\ .$$

The displacement of any point M under the loading is the sum of the deflections caused by both the strips AM and MB:

$$\zeta_M=\frac{1.558p}{C}\left(\sqrt{a+x}+\sqrt{a-x}\right),\quad -a<x<a\ .$$

Under the middle line of the strip, we obtain

$$\zeta_0=\frac{1.558p}{C}2\sqrt{a}\;;$$

which is larger by 41 per cent than the deflection under the borders of the strip.

At any point N outside the loaded area, there is $\zeta_N=\frac{1558p}{C}\left(\sqrt{|x|+a}-\sqrt{|x|-a}\right), |x|>a.$

In the boundary plane, there are no displacements in the x -direction.

ii) Horizontal forces of intensity $\bar{p}$ distributed uniformly over the same strip, produce at points of the boundary plane displacements only in the x-direction. These are as follows:

$$\xi_M=\frac{3.505\bar{p}}{C}\left(\sqrt{a+x}+\sqrt{a-x}\right),\quad -a<x<a\,.$$

$$\xi_N=\frac{3.505\bar{p}}{C}\left(\sqrt{|x|+a}-\sqrt{|x|-a}\right),\quad |x|>a\,.$$

iii) Superposing the effects of both the horizontal and the vertical component of an inclined pressure, the settlement of the boundary may be represented by the curve s shown in Fig. 4 for the case $p=3\bar{p}$. All points of the boundary move in the same direction which is, however, not parallel to the direction of acting forces. This phenomenon is due to the type of heterogeneity expressed by the law $E=C\sqrt{z}$. The greatest displacement is that of the middle line of the strip:

$$s_0=\frac{1558p}{C}2\sqrt{a}\sqrt{1+\left(\frac{3}{4}\right)^2}=\frac{3.895p\sqrt{a}}{C}\,.$$

According to the preceding results, a horizontal load produces at a point of the boundary a displacement ξ which is 2.25 times greater than the deflection ξ caused by a vertical load of the same intensity. In fact, remarkable horizontal movements of bridge abutments have been observed in many cases.

The methods of the consolidation theory offer no possibility of estimating the horizontal component of settlement which is, however, very important in bridge construction.

5. LOAD DISTRIBUTED UNIFORMLY OVER A CIRCULAR AREA.

Let the radius of the loaded area be a and the intensity of the load be p per square unit. The elemental load distributed over a circle with the radius r is $dP=p.2\pi r\,dr$.

According to Eqs.(7) and (8), this load produces in a point of the z-axis the vertical displacement

$$d\zeta=du\cos\varphi-dv\sin\varphi=\frac{pr}{C(r^2+z^2)^{3/2}}\left(\frac{7}{3}\cos^2\varphi+\frac{14}{15}\sin^2\varphi\right)dr,$$

the vertical stress component

$$d\sigma_z=d\sigma_R\cos^2\varphi=-\frac{7pr}{2(r^2+z^2)}\cos^{7/2}\varphi\,dr$$

and the horizontal stress component

$$d\sigma_r=\frac{1}{2}d\sigma_R\sin^2\varphi=-\frac{7pr}{4(r^2+z^2)}\sin^2\varphi\cos^{3/2}\varphi\,dr\;.$$

Integrating from $r=0$ to $r=a$ we obtain

i) the vertical displacements of the points in the z-axis

$$\zeta=\frac{14p\sqrt{z}}{15C}\left(2\sqrt{\frac{s}{z}}-1-\sqrt{\frac{z^3}{s^3}}\right),\quad s=\sqrt{a^2+z^2}\;;$$

ii) the stress components at the same points

$$\sigma_z=-p\left[1-\left(\frac{z}{s}\right)^{7/2}\right],\quad \sigma_r=-\frac{p}{6}\left[4+3\left(\frac{z}{s}\right)^{7/2}-7\left(\frac{z}{s}\right)^{3/2}\right]\,.$$

In the middle of the loaded area, there is

$$\zeta_0=\frac{28p\sqrt{a}}{15C},\quad \sigma_z=-p,\quad \sigma_r=-\frac{2}{3}p\,.$$

At a point situated on the z-axis at a depth $z=4a$ below the surface, there is

$$\zeta_{(z=4a)}=\zeta_0\left(\sqrt[4]{17}-1-\frac{8}{\sqrt[4]{17^3}}\right)=0.075\zeta_0,\quad \sigma_z=-0.100p,\quad \sigma_r=-0.009p$$

Thus, 0.925 of the magnitude of the deflection ζ_0 is due to the compression of the upper mass layer, the thickness of which equals double the diameter of the loaded circle. Not more than about a thirteenth of the deflection is due to the deformation of deeper parts of the solid.

When, however, the shape of the loaded portion of the surface is a rectangular one, the part of the deflection originating in the compression of the surface layer depends on the proportion of both the sides of the rectangle. The longer the rectangle is, the more the deeper parts of the body are effected. In the case of an infinitely long strip, at last, only about 0.63 of the settlement is caused by the compression of the material located within the depth equal to double the width of the strip.

This may be one of the sources of error in settlement estimating, as often only a relatively shallow bulb of pressure is taken into consideration.

Conclusion.

Observations and theoretical investigations indicate that a suitable representation of a regularly stratified foundation soil may be obtained by thinking of it as a semi-infinite solid with a variable modulus of elasticity. The assumption $E=Cz$ yields infinite values of settlement even under a distributed load, whereas the results of the investigation of a solid with $E=C\sqrt{z}$ are in reasonable agreement with observed phenomena. As they can be represented by simple closed formulae, the may be of use for foundation practice.

REFERENCE:

1) K. Hruban, The Semi-infinite Solid with Variable Modulus of Elasticity. Bulletin international de l' Académie tchèque des Sciences, XLVI, (1944), No. 13. In this paper, several other solutions are derived.

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