SECTION X

GROUNDWATER PROBLEMS

SUB-SECTION X a

GENERAL GROUNDWATER INVESTIGATIONS

X a 5

GEOMETRIC RULES GOVERNING SUBSOIL WATER FLOW.

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1. EQUALITY OF THE SECOND CURVATURE OF BOTH FIELD LINES.

Subsoil water flow is mostly treated as a plane potential flow, free of sources outside of special points or lines. The influence of friction and capillarity is neglected. This method renders a good approximation sufficient for a good number of practical purposes.

The diagram representing such a flow shows in most cases the stream lines and the so called level lines. Both, stream lines and level lines, will be called in this paper "field lines". The stream lines are curves the tangent lines of which determine in every point the direction of the velocity of the fluid. The level lines are the orthogonal trajectories of the stream lines. They are representing the loci of constant velocity potential U. The streaming field, that is the field of velocities, is forming a source and whirl free vector field, or - as it is often called - a Laplacian field. It is characterized by the differential equation of Laplace

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (1)$$

Stream lines and level lines of such a plane source free potential flow can be changed one with the other. By this we get an other plane source free potential flow. We call this new streaming field the conjugated field in relation to the first.

Careful observation of correct pictures representing source free potential flows could lead to the establishment of certain geometric rules fulfilled by the two families of curves. But till 1934 few attention was called upon this rules, 1) which in the following will be derived and demonstrated.

Very often the stream lines of a subsoil water flow, determined by certain boundary conditions, are not calculated but drawn by intuition. For this very purpose the knowledge of geometric rules fulfilled by the two families of curves will be of value.

In the surrounding of anyone not singular point O of the streaming field the velocity potential U and the so called streaming function V (constant for all points of one stream line) may be developed as an integer rational function of the coordinates as far as to the third degree. O may be the origin of the coordinate system.

$$\left.\begin{aligned} U = {} & U_0 + U_1 x + U_2 y + \tfrac{1}{2}\left(U_{11}x^2 + 2U_{12}xy + U_{22}y^2\right) + \\ & + \tfrac{1}{6}\left(U_{111}x^3 + 3U_{112}x^2y + 3U_{122}xy^2 + U_{222}y^3\right) \\ V = {} & V_0 + V_1 x + V_2 y + \tfrac{1}{2}\left(V_{11}x^2 + 2V_{12}xy + V_{22}y^2\right) + \\ & + \tfrac{1}{6}\left(V_{111}x^3 + 3V_{112}x^2y + 3V_{122}xy^2 + V_{222}y^3\right) \end{aligned}\right| \qquad (2)$$

Between the potential U and the streaming function V the so called Cauchy Riemann Equations

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0 \qquad (3)$$

are existing. The consequence of this equations is, that U as well as V are fulfilling the Laplacian differential equation (1).

The Cauchy Riemann Equations (3) render the identities

$$\begin{aligned} & U_1 + U_{11}x + U_{12}y + \tfrac{1}{2}\left(U_{111}x^2 + 2U_{112}xy + U_{122}y^2\right) \equiv \\ \equiv {} & V_2 ++ V_{12}x + V_{22}y + \tfrac{1}{2}\left(V_{112}x^2 + 2V_{122}xy + V_{222}y^2\right) \\ & U_2 + U_{12}x + U_{22}y + \tfrac{1}{2}\left(U_{112}x^2 + 2U_{122}xy + U_{222}y^2\right) + \\ + {} & V_1 + V_{11}x + V_{12}y + \tfrac{1}{2}\left(V_{111}x^2 + 2V_{112}xy + V_{122}y^2\right) \equiv 0 \end{aligned} \qquad (4)$$

The equations between the coefficients, which are the consequence of these two identities, reduce the number of coefficients to only two for every degree. We will set

$$\left.\begin{aligned} & U_1 = V_2 = A_1, \\ & U_{11} = V_{12} = -U_{22} = A_2, \\ & U_{111} = V_{112} = -U_{122} = -V_{222} = A_3, \\ & U_2 = -V_1 = B_1, \\ & U_{12} = -V_{11} = V_{22} = B_2, \\ & U_{112} = -V_{122} = -U_{222} = -V_{111} = B_3 \end{aligned}\right| \qquad (5)$$

If we suppose that in the origin O the x-axis is tangent line of the U-line, the y-axis, tangent line of the V-line, then $A_1 = 0$. In the surrounding of O potential and streaming function can be expressed in the following form

$$\left.\begin{aligned} U &= U_0 + B_1 y + \tfrac{1}{2}\left[A_2\left(x^2 - y^2\right) + 2B_2 xy\right] + \\ &+ \tfrac{1}{6}\left[A_3 x\left(x^2 - 3y^2\right) + A_3 y\left(3x^2 - y^2\right)\right] \\ V &= V_0 - B_1 x + \tfrac{1}{2}\left[-B_2\left(x^2 - y^2\right) + 2A_2 xy\right] + \\ &+ \tfrac{1}{6}\left[-B_3 x\left(x^2 - 3y^2\right) + A_3 y\left(3x^2 - y^2\right)\right] \end{aligned}\right\} \quad (6)x)$$

The U-line and the V-line going through the origin O

$$U = U_0 \quad , \quad V = V_0 \qquad (7)$$

may be approximated in the surrounding of O by parabolas of third order: the U-line by a parabola with the axis parallel to the y-axis, the V-line by a parabola with the axis parallel to the x-axis. The equation of these curves - the tangent lines of which in O are the x-axis and the y-axis respectively - can be written as follows:

$$y = \tfrac{1}{2}C_2 x^2 + \tfrac{1}{6}C_3 x^3 \;,\quad x = \tfrac{1}{2}D_2 y^2 + \tfrac{1}{6}D_3 y^3 \qquad (8)$$

We consider x in the equation of the U-line and y in the equation of the V-line as the independent variable. In dealing with the U-line we express the first three derivatives of y with respect to x by y', y'', y''', in dealing with the V-line the first three derivatives of x with respect to y by x', x'' and x'''. In any point of these curves

$$K_U = \frac{y''}{\left(1 + y'^2\right)^{3/2}} \;,\quad K_V = \frac{x''}{\left(1 + x'^2\right)^{3/2}} \qquad (9)$$

are the curvature of the U- and V-line respectively. If we call ds_U and ds_V the line elements of these curves

$$\frac{dK_U}{ds_U} = \frac{y'''\left(1 + y'^2\right) - 3y'y''^2}{\left(1 + y'^2\right)^3} \;,\quad \frac{dK_V}{ds_V} = \frac{x'''\left(1 + x'^2\right) - 3x'x''^2}{\left(1 + x'^2\right)^3} \qquad (10)$$

are the rates of change of the curvature of these curves.

The U-line possesses in the origin the values $y = 0$, $y' = 0$, $y'' = C_2$, $y''' = C_3$ and the V-line the values $x = 0$, $x' = 0$, $x'' = D_2$, $x''' = D_3$.

Therefore

$$K_U = C_2 \;,\; \frac{dK_U}{ds_U} = C_3 \;,\; K_V = D_2 \;,\; \frac{dK_V}{ds_V} = D_3 \qquad (11)$$

If we substitute in the first and in the second equation of (6) respectively the first respectively the second equation of (7) and (8), we get the identities

$$\left.\begin{aligned} 0 &= B_1\left(\tfrac{1}{2}C_2 x^2 + \tfrac{1}{6}C_3 x^3\right) + \tfrac{1}{2}\left(A_2 x^2 + B_2 C_2 x^3\right) + \tfrac{1}{6}A_3 x^3 \\ 0 &= -B_1\left(\tfrac{1}{2}D_2 y^2 + \tfrac{1}{6}D_3 y^3\right) + \tfrac{1}{2}\left(B_2 y^2 + A_2 D_2 y^3\right) - \tfrac{1}{6}A_3 y^3 \end{aligned}\right\} \qquad (12)$$

The consequence of these identities are the equations

$$\left.\begin{aligned} &\tfrac{1}{2}B_1 C_2 + \tfrac{1}{2}A_2 = 0 \;, \quad -\tfrac{1}{2}B_1 D_2 + \tfrac{1}{2}B_2 = 0 \\ &B_1 C_3 + \tfrac{1}{2}B_2 C_2 + \tfrac{1}{6}A_3 = 0 \;, -\tfrac{1}{6}B_1 D_3 + \tfrac{1}{2}A_2 D_2 - \tfrac{1}{6}A_3 = 0 \end{aligned}\right\} \qquad (13)$$

with the solutions

$$\left.\begin{aligned} &C_2 = -\frac{A_2}{B_1} = K_U \;,\quad D_2 = +\frac{B_2}{B_1} = K_V \\ &C_3 = D_3 = \frac{3A_2 B_2 - A_3 B_1}{B_1^2} = \frac{dK_U}{ds_U} = \frac{dK_V}{ds_V} = K' \end{aligned}\right\} \qquad (14).$$

The equality of the second parameters C_3 and D_3 of the two parabolas of third order, approximating the level line and the stream line in the surrounding of O, is the expression of the following law:

"In every point of the field of a plane source free potential streaming the level line and the stream line are possessing equal rate of change of curvature". 2)

A field point can be the vertex of both field lines (level and stream line) but never be the vertex of only one of them.

2. THE TYPES OF CROSSING OF THE FIELD LINES IN THE STREAMING FIELD.

$K_U > 0$, means that the U-line in the origin is concave upward, $K_U < 0$, that it is concave downward (Fig. 1 and 2). $K_V > 0$, means that the V-line in the origin is concave to the right side, $K_V < 0$ that it is concave to the left side. Disregarding the sign of curvature we will call the part of every curve in which the magnitude of the curvature is larger, the "stronger curvatured part", the part of the curve in which the magnitude of the curvature is smaller, the "less curvatured part". Every point of a field line which is neither a vertex nor a point of inflection divides in its surrounding the curve in a stronger curvatured and in a less curvatured part.

The different types of orthogonal crossing of the two lines (U-line and V-line) of a source free potential flow are characterized by the three parameters K_U, K_V and $K' = \frac{dK_U}{ds_U} = \frac{dK_V}{ds_V}$.

If all three parameters are positive the type of crossing is represented by Fig. 1a. Two of the four parts of the two curves are lying in the first quadrant. They are the stronger curvatured parts of the two curves and turn their concave side to each other.

The $2^3 = 8$ parts of Fig. 1 are representing the types of crossing dependent on the sign of the three parameters. In the upper line of figures $K' > 0$, in the lower line $K' < 0$.

In all combinations in which $K_U K_V K' > 0$

x) The reader familiar with the elements of the theory of functions of a complex variable will recognize that the terms multiplied with one coefficient are the real and the imaginary part respectively of $(x+iy)^n$. $(n=1,2,3)$.

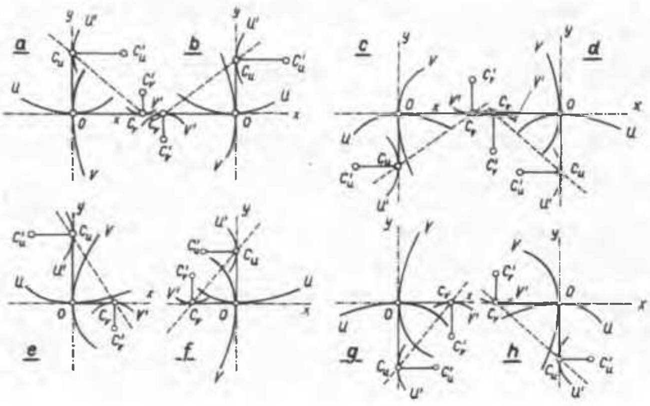

FIG. 1

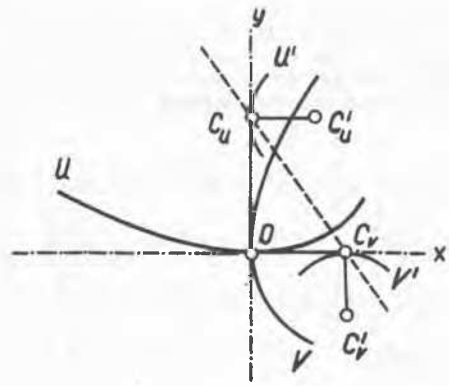

FIG. 2

(Fig. 1a, d, f, g) the two stronger curvatures parts of the two curves are lying in one quadrant and turn their concave sides to each other. If $K_U K_V K' < 0$ (Fig. 1b, c, e, h) the two less curvatured parts of the curves are lying in one quadrant and turn their concave sides to each other. The quadrant in which two of the four parts of the curves are lying is the quadrant crossed bu the Cesàro line, that is the straight line joining the two centers of curvature C_U and C_V of the U- and of the V-line. 3)

The curves U' and V' in the eight parts of Fig. 1 are elements of the evolutes of the U-line and of the V-line respectively. C_U' and C_V' are the centers of curvature of these evolutes, or the so called "second centers of curvature" of the U-line and of the V-line.

If we follow the trace of lines $C_U' C_U O C_V C_V'$, we turn three times through a right angle. In the figures a), d), f), g), ($K_U K_V K' > 0$) we turn three times in the same sense. In the figures b), c), e), h), ($K_U K_V K' < 0$) the trace of lines possesses the form of a "W" and the sense of turning through a right angle is allways alternated. According to the form of this characteristic trace of lines we will call the first type of crossing where the stronger curvatured parts of the curves turn their concave sides to each other "monotonic crossing", the second type, where the less curvatured parts of the curves are turning their concave sides to each other, "alternating crossing".

In any two families of curves which are orthogonal trajectories one to another a third type of crossing may occur. In general

$K_U' = \frac{dK_U}{dS_U}$ and $K_V' = \frac{dK_V}{dS_V}$ must not be equal and can also possess different signs. The type of crossing is influenced by the signs of the four parameters K_U, K_V, K_U', K_V'. By that consideration we get $2^4 = 16$ combinations instead $2^3 = 8$. We don't need to consider the cases in which $K_U' K_V' > 0$, because they are in qualitative regard identical with the special case, $K_U' = K_V'$. But we have to consider the cases where $K_U' K_V' < 0$. If $K_U > 0$, $K_V > 0$, $K_U' > 0$, $K_V < 0$, we get the crossing represented in Fig. 2. There too, two of the four parts of the two curves are lying in the same quadrant (the first) which is crossed by the Cesàro-line. But one of these two parts is the stronger, the other the less curvatured part of the curve to which it belongs, and these parts are turning one against the other their concave sides. The stronger curvatured parts of the two curves are turning one against the other different curvatured sides, convave and convex.

Starting from the diagram in Fig. 2, we can get three new pictures by turning the curves three times through a right angle around O, and four new by reflecting the curves of Fig. 2 around one axis and than turning it three times through a right angle around O. In that manner we get all the 8 combinations in which $K_U' K_V' < 0$.

If in these diagrams we are following again the trace $C_U' C_U O C_V C_V'$, two turns through a right angle following one another, are possessing the same sense: the third turn possesses the opposite sense. We will call this type of crossing of the two field lines according to the form of the characteristic trace of lines "indefinite crossing".

In the case of "monotonic crossing" the two second centers of curvature are lying in the same quadrant, in the case of "alternating crossing" in opposite quadrants, and in the case of "indefinite crossing" in neighbouring quadrants.

In the streaming field representing a plane source free potential flow, and therefire coming into consideration for representing a subsoil water flow, only the monotonic and the alternating crossing of level line and stream line can happen. An indefinite crossing of level line and stream line does never happen in such a flow. It does never occur in a plane Laplacian field.

3. RELATION BETWEEN THE POSITIONS OF THE SECOND CENTERS OF CURVATURE OF BOTH FIELD LINES.

The consequence of the equality $K_U' = K_V'$ is, that there is existing in every point a relation between the curvature elements of second order of the two field lines.

If we call φ the angle of inclination of the U-line at any point against the positive x-axis, $r_U = \frac{1}{K_U}$, the radius of curvature of the U-line ds_U', the element of the evolute U' of the U-line, r_U', the radius of curvature of the evolute, then

$$d\varphi = \frac{dS_U}{r_U} = \frac{dS_U'}{r_U'} = -\frac{dr_U}{r_U'} \qquad (15)$$

Analoguously we call φ the angle of inclination of the V-line, dS_V the element of the evolute V', r'_V the radius of curvature of the evolute. Then

$$d\varphi = \frac{dS_V}{r_V} = \frac{dS'_V}{r'_V} = -\frac{dr_V}{r'_V} \qquad (15')$$

Therefore we get

$$r'_U = -\frac{dr_U}{d\varphi} = -\frac{dr_U}{dS_U}\cdot\frac{ds_U}{d\varphi} = -r_U\frac{dr_U}{dS_U} = +K'r_U^3 \qquad (16)$$

and in the same manner

$$r'_V = +K'r_V^3 \qquad (16')$$

With respect to the system x, y (Fig. 3) the coordinates of C'_U are $x=r'_U$, $y=r_U$ and the coordinates of C'_V, $x=r_V$, $y=r'_V$.

$$x = K'y^3 \qquad (17)$$

Is the equation of a cubic parabola p_U of which O is the point of inflection and the y-axis the tangent line in it. Upon this parabola is lying the second center of curvature C'_U of the U-line. If we reflect this parabola by one of the straight line bysecting the angles between the coordinate axis, we get the cubic parabola p_V with the equation

$$y = K'x^3 \qquad (17')$$

upon which C'_V, the second center of curvature of the V-line must be situated.

Both cubic parabolas p_U and p_V are lying in the same quadrant in the case of monotonic crossing and in opposite quadrants in the case of alternating crossing.

4. EXAMPLES FOR THE APPEARANCE OF BOTH POSSIBLE TYPES OF CROSSING IN THE STREAMING FIELDS.

The field in which all points are vertices of both field lines is the field produced by one positive and one negative point source of equal intensity. The stream lines are the family of circles going through these two points, the level lines, the family of circles which are the orthogonal trajectories. The conjugated field would be produced by two whirls of equal magnitude but opposite sense in the two points where the sources of the first field are situated.

Disregarding this special case there are existing three kinds of fields of a source free potential flow.

a) The monotonic field. All points are points of monotonic crossing of both field lines. Only exeptionally there can appear loci of vertex crossing. An example for such a field is represented by the system of confocal ellipses and hyperbolas as field lines. In Fig. 4 there is only drawn one ellipse and one hyperbola of the system. This field can be produced by the streaming of the fluid from one side of a thin vertical wall to the other side if a vertical strip is cut out from the wall. In every horizontal plane the stream lines are the family of confocal hyperbolas, the level lines the family of confocal ellipses. The foci F_1 and F_2 are the points of intersection of the vertical straight lines bordering the strip with the horizontal plane. In Fig. 4 is to recognize that always the stronger curvatured parts of both field lines are turning one against the other their concave side. The two coordinate axis are loci of vertex crossing.

If a fluid is streaming around an elliptic cylinder we have the conjugated streaming field. In this case the ellipses are the stream lines, the hyperbolas the level lines.

b) The alternating field. All points are points of alternating crossing, only exceptionally their can appear loci of vertex crossing. Such a field is the streaming field in the right angle formed by two vertical walls perpendicular one to another (Fig. 5). If we choose the line of intersection of the vertical walls

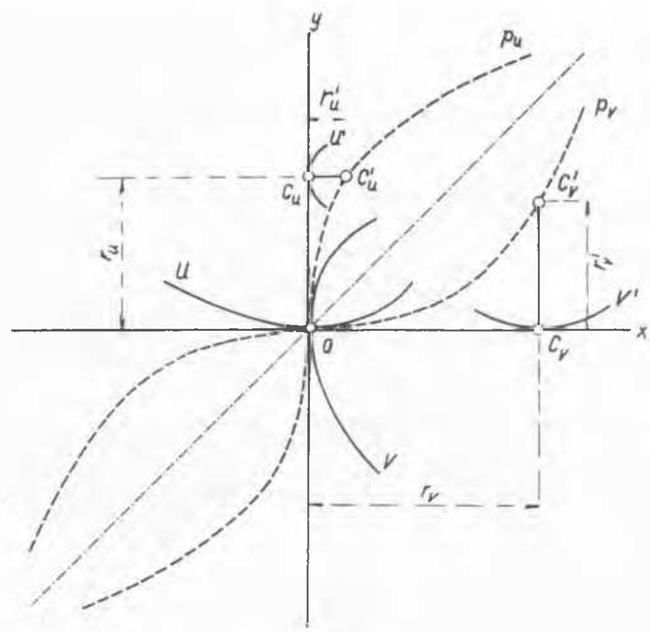

FIG. 3

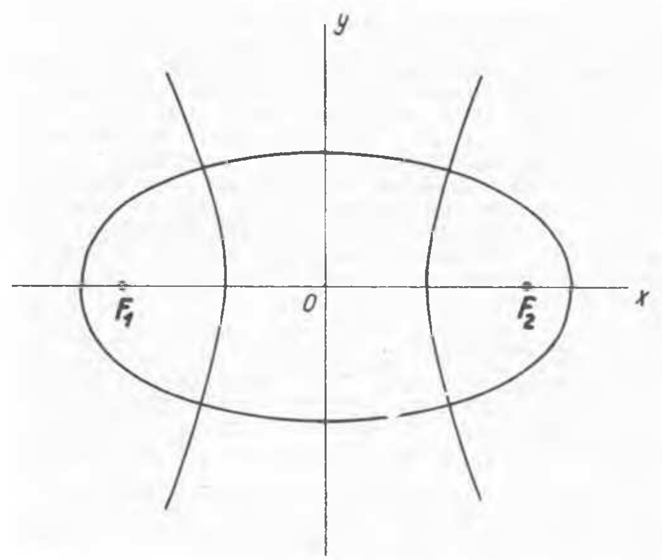

FIG.4

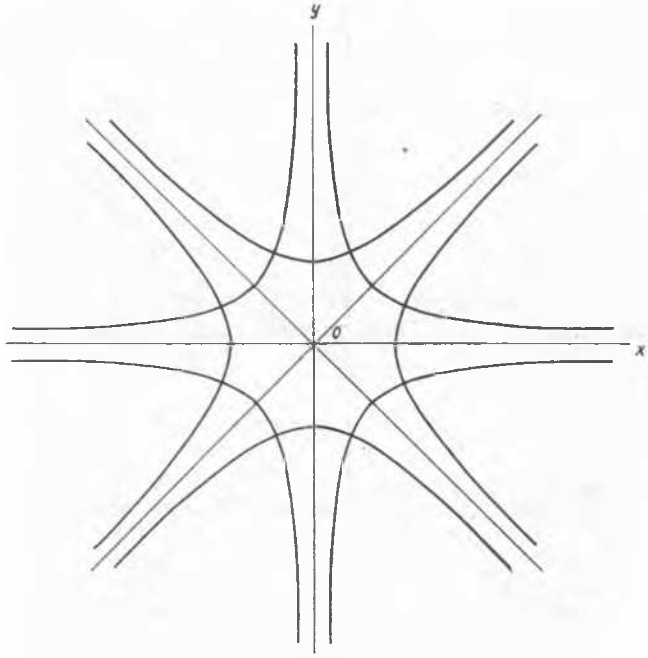

FIG.5

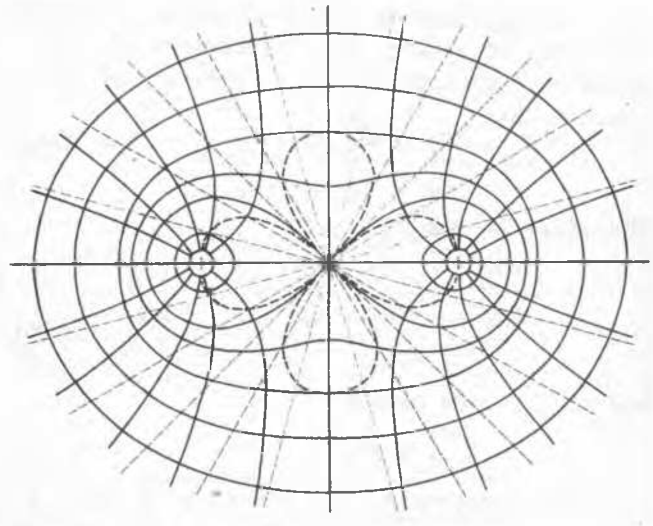

FIG.6

with the horizontal plane as the coordinate axis x, y, then the equi-lateral hyperbolas xy=constant are the stream lines and the equi-lateral hyperbolas $x^2 - y^2$ = const. are the level lines. The coordinate axis and the straight lines bysecting the right angles between them are loci of vertex crossing.

If the streaming would happen in the angle between the planes y = +x and y = -x the family of equilateral hyperbolas xy=const. would be the level lines, the family of equilateral hyperbolas $x^2 - y^2$ = constant the stream lines.

c) The mixed field. There are appearing regions of monotonic crossing and regions of alternating crossing, both separated by loci of vertex crossing. Besides these loci can also appear loci of vertex crossing which don't separate regions of different types of crossing.

Such a field would be produced p.e. by two parallel thin vertical tubes, perforated or porous, with equal productivity, or - if we consider only the relations in one horizontal plane - by two point sources of equal productivity. We will call these points the foci, the straight line through these points the principal axis, the line of symmetry between the foci the secondary axis. The field conjugated to the field just characterized would be produced by two whirls of equal intensivity and the same sense, situated in the two foci.

In the first field the family of equilateral hyperbolas going through the foci are forming the stream-lines: the family of the so called Cassini-curves - which are the loci of all points with constant product of the distances from the two foci - are the level lines. (Fig. 6) We will call 2e the distance of the foci and c^2 the constant product of the distances of any point of the Cassini-curve from the foci. If $c^2 < e^2$, the Cassini-curve consists in two separated branches, everyone forming a closed oval around one of the foci. In the case $c^2 = e^2$ the curve becomes the lemniscate (we call it L) with the form of a lying "8", the double point of which is the point bysecting the line segment between the foci. If $e^2 < c^2 < 2e^2$, the level lines are forming closed curves surrounding both focis without double points but with four points of inflection. If $c^2 \geq 2e^2$, the level lines are forming one oval surrounding both foci.

The two axis are loci of vertices of the family of Cassini-curves. The locus of the vertices of the hyperbolas is a lemniscate. This lying lemniscate L_1 - indicated in the figure by an interrupted line - originates by reducing the lemniscate L from the origin in the ratio $1:\sqrt{2}$. The points of these curve are also vertices of the Cassini-curves consisting in two branches. The points of intersection of the Cassini-curves with the principal axis are vertices of largest curvature, the points of intersection with the lemniscate L_1 are vertices of smallest curvature. The region of monotonic crossing is the region outside of the lemniscate L_1. The interior of the loops of the lemniscate L_1 is the region of alternating crossing of the two field lines.

By turning the lemniscate L_1 by a right angle around the center of the figure we get the congruent lemniscate L_2 - also indicated in Fig. 6 by an interrupted line. This upright lemniscate L_2 is the locus of the points of inflection of Cassini-curves. But these points are ordinary points of the equilateral hyperbolas.

5. A VECTOR DETERMINED BY THE PICTURE OF THE STREAMING FIELD, CHARACTERISTIC FOR EVERY POINT IN IT.

There may be drawn one family of field lines - we will assume the level lines U - but there may be written no scale in the picture. Therefore we can not read in the pictures the gradient of U, that means the velocity of streaming. We may also not know if two neighboured level lines are representing always the same difference of the potential. But nevertheless it is possible to determine the loci of points where the gradient, which represents the velocity of streaming, possesses the same magnitude.

The x and y component of the velocity are, disregarded terms of higher than the first degree with respect to x and y,

$$v_x = \frac{\partial U}{\partial x} = A_2 x + B_2 y \ , \ v_y = \frac{\partial U}{\partial y} = B_1 + B_2 x - A_2 y \quad (18$$

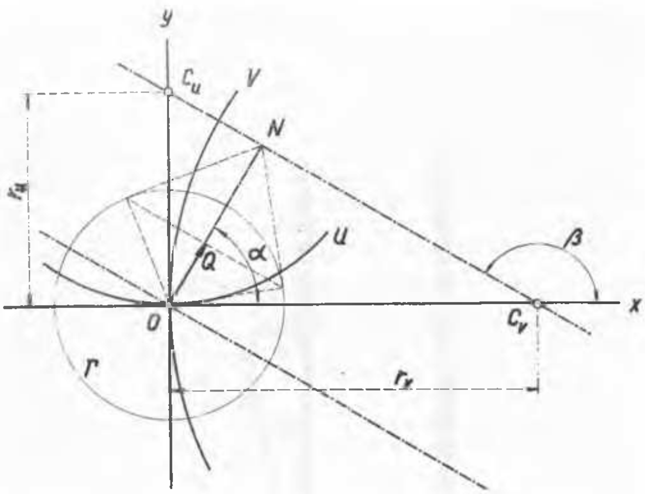

FIG.7

and the square of the velocity, with the same neglection

$$v^2 = v_x^2 + v_y^2 = B_1^2 + 2B_1B_2x - 2B_1A_2y \qquad (19)$$

In the origin O the velocity v possesses the same magnitude as B_1 and remains in the same direction constant as v^2. Perpendicular to this direction happens the quickest change of these values, that is in the direction of the gradient of v^2 and of the gradient of the speed v. The components of the gradient of v^2 are

$$\frac{\partial(v^2)}{\partial x} = 2B_1B_2 \quad , \quad \frac{\partial(v^2)}{\partial y} = -2B_1A_2 \qquad (20)$$

By the ratio of these components

$$\frac{\partial(v^2)}{\partial y} : \frac{\partial(v^2)}{\partial x} = -\frac{A_2}{B_2} = +\frac{K_U}{K_U} = +\frac{r_U}{r_U} = \operatorname{tg}\alpha \qquad (21)$$

is given the angle α (Fig. 7) between the positive x-axis and the direction of grad (v^2). The direction perpendicular to this direction, that is the direction in which the magnitude of v remains constant, is parallel to the Cesàro-line C_UC_V belonging to the point O, because we see in Fig. 7 that the angle between the positive x-axis and the Cesàro-line is given by

$$\operatorname{tg}\beta = -\frac{r_U}{r_V} \qquad (22)$$

We can not recognize in the pictures of the level lines the gradient of the magnitude of v, but we can recognize its ratio to v. We develop

$$\text{grad ln } v = \frac{\text{grad } v}{v} = \frac{\text{grad } (v^2)}{2v^2} \qquad (23)$$

The components of this gradient are

$$\left.\begin{aligned} \frac{\partial}{\partial x}\ln v &= \frac{1}{2v^2}\cdot\frac{\partial(v^2)}{\partial x} = \frac{B_2}{B_1} = K_V = \frac{1}{r_V} \\ \frac{\partial}{\partial y}\ln v &= \frac{1}{2v^2}\cdot\frac{\partial(v^2)}{\partial y} = -\frac{A_2}{B_1} = K_U = \frac{1}{r_U} \end{aligned}\right\} \qquad (24)$$

The direction of this gradient is the same as the direction of gradient v, characterized by the angle α and perpendicilar to the Cesàro-line. The magnitude can be found by the equation

$$(\text{grad ln } v)^2 = \frac{1}{r_V^2} + \frac{1}{r_U^2} = \frac{r_U^2 + r_V^2}{r_U^2 r_V^2} \qquad (25)$$

Therefore

$$\left|\text{grad ln } v\right| = \frac{\sqrt{r_U^2 + r_V^2}}{r_U^2 r_V^2} = \frac{1}{p} \qquad (26)$$

where p is the length of the perpendicular ON, drawn from O to the Cesàro-line. Therefore the vector grad ln v can be found by a simple construction. We draw the unit circle around the field point O and determine the pol Q of the Cesàro-line. The vector OQ is representing grad ln v. This vector is independent on the values of the potential belonging to the different level lines drawn in the pictures also independent from the direction in which the potential is increasing. The vector belonging to the point O would also be determined if only the curvature elements of the U-line and of the V-line going through the point O would be drawn. But beyond it, this field vector, characterizing the field in the surrounding in the field point - is independent on the fact whether the family of curves drawn in the pictures are the level or the stream lines.

REFERENCES

1) A. Basch, Zur Geometrie der Skalar und Vektor Felder, insbesondere des Laplaceschen Feldes. Monatshefte für Mathematik und Physik, 41, 1934, p. 300 - 321.
2) R. v. Lilienthal, Grundlage einer Krümmungslehre der Curvenscharen. Leipzig,1896, p. 9.
3) E. Cesàro, Vorlesingen über natürliche Geometrie. (German edition by G. Kowalevsky) Leipzig 1901. In par. 116 is shown that the straight line mentioned above is the locus of the centers of curvature for all isogonal trajectories of the family U = const, drawn through the point O.

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