13. Since the representation is only approximate, the arbitrary constants have to be determined in accordance with whatever further hypothesis one chooses to adopt. Personally I adopted the 'principle of least squares' for reducing error to a minimum. It leads to a set of linear equations for the determination of the arbitrary constants, and the solution can then be completed. One case is worked out in detail.

14. Admittedly the method is neither short nor easy, and it is open to question whether the final form of the results is strictly defensible. Personally I merely regard them as first tentatives and hope for something better from our other avenues of research. The results are too new to have been tried out in practice; they are here made public for what they are worth to whoever chooses to use them.

No. E-3

TANGENTIAL STRESSES UNDER A SPREAD FOUNDATION

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Limiting Curves. Loading tests reveal the fact that under a loaded circular disk there are zones of vertical pressures (Zones I and II, Fig. 1) and a so-called "stressless zone" (German "spannungslos", Zone III, Fig. 1). Zones I and II are separated from Zone III by a characteristic surface of rotation. Its meridional section as shown in Fig. 1 will be termed "limiting curve" throughout this paper. In a natural earth mass under the action of its own weight only, the ratio of the horizontal (minor) pressure to the vertical (major) is K, "coefficient of pressure at rest" (terminology of Dr. Terzaghi). The writer believes that the condition controlling this state of "limit equilibrium", should be generalized. In Fig. 2 the stress, s, caused by a concentrated force, F, strikes a plane, MN, and makes an acute angle, α, with the normal to this plane. Pressures $s.\cos^2\alpha$ (minor) and $s.\sin^2\alpha$ (major) act normally and parallel to Plane MN, respectively. There will always be compression at Point O, if Equation (1) is satisfied. If $\cot^2\alpha =$ K, the stressed condition at Point O may resolve in shear, but only under condition that the direction of the shear be upwards (Fig. 2a) and not downwards (Fig. 2b). Thus the loaded earth mass would be divided into two zones: "zone of compressions" (Zones I and II, Fig. 1) and "zone of shears" (Zone III, Fig. 1). The "zone of compressions" would be bounded by the locus of the points where the direction of the shear is horizontal; in other words, where a stress, s, coming from the origin, A (Fig. 3) strikes the horizontal plane under the angle $90^\circ - \alpha$. In such a case, $\theta = \alpha$ if θ designates the angle formed by a stress with the vertical. This locus is the straight line, AO, making an angle $\theta = \alpha$ with the vertical.

If the load is uniformly distributed around a circular disk, the "limiting curve" would be a curve having the straight line, AO, for asymptote. This is because at a distance of more than two widths from the loaded area, the latter acts practically as a concentrated load.

Because of the uncertainty as to the actual load distribution along the base of the loading disk, the "limiting curve" in the neighborhood of the loaded portion may be traced approximately by transforming conformally the straight line, AO, into a hyperbola N'N" passing through the edges of the loaded portion N'N', 2b wide (Fig. 3 and 4). If the radius of the circle to be "flattened out", were b, the transformation formulas to be applied would be those in (2). The symbols ρ and θ in equations (2) are polar coordinates of the points on the straight line, AO, and x and z are orthogonal coordinates of those on the hyperbola, N'N". However, since the radius of the circle in Fig. 4 is $b/\sin\theta$, and not b, the value b^2 in Formulas (2) must be replaced by $b^2/\sin^2\theta$. By squaring and deducting, Formula (3) would be obtained. Since the values of the coefficient of pressure at rest, K, and of the concentration factor, n, are interconnected (Proc., Am.Soc.C.E., October, 1935, page 1255) as shown by Equation (4), the final equation of the limiting curve is given by (5).

Fig. 5 represents the locus (5) traced for both cases of n = 6 (heavy solid line T_s) and of n = 3 (heavy dotted line T_c). The curve T_s corresponds to average course sands, and the equilateral hyperbola T_c to elastic isotropic bodies and may be applied in the case of clays. Curves G; KS; and P, approximately represent experimental limiting curves obtained by Messrs. Goldbeck; Kögler and Scheidig (Curves G and KS are plotted from Fig. 8, page 420, Die Bautechnik, 1927); and Press (Curve P is plotted from Fig. 1 and 2, pages 569-570, Die Bautechnik, 1934) respectively.

The writer believes that vertical pressures in Zone III (Fig. 1) are negligible if present at all and that the stressed condition in that zone practically resolves into pure shear, so that apparatus for measuring vertical pressure could not reflect the situation. An actual shear failure in Zone III occurs, however, only if the shearing resistance is not able to stand the stress; and when Zone III is broken, material in Zones I and II also starts to move following the shearing surfaces.

Two dimensional stress distribution only will be considered on continuation.

Angle of Mutual Intersection of the Shearing Surfaces (Slip Lines); Value of the Shearing Stress. At Point O (Fig. 6a) there are two planes, mn, and m'n', where condition (1) is not satisfied. They form angles $(90^\circ - \alpha)$ with the direction of the stress and are tangent to the shearing surfaces at Point, O. In the case of a distributed load, the shearing surface form angles $(90^\circ - \alpha)$ with the direction of the major principal stress, s, since the direction of the latter is that of the resultant of all forces

FIG. 1

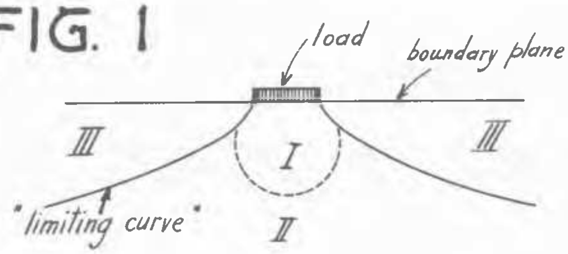
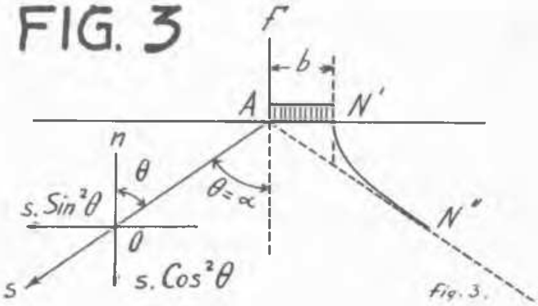


FIG. 3



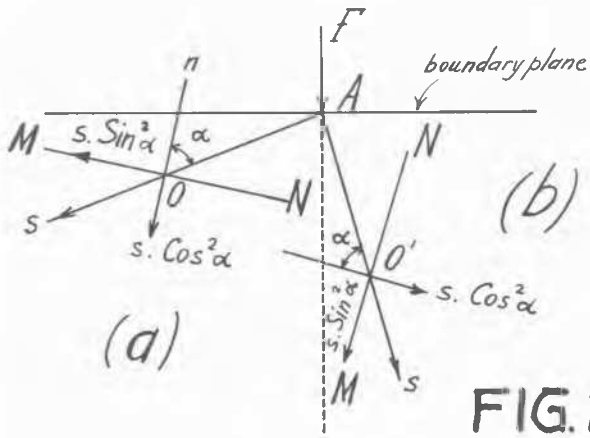


FIG. 2

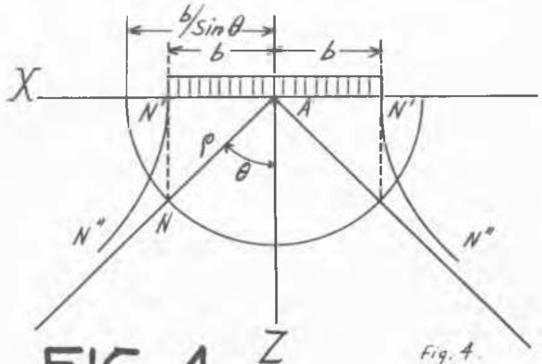
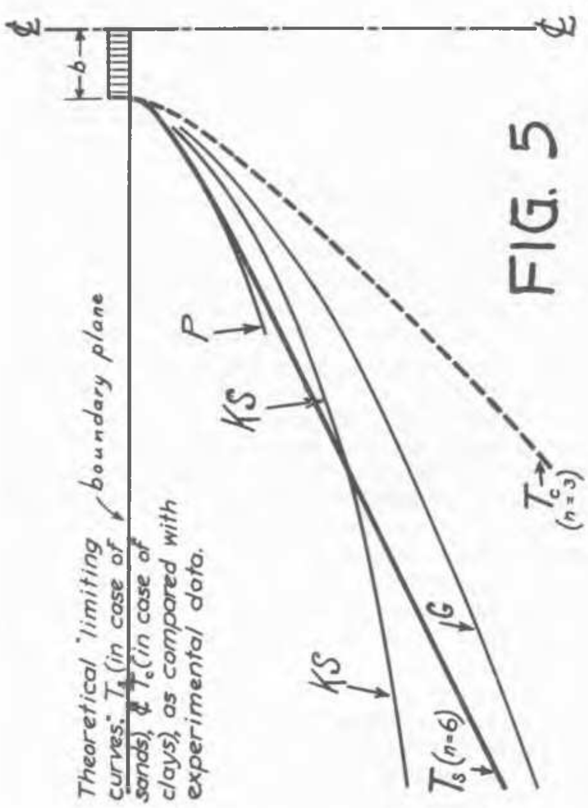
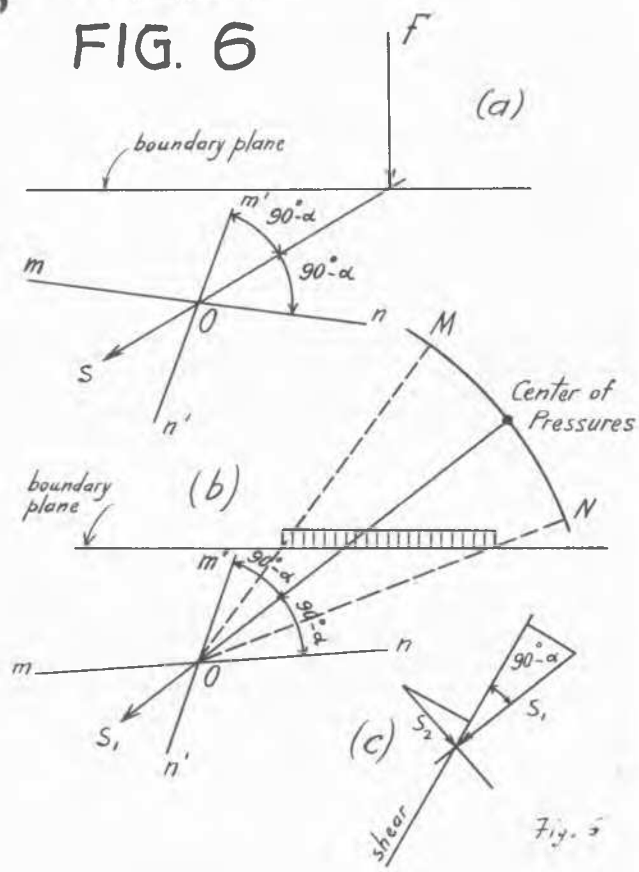


FIG. 4

FIG. 6





This figure shows the angle formed by the slip lines at a point; value of the shearing stress.

$$\frac{s.\,Cos^2\alpha}{s.\,Sin^2\alpha} = K; \quad Cot^2\alpha = K \quad \ldots\ldots (1)$$

$$\left.\begin{aligned} x &= \tfrac{1}{2}\left(\rho + \frac{b^2}{\rho}\right).\,Cos\,(90^\circ - \theta) \\ z &= \tfrac{1}{2}\left(\rho - \frac{b^2}{\rho}\right).\,Sin\,(90^\circ - \theta) \end{aligned}\right\} \ldots (2)$$

$$x^2 - z^2.\,tan^2\theta = b^2 \quad \ldots\ldots (3)$$

$$n = 2 + \frac{1}{K}; \quad tan^2\theta = n - 2 \quad \ldots\ldots (4)$$

$$x^2 - z^2(n-2) = b^2 \quad \ldots\ldots (5)$$

$$\tau_{max} = s_1.\,Sin\,\theta.\,Cos\,\theta - s_2.\,Cos\,\theta.\,Sin\,\theta =$$

$$= \tfrac{1}{2}(s_1 - s_2).\,Sin\,2\theta \quad \ldots\ldots (6)$$

$$\tau_{max} = \tfrac{1}{2}(s_1 - s_2) \quad \ldots\ldots (7)$$

$$\rho = a_1.\,e^{\frac{\pi}{2}-\theta}; \quad \rho = a_2.\,e^{-(\frac{\pi}{2}-\theta)} \quad \ldots\ldots (8)$$

$$b = a_1.\,e^{\frac{\pi}{2}-\theta_0}; \quad b = a_2.\,e^{-(\frac{\pi}{2}-\theta_0)} \quad \ldots\ldots (9)$$

$$\rho = b.\,e^{\theta_0-\theta}; \quad \rho = b.\,e^{-(\theta_0-\theta)} \quad \ldots\ldots (10)$$

$$\left.\begin{aligned} x &= \tfrac{1}{2}\left[b.\,e^{\pm(\theta_0-\theta)} + \frac{b^2}{b.\,e^{\pm(\theta_0-\theta)}}\right].\,Sin\,\theta \\ z &= \tfrac{1}{2}\left[b.\,e^{\pm(\theta_0-\theta)} - \frac{b^2}{b.\,e^{\pm(\theta_0-\theta)}}\right].\,Cos\,\theta \end{aligned}\right\} (11)$$

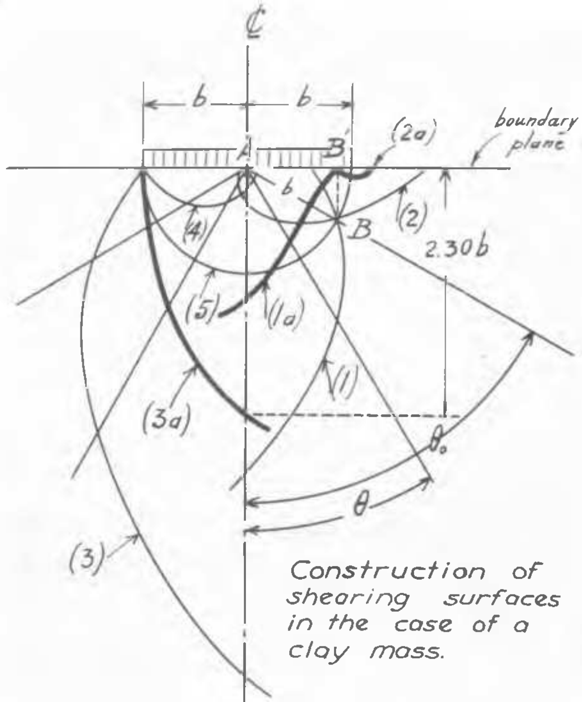
$$x = b.\,Cosh\,(\theta_0-\theta).\,Sin\,\theta; \quad z = b.\,Sinh\,(\theta_0-\theta).\,Cos\,\theta \quad (12)$$

$$x = b.\,Cosh\,(\tfrac{\pi}{2}-\theta).\,Sin\,\theta; \quad z = b.\,Sinh\,(\tfrac{\pi}{2}-\theta).\,Cos\,\theta \quad (13)$$

$$z = b.\,Sinh\,\tfrac{\pi}{2} = 2.30\,b \quad \ldots\ldots (14)$$

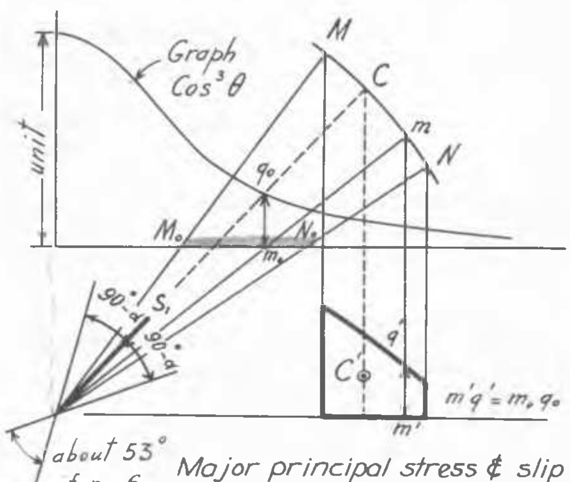
$$s = \frac{3F}{\pi\rho}.\,Cos^4\theta \quad \ldots\ldots (15)$$

$$s = \frac{2F}{\pi\rho}.\,Cos\,\theta \quad \ldots\ldots (16).$$



Construction of shearing surfaces in the case of a clay mass.

FIG. 7



Major principal stress & slip lines in case of a sand mass.

FIG. 8

acting at the cylindrical projection, MN, of the loaded portion, M_oN_o. In the case of clays the shearing surfaces practically bisect the angle between the principal stresses, s_1 and s_2, as required by the theory of elasticity. in the case of sand, the value of the concentration factor, n, is about 6, and the angle formed by the shearing surface with the major principal stress is about $23\frac{1}{2}°$. This value checks with the experiments of the writer ("Civil Engineering", New York , October, 1933, page 574 et seq.).

The value of the shearing stress, t, may be determined as shown in Fig. 6(c), and this leads to Formula (6). Only in the case of clays, when Sin 2 Θ = 1, may the well known elastic formula (7) be applied.

Shape of the Shearing Surface (Slip Lines) In the Case of a Clay Mass. In the case of action of a concentrated force, F, the shearing surfaces are logarithmic spirals (8). The value of the parameters, a_1 and a_2, may be determined by letting the curves (8) pass through a point, B, (Fig. 7) with coordinates $\rho = b$; $\Theta = \Theta_o$. This furnishes the conditions (9); and using them, the equations of the logarithmic spirals (8) may be represented under the form (10). Applying transformation formulas (2), the straight stress directions radiating from the origin, A, would be transformed into hyperbolas, or lines of major principal stresses under a load uniformly distributed at a distance, 2b. Consequently the logarithmic spirals (10) would be transformed into slip lines (11) or (12). Equations (12) represent the loci in question in parametric form. In the process of conformal transformation, Point B reaches the position, B'; sections of the logarithmic spirals (1 and 2) outside the circle (5) of a radius, b, with A for center, are transformed into slip lines 1(a) and 2(a), answering Equations (12). To use Equations (12) it is necessary to determine first the constant, Θ_o, from this condition: Sin Θ_o should be equal to the ratio AB':AB = AB':b, where B' is the point at the boundary of the mass from which the slip lines in question have to emanate. In tracing the slip lines it should be remembered that Θ is an angle corresponding to the spiral (10), and not to the spiral (12). In Fig. 7 (heavy line, left part) a slip line, (3a), passing through the edge of the loaded portion is represented. Original spirals are (3) and (4). In this case $\theta_o = \pi/2$ and Equations (12) become (13). The point of intersection with the center line of the loaded portion is located at a depth of 2.30 b. (Equation 14.)

Shear Directions Within a Sandy Mass. In the case of the action of a concentrated force, F, at the boundary of a sand mass, the value of the plane stress, s, at a point (ρ, θ) may be expressed by Formula (15). Comparing it with Michell's radial distribution Formula (16), it may be concluded that equal stress, s, at the point (ρ, θ) of an isotropic elastic mass would be produced, if the loaded portion, M_oN_o, were loaded with a variable load $3/2p.\cos^3\theta$. Thus the problem is reduced to that solved in a paper printed elsewere in this volume (D. P. Krynine. Determination of Stresses Within a Two Dimensional Elastic and Isotropic Earth Mass.). A graphical solution is given in Fig. 8: the direction of the major principal stress, s, is found, and the slip lines forming an angle 90° - α with it (Equation 1) are traced.

Conclusions. 1. There is a "zone of compressions" and a "zone of shears" within a semi-infinite earth mass loaded symmetrically at the boundary. The "limiting curve" separating them in a meridional section is practically a hyperbola.

2. The slip lines form equal angles with the direction of the principal stress; and the value of this angle is a function of the "concentration factor." In the case of clays, slip lines under a loaded portion of a two dimensional mass can be obtained from those under a concentrated load using the method of conformal representation.

3. The problem of finding shear directions within a uniformly loaded two dimensional sand mass can be reduced to that of a non-uniformly loaded elastic and isotropic mass.

No. E-4

DETERMINATION OF STRESSES WITHIN A TWO DIMENSIONAL ELASTIC AND ISOTROPIC EARTH MASS

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A graphical method for determining both normal and tangential stresses within a two dimensional elastic and isotropic earth mass, is given in this paper. The method can be applied in the case of plane stress distribution within any elastic isotropic body.

Notations.

a - arm or distance from the centroid of the area, F, to Point, O.
α - "angle of visibility" of the foundation
A - half, the area of a circle of a radius, R
b - half the width of the loaded foundation
C - center of pressures or the point of application of the resultant of the forces acting at the auxiliary arc, M'N'
CO - direction of the principal stress
f - fraction to show the distance from the center of a loaded foundation in terms of half its width, b
F_o - loading area formed by plotting vertical ordinates, p, at each point of the foundation, M_oN_o
F and F_1 - loading areas formed by plotting vertical ordinates p, and $\frac{p}{\cos\theta}$, respectively, at each point of the horizontal projection, M'N', of the auxiliary arc, MN
M - moment of the area, F, about point, O
dM - elementary moment
M_oN_o - loaded foundation placed at the horizontal boundary of the earth mass
m_on_o - small element of the loaded foundation, M_oN_o
MN - auxiliary arc or the projection of the foundation, M_oN_o, at the circumference of an arbitrary radius, R (center at O)
mn - small element of the auxiliary arc
M'N' - horizontal projection of the auxiliary arc, MN
m'n' - small element of the projection, M'N'
O - point where stresses are to be determined
p - unit load (variable) acting at the foundation, M_oN_o
p_o - average unit load at the foundation M_oN_o or an arbitrary standard unit load
p_z - vertical pressure at a point
dp_z - elementary vertical pressure at a point
ρ - radius vector
R - arbitrary radius of the auxiliary arc, MN
s - stress or sum of principal stresses
ds - elementary stress
τ_{xy} - shear stress acting at the horizontal plane
θ - vertical angular distance
I and II - ordinates