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No. E-10

## STRESS DISTRIBUTION IN ELASTIC SOLIDS

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Note: Bracketed numbers [ ] refer to the publications listed in the bibliography. Numbers in parenthesis ( ) designate formulas. A table at the end of the paper indicates the sources of all formulae.

Introduction. The distribution of stresses in elastic bodies has for many years received the attention of mathematicians and physicists, and their investigations have resulted in the solutions, both explicit and implicit, of a great number of problems in this field. [1] Among these solutions is the case wherein loads are distributed over portions of the plane surface of an elastic solid of semi-infinite extent. [2] The results of this particular analysis have been used in determining the stresses produced in the underground by structures erected on the surface of the earth.

The theoretical contributions in this field are scattered through numerous publications, many of which are not readily accessible to the majority of engineers. For this reason Professor A. Casagrande has suggested to the author that a collection of those results most applicable to foundation problems be prepared for publication. While no claim of completeness is made, it is hoped that this presentation will suffice to fill adequately the needs of those engaged in the analysis of foundations. For those desiring to inquire further into this subject, the necessary references are furnished.

It may not be superfluous to suggest that some knowledge of the theory of elasticity will insure intelligent application of the material presented in this paper. On this score it is well to point out that the assumptions of elastic theory regarding the nature of the stressed medium are never fulfilled in soils, and so the analyses of stresses produced by loads on the surface may in some instances be merely qualitative approximations. The discrepancies between the elastic properties of natural soil beds and the properties of the theoretical medium together with the methods which may be used in taking these differences into account in practice cannot be adequately treated in a brief paper. The reader is referred to [9], [10], [11].

Attention is called to graphical solutions and influence tables published in recent years, which simplify the evaluation of several important formulae. These aids are noted in the bibliography.

The writer is indebted to Professor Casagrande for many suggestions concerning the manner of presenting the material. The aid of Mr. J. D. Watson in preparing the material for publication is gratefully acknowledged.

## Generalities

Term	Definition
Elastic body	A body which recovers its dimensions of the unstressed state when applied stresses are removed. Materials behave elastically only when the stresses do not exceed certain values, the "elastic limits."
Hooke's Law	A material obeys Hooke's Law if it is elastic and if the stress-strain relationship is linear. Such a material is commonly termed "perfectly elastic."
Homogeneous	A mass is homogeneous if the specific physical properties of all portions of the mass (regardless of their size) are the same.
Isotropic	If certain properties of a material are the same in all directions at any point, the material is isotropic with respect to those properties.
Aeolotropic (Anisotropic)	If certain properties of a material vary with direction at any point, the material is aeolotropic.

Hereafter, the term "elastic solid" will be used to designate a homogeneous body which obeys Hooke's Law when subjected to any stresses which may be applied.

Generalization of elastic theory to include cases of heterogeneous and aeolotropic substances leads to exceedingly complex results, and practical solutions of only a few special cases have been obtained. These will be considered independently, after due attention has been given to the stress distribution in an elastically isotropic solid. For this latter material the expressions for the components of stress and strain at any point ( $x, y, z$ ) of a solid are functions of the distribution of the applied load, of  $x, y,$  and  $z,$  and of two elastic constants.

Symbols.

- $x, y, z:$  Rectangular coordinates.
- $r, \theta:$  Polar coordinates in  $xy$  plane.
- $R, \theta, \psi:$  Spherical coordinates.
- $P:$  Total applied force.
- $p:$  Force applied per unit of area.

- $\bar{p}$  : Force applied per unit of length.
- $\sigma$  : Normal stress in the direction denoted by the subscript. (Compressive stresses are positive; tensile, negative).
- $\tau$  : Shearing stress. Of the two subscripts, the first denotes the direction of the normal to the plane on which  $\tau$  acts, and the second denotes the direction in which  $\tau$  acts.
- $\Delta$  : Total displacement of a point in the direction signified by the subscript.
- $\epsilon$  : Strain at any point in the direction denoted by the subscript.

Elastic Constants.

- $m$ : Poisson's number.
- $\nu = \frac{1}{m}$ , Poisson's ratio.
- $E$ : Modulus of elasticity in tension or compression.
- $G = \frac{mE}{2(m+1)}$ : " " " " shear = modulus of rigidity.
- $\lambda, \mu$ : Lamé's constants. These appear frequently in the literature, their values being given by  $\mu = G$ , and  $\lambda = \frac{mE}{(m+1)(m-2)}$

Uniformly loaded rectangular area. The general expressions for the stresses and strains produced by loads distributed over rectangular areas are cumbersome and need not be given here. They may be found in a paper by A. E. H. Love in the Phil. Trans., Roy. Soc. London; Series A, vol. 228, 1929.

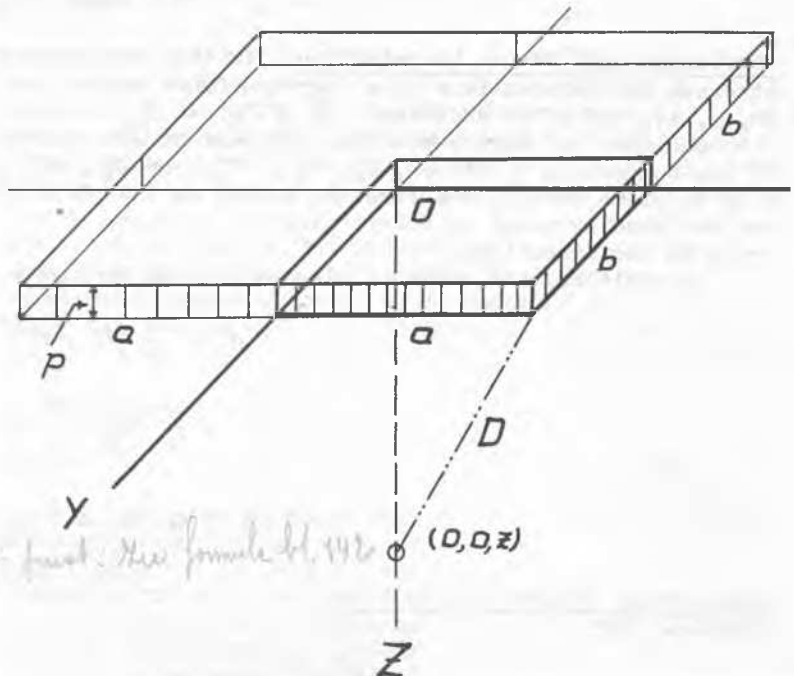
The vertical stress at any point directly beneath the corner of a rectangle whose dimensions are  $a$  and  $b$  is given by

$$\sigma_z = \frac{p}{2\pi} \left[ \frac{abz}{D} \cdot \frac{a^2 + b^2 + 2z^2}{D^2z^2 + a^2b^2} + \sin^{-1} \frac{a \cdot b}{\sqrt{a^2 + z^2} \sqrt{b^2 + z^2}} \right]$$

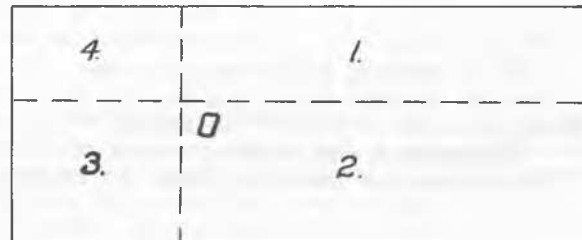
where  $D^2 = a^2 + b^2 + z^2$ .

Hence the vertical stress at any point directly beneath the center of a rectangle whose dimensions are  $2a$  and  $2b$  is

$$(ii) \quad \sigma_z = \frac{2p}{\pi} \left[ \frac{abz}{D} \cdot \frac{a^2 + b^2 + 2z^2}{D^2z^2 + a^2b^2} + \sin^{-1} \frac{ab}{\sqrt{a^2 + z^2} \sqrt{b^2 + z^2}} \right]$$



Influence tables based upon formula (i) have been compiled by Newmark (N. M. Newmark. Simplified Computation of Vertical Pressures in Elastic Foundations. Circular 24, Eng. Exp. Sta., Univ. of Illinois. 1935. A graphical means of determining stresses and settlements beneath rectangular areas was published by Steinbrenner in 1934 in "Die Strasse" [18]. This publication is not generally available in this country.) We may find the stress at any point beneath a rectangular area by dividing the area into smaller rectangles such that the point beneath which we are seeking the stresses is a corner common to all the smaller rectangles. Formula (i) can then be applied to each of these smaller rectangles and the sum of the individual results yields the total stress.



Uniformly Loaded Circular Area. The general expressions for the stresses and strains due to loads distributed over circular areas involve elliptic integrals and will not be given here. They are to be found in the above-mentioned paper by Love. [13]

At any point directly beneath the center of a uniformly loaded area of radius,  $a$ , the formulae

for the stresses reduce to

$$\sigma_z = p \left[ 1 - \frac{z^3}{(a^2 + z^2)^{3/2}} \right]$$

(iii)

$$\sigma_r = \sigma_\theta = \frac{p}{2} \left[ \frac{m+2}{m} - 2 \frac{m+1}{m} \left\{ \frac{z}{\sqrt{a^2 + z^2}} \right\} + \left\{ \frac{z}{\sqrt{a^2 + z^2}} \right\}^3 \right]$$

$z$  denotes as usual the depth of the point beneath the surface.

At the center of the loaded area:

(iv)

$$\Delta z = \frac{2}{E} \cdot \frac{m^2 - 1}{m^2} pa$$

At the edge of the loaded area:

$$\Delta z = \frac{4}{\pi E} \cdot \frac{m^2 - 1}{m^2} pa$$

Two Dimensional Stress Distribution. If the coordinate axes can be chosen in such a manner that the stresses and deformations in a body are functions of only two, ( $x$  and  $z$ ), of the three variables, ( $x, y, z$ ), and if in addition,  $\delta_y = \tau_{xy} = \tau_{yz} = 0$ , then the body is in a state of "plane strain". The conditions of stress are then the same at all points of any line which is parallel to the  $y$ -axis. If the components of stress,  $\sigma_x, \sigma_z, \tau_{xz}$ , or  $\sigma_r, \sigma_\theta, \tau_{r\theta}$ , are known at any point in a body subject to plane strain, the stresses acting on any plane through the point, and parallel to the  $y$ -axis can be found by means of Mohr's circle of stress, or by the corresponding equations of internal equilibrium in two dimensions.

In addition, for cases of plane strain as defined above, we have:

$$\sigma_y = \frac{1}{m} (\sigma_x + \sigma_z)$$

$$\delta_x = \frac{1}{E} \frac{m+1}{m^2} \left[ (m-1) \sigma_x - \sigma_z \right]$$

$$\delta_z = \frac{1}{E} \frac{m+1}{m^2} \left[ (m-1) \sigma_z - \sigma_x \right]$$

Equations of statical equilibrium for interior points of a body subject to plane strain (or plane stress). Fig. 2.

$$\sigma = \sigma_x \cos^2 \epsilon + \sigma_z \sin^2 \epsilon - \tau_{xz} \sin 2\epsilon$$

$$= \frac{1}{2} (\sigma_x + \sigma_z) + \frac{1}{2} (\sigma_x - \sigma_z) \cdot \cos 2\epsilon - \tau_{xz} \sin 2\epsilon$$

$$\tau = \frac{1}{2} (\sigma_x - \sigma_z) \cdot \sin 2\epsilon + \tau_{xz} \cdot \cos 2\epsilon$$

$\epsilon$  is defined as the angle between  $\sigma$  and the  $x$ -axis. By eliminating  $\epsilon$  from the above equations,  $\tau$  can be written as a function of  $\sigma$ . The graph of this function is a circle, commonly called "Mohr's circle of Stress" or simply "circle of stress."

The maximum and minimum values of  $\sigma$  are known as the principal stresses, and the maximum value of  $\tau$  is called the principal shearing stress. These stresses are given by:

$$\sigma_1 = \frac{1}{2} (\sigma_x + \sigma_z) + \sqrt{\frac{1}{4} (\sigma_x + \sigma_z)^2 + \tau_{xz}^2}$$

$$\sigma_2 = \frac{1}{2} (\sigma_x + \sigma_z) - \sqrt{\frac{1}{4} (\sigma_x + \sigma_z)^2 + \tau_{xz}^2}$$

$$\tau_{\max} = \frac{1}{2} (\sigma_1 - \sigma_2) = \sqrt{\frac{1}{4} (\sigma_x + \sigma_z)^2 + \tau_{xz}^2}$$

$\sigma_1$  and  $\sigma_2$  occur on mutually perpendicular planes (the principal planes) such that

$$\epsilon = \epsilon_o = \tan^{-1} \frac{2T_{xy}}{\sigma_x - \sigma_y}$$

and  $T_{\max}$  occurs upon planes inclined at angles of  $45^\circ$  to the principal planes. Furthermore, if  $\epsilon_o$  is the angle which  $\sigma_1$  makes with the x-axis, then

$$\sigma_x = \frac{1}{2} (\sigma_1 + \sigma_2) + \frac{1}{2} (\sigma_1 - \sigma_2) \cos 2\epsilon_o$$

$$\sigma_y = \frac{1}{2} (\sigma_1 + \sigma_2) - \frac{1}{2} (\sigma_1 - \sigma_2) \cos 2\epsilon_o$$

$$T_{xy} = \frac{1}{2} (\sigma_1 - \sigma_2) \sin 2\epsilon_o$$

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#### SOURCES OF FORMULAE

Number of formula	Reference
(1)	[2], [5].
(1a)	[2], [3], [4], [5].
(i), (ii), (iii), (iv)	[13]
(2)	[16], [4], [5].
(2a)	[16].
(3)	[7], [4].
(4)	[7], [8].
(5), (6), (7), (8), (9), (10)	[7]
(11), (12), (13)	[19].

CASE I.

SINGLE CONCENTRATED LOAD,  $P$ , APPLIED AT O

$$\sigma_z = \frac{3P}{2\pi} \cdot \frac{z^3}{R^5} = \frac{3P}{2\pi R^3} \cdot \cos^3 \psi.$$

$$\sigma_x = \frac{P}{2\pi} \left[ 3 \frac{x^2 z}{R^5} - \frac{m-2}{m} \left\{ \frac{x^2 - y^2}{R^2 (R+z)} + \frac{y^2 z}{R^3 r^2} \right\} \right].$$

$$\sigma_y = \frac{P}{2\pi} \left[ 3 \frac{y^2 z}{R^5} - \frac{m-2}{m} \left\{ \frac{y^2 - x^2}{R^2 (R+z)} + \frac{x^2 z}{R^3 r^2} \right\} \right].$$

$$\sigma_r = \frac{P}{2\pi} \left[ 3 \frac{r^2 z}{R^5} - \frac{m-2}{m} \left\{ \frac{R-z}{R r^2} \right\} \right].$$

$$\sigma_\theta = \frac{P}{2\pi} \left[ \frac{m-2}{m} \left\{ \frac{1}{r^2} - \frac{z}{R^3} - \frac{z}{R r^2} \right\} \right].$$

$$\tau_{rz} = \frac{3P}{2\pi} \cdot \frac{r z^2}{R^5} \quad \tau_{zx} = \frac{3P}{2\pi} \cdot \frac{z^2 x}{R^5} \quad \tau_{zy} = \frac{3P}{2\pi} \cdot \frac{z^2 y}{R^5}.$$

$$\tau_{r\theta} = \tau_{z\theta} = 0.$$

$$\Delta z = \frac{P}{2\pi E} \cdot \frac{m+1}{m} \left[ \frac{z^2}{R^3} + \frac{m-1}{m} \cdot \frac{z}{R} \right].$$

$$\Delta x = \frac{P}{2\pi E} \cdot \frac{m+1}{m} \left[ \frac{xz}{R^3} - \frac{m-2}{m} \cdot \frac{x}{R(R+z)} \right].$$

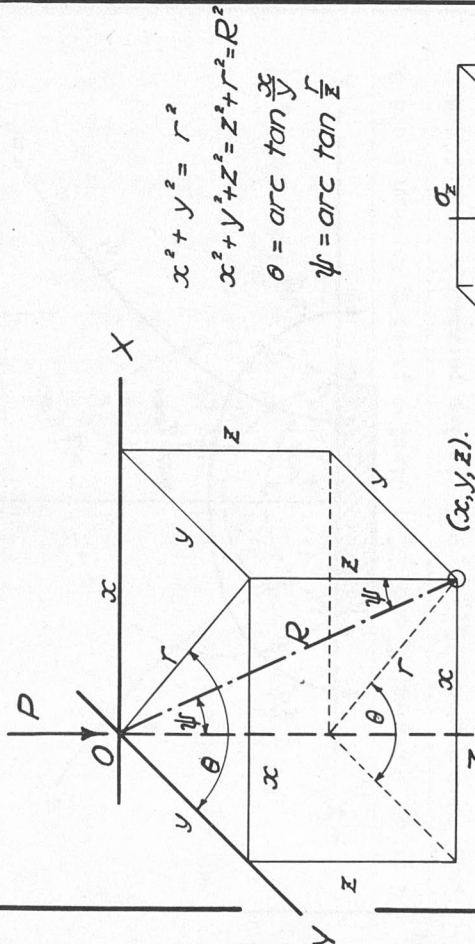
$$\Delta y = \frac{P}{2\pi E} \cdot \frac{m+1}{m} \left[ \frac{yz}{R^3} - \frac{m-2}{m} \cdot \frac{y}{R(R+z)} \right].$$

$$\Delta r = \frac{P}{2\pi E} \cdot \frac{m+1}{m} \left[ \frac{rz}{R^3} - \frac{m-2}{m} \cdot \frac{r}{R(R+z)} \right].$$

$$\delta_z = \frac{\partial \Delta z}{\partial z} = \frac{P}{2\pi E} \cdot \frac{m+1}{m} \left[ 3 \frac{r^2 z}{R^5} - \frac{3m-2}{m} \cdot \frac{z}{R^3} \right].$$

$$\delta_r = \frac{\partial \Delta r}{\partial r} \quad \delta_x = \frac{\partial \Delta x}{\partial x} \quad \delta_y = \frac{\partial \Delta y}{\partial y}.$$

COORDINATE SYSTEM



$$x^2 + y^2 = r^2$$

$$x^2 + y^2 + z^2 = z^2 + r^2 = R^2$$

$$\theta = \arccos \frac{z}{R}$$

$$\psi = \arccos \frac{r}{R}$$

FIG. 1.

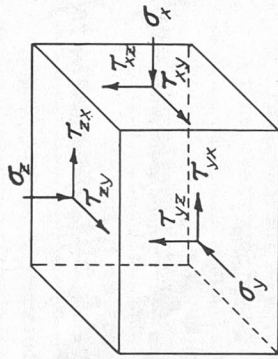


FIG. 1a.

The  $xy$  plane represents the surface of a semi-infinite, elastically isotropic solid. This solid extends indefinitely in the direction of the  $z$ -axis. Unless otherwise noted, formulas will relate to loads distributed upon the surface of this solid. Fig. 1a shows the stresses acting upon the faces of an infinitesimal cube, situated at  $(xyz)$ , whose faces are parallel to the coordinate planes.

CASE III.  
UNIFORMLY LOADED STRIP OF WIDTH,  $2b$ , AND INFINITE LENGTH

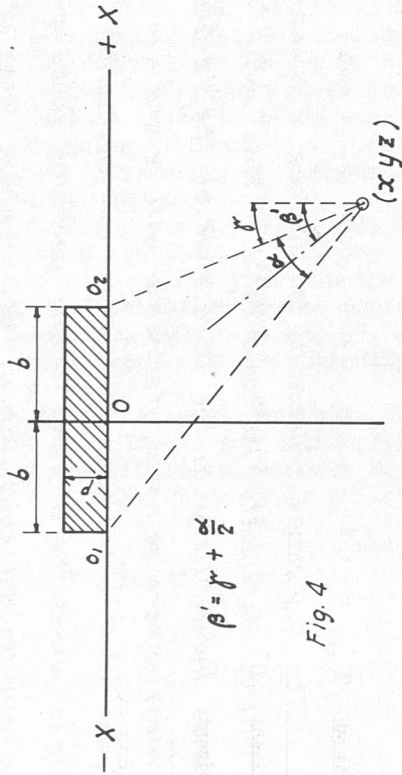


Fig. 4

$$(3) \left\{ \begin{aligned} \sigma_x &= \frac{P}{\pi} [\alpha - \sin \alpha \cdot \cos 2\beta'] \\ \sigma_z &= \frac{P}{\pi} [\alpha + \sin \alpha \cdot \cos 2\beta'] \\ \tau_{xz} &= \frac{P}{\pi} \sin \alpha \cdot \sin 2\beta' \end{aligned} \right. \quad \left\{ \begin{aligned} \sigma_y &= \frac{2P}{\pi m} \cdot \alpha \\ \delta_z &= \frac{P}{\pi E} \frac{m+1}{m^2} [(m-2)\alpha + m \sin \alpha \cdot \cos 2\beta'] \\ \delta_x &= \frac{P}{\pi E} \frac{m+1}{m^2} [(m-2)\alpha - m \sin \alpha \cdot \cos 2\beta'] \end{aligned} \right.$$

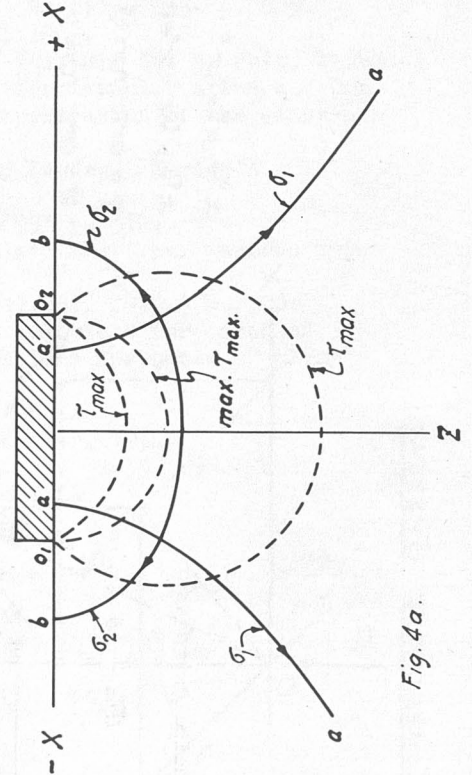


Fig. 4a.

CASE IV.  
UNIFORMLY DISTRIBUTED LOAD COVERING HALF THE INFINITE PLANE.

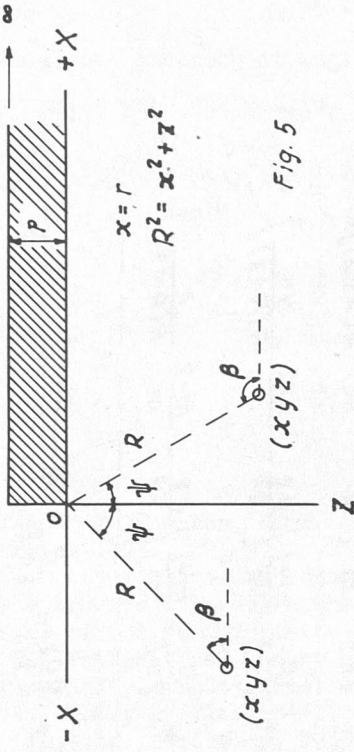


Fig. 5

$$(4) \left\{ \begin{aligned} \sigma_x &= \frac{P}{\pi} \left( \beta - \frac{xz}{R^2} \right) \\ \sigma_z &= \frac{P}{\pi} \left( \beta + \frac{xz}{R^2} \right) \\ \tau_{xz} &= -\frac{P}{\pi} \cdot \sin^2 \beta \end{aligned} \right. \quad \left\{ \begin{aligned} \sigma_y &= \frac{2P}{\pi m} \beta \\ \delta_z &= \frac{P}{\pi E} \frac{m+1}{m} \left[ \frac{m-2}{m} \beta + \frac{xz}{R^2} \right] \\ \delta_x &= \frac{P}{\pi E} \frac{m+1}{m} \left[ \frac{m-2}{m} \beta - \frac{xz}{R^2} \right] \end{aligned} \right.$$

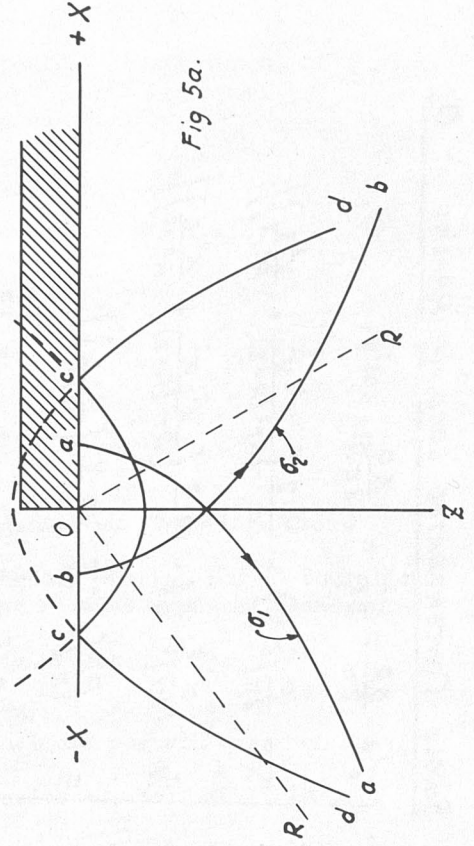


Fig. 5a.

Principal stresses: Cf. Fig. 4a

$$\sigma_1 = \frac{P}{\pi}(\alpha + \sin \alpha) \quad \sigma_2 = \frac{P}{\pi}(\alpha - \sin \alpha)$$

$$T_{max} = \frac{P}{\pi} \sin \alpha$$

Loci of constant $\sigma_1$ and $\sigma_2$	Circles passing through 0 and 0.
Loci of constant $T_{max}$	Circles passing through 0 and 0.
Trajectories of $\sigma_1$	Family of confocal hyperbolas, (aa)*. Foci at 0 and 0.
Trajectories of $\sigma_2$	Family of confocal ellipses, (bb). Foci at 0 and 0.
Trajectories of $T_{max}$	Two orthogonal families: equi-angular spirals intersecting the ellipses under $45^\circ$ .

Maximum  $T_{max} = P/\pi$ , occurs at all points of the semi-circle through 0 and 0.

Maximum  $\sigma_1 = P$ , occurs at points  $(x, 0)$ ,  $-b < x < b$ .

Minimum  $\sigma_2 = 0$ , " "  $(x, 0)$   $-b > x > b$ .

\*These curves bisect the angle,  $\alpha$ , at all points.

Principal stresses:  $\sigma_1 = \frac{P}{\pi}(\beta + \sin \beta)$

Cf. Fig. 5a  $\sigma_2 = \frac{P}{\pi}(\beta - \sin \beta)$

$$T_{max} = \frac{P}{\pi} \sin \beta$$

Loci of constant $\sigma_1$ and $\sigma_2$	Radial lines, OR.
Loci of constant $T_{max}$	Radial lines, OR.
Trajectories of $\sigma_1$	Family of confocal parabolas, (aa)*. Focus at 0. Horizontal axis.
Trajectories of $\sigma_2$	Family of confocal parabolas, (bb). Focus at 0. Horizontal axis.
The two families are orthogonal.	

Trajectories of $T_{max}$	Two orthogonal families of parabolas, (cc) and (cd). Focus at 0. Vertical axis.
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Maximum  $T_{max} = P/\pi$ , occurs at points  $(0, z)$ ,  $z > 0$ .

Maximum  $\sigma_1 = P$ , " " "  $(x, 0)$   $x > 0$ .

Minimum  $\sigma_2 = 0$ , " " "  $(x, 0)$   $x < 0$ .

\*These curves bisect the angle,  $\beta$ , at all points.

PLANE STRAIN

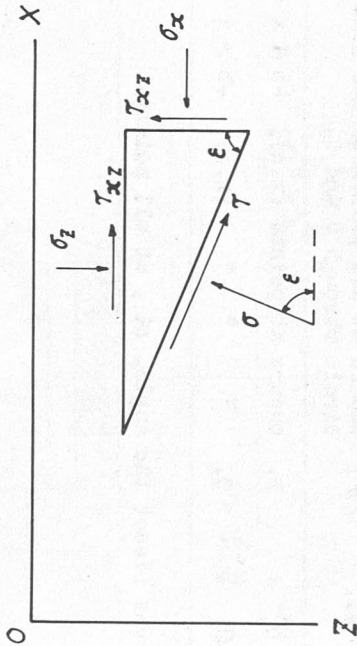


Fig. 2. Equilibrium of a two-dimensional element.

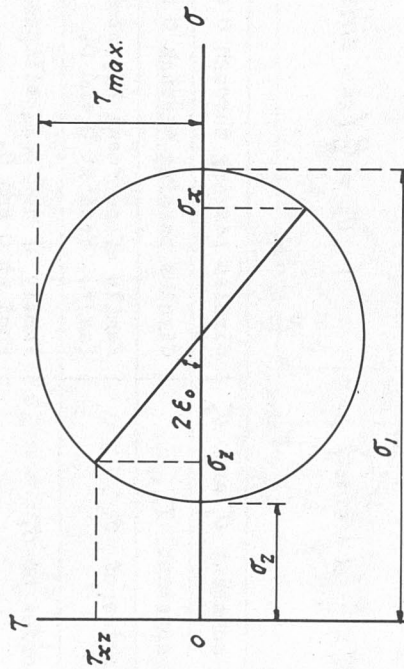


Fig. 2a Circle of Stress.

CASE II.

UNIFORMLY LOADED LINE OF INFINITE LENGTH

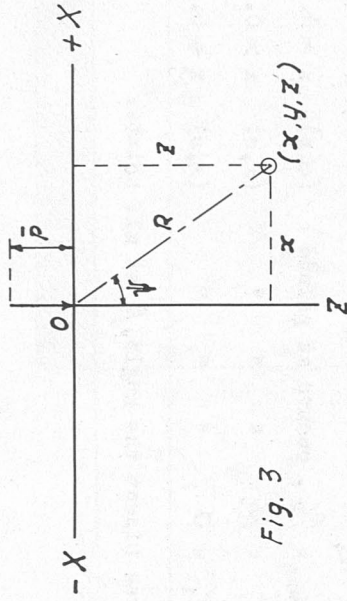


Fig. 3

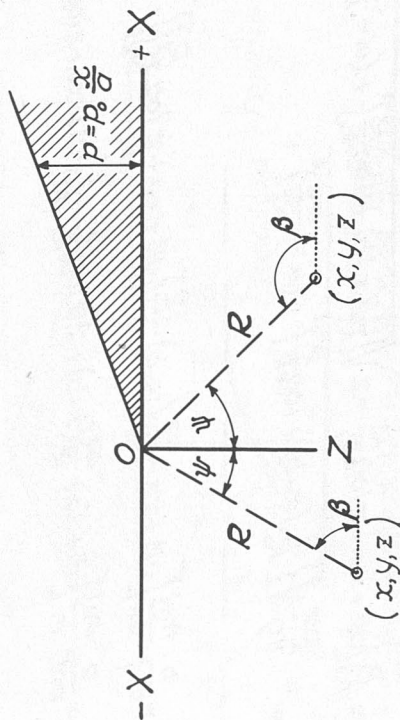
$$(2) \quad \begin{cases} \sigma_x = \frac{2\bar{p}}{\pi} \cdot \frac{x^2 z}{R^4} \\ \sigma_z = \frac{2\bar{p}}{\pi} \cdot \frac{z^3}{R^4} \\ \tau_{xz} = \frac{2\bar{p}}{\pi} \cdot \frac{x z^2}{R^4} \end{cases} \quad \begin{cases} \sigma_y = \frac{2\bar{p}}{\pi m} \cdot \frac{z}{R^2} \\ \tau_{xz} = \frac{2\bar{p}}{\pi} \cdot \frac{x z^2}{R^4} \end{cases}$$

$$(2a) \quad \begin{cases} \sigma_z = \frac{2\bar{p}}{\pi E} \cdot \frac{m+1}{m} \left[ \frac{z^3}{R^4} - \frac{1}{m} \cdot \frac{z}{R^2} \right] \\ \sigma_x = \frac{2\bar{p}}{\pi E} \cdot \frac{m+1}{m} \left[ \frac{x^2 z}{R^4} - \frac{1}{m} \cdot \frac{z}{R^2} \right] \\ \sigma_r = \frac{2\bar{p}}{\pi E} \cdot \frac{m^2-1}{m^2} \cdot \frac{z}{R^2} \\ \sigma_\psi = -\frac{2\bar{p}}{\pi E} \cdot \frac{m+1}{m^2} \cdot \frac{z}{R^2} \end{cases}$$

(2b) { Principal Stresses:  $\sigma_1 = \sigma_r = \frac{2\bar{p}}{\pi} \cdot \frac{z}{R^2}$      $\sigma_2 = \sigma_\psi = 0$ .  
 $\tau_{max} = \frac{\bar{p}}{\pi} \cdot \frac{z}{R^2}$  .

- Loci of constant  $\sigma_1, \sigma_2$  and  $\tau_{max}$     Circles tangent to x-axis at O.
- Trajectories of  $\sigma_1$     Radial lines through O.
- Trajectories of  $\sigma_2$     Family of semi-circles, centers at O.
- Trajectories of  $\tau_{max}$     Two orthogonal families of equiangular spirals intersecting the radial lines under  $45^\circ$ .

CASE V.



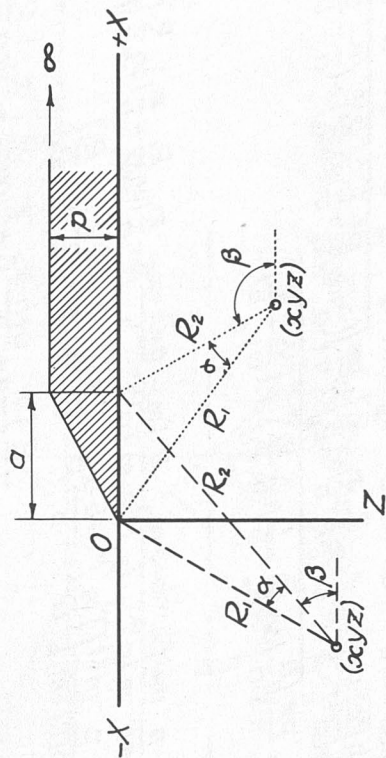
$$(5) \quad \left\{ \begin{aligned} \sigma_x &= \frac{P_0}{\pi a} [x\beta - z - 2z \log_e R] & \sigma_z &= \frac{P_0}{\pi a} [x\beta + z] \\ \sigma_y &= \frac{2P_0}{\pi a m} [x\beta - z \log_e R] & \tau_{xz} &= \frac{P_0}{\pi a} z\beta \end{aligned} \right.$$

Principal Stresses:

$$(5a) \quad \left\{ \begin{aligned} \sigma_1 &= \frac{P_0}{\pi a} (x\beta - z \log_e R) + \frac{P_0 z}{\pi a} \sqrt{(1 + \log_e R)^2 + \beta^2} \\ \sigma_2 &= \frac{P_0}{\pi a} (x\beta - z \log_e R) - \frac{P_0 z}{\pi a} \sqrt{(1 + \log_e R)^2 + \beta^2} \\ \tau_{max} &= \frac{P_0 z}{\pi a} \sqrt{(1 + \log_e R)^2 + \beta^2} \end{aligned} \right.$$

$$(5b) \quad \left\{ \begin{aligned} \delta_z &= \frac{P_0}{\pi a E} \cdot \frac{m+1}{m} [x\beta + z - \frac{2}{m} (x\beta - z \log_e R)] \\ \delta_x &= \frac{P_0}{\pi a E} \cdot \frac{m+1}{m} [x\beta - z - 2z \log_e R - \frac{2}{m} (x\beta - z \log_e R)] \end{aligned} \right.$$

CASE VI.



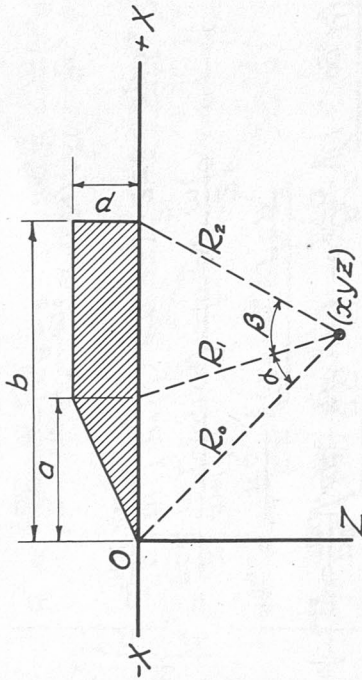
$$(6) \quad \left\{ \begin{aligned} \sigma_x &= \frac{P}{\pi a} [a\beta + \alpha\alpha + 2z \log_e \frac{R_2}{R_1}] & \sigma_z &= \frac{P}{\pi a} [a\beta + \alpha\alpha] \\ \sigma_y &= \frac{2P}{\pi a m} [a\beta + \alpha\alpha + z \log_e \frac{R_2}{R_1}] & \tau_{xz} &= -\frac{P}{\pi a} \cdot z\alpha \end{aligned} \right.$$

Principal Stresses:

$$(6a) \quad \left\{ \begin{aligned} \sigma_1 &= \frac{P}{\pi a} [a\beta + \alpha\alpha + z \log_e \frac{R_2}{R_1}] + z \sqrt{\log_e^2 \frac{R_2}{R_1} + \alpha^2} \\ \sigma_2 &= \frac{P}{\pi a} [a\beta + \alpha\alpha + z \log_e \frac{R_2}{R_1}] - z \sqrt{\log_e^2 \frac{R_2}{R_1} + \alpha^2} \\ \tau_{max} &= \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_2}{R_1} + \alpha^2} \end{aligned} \right.$$

$$(6b) \quad \left\{ \begin{aligned} \delta_z &= \frac{P}{\pi a E} \cdot \frac{m+1}{m^2} [(m-2)(a\beta + \alpha\alpha) - 2z \log_e \frac{R_2}{R_1}] \\ \delta_x &= \frac{P}{\pi a E} \cdot \frac{m+1}{m^2} [(m-2)(a\beta + \alpha\alpha) + 2z \log_e \frac{R_2}{R_1}] \end{aligned} \right.$$

CASE VII.



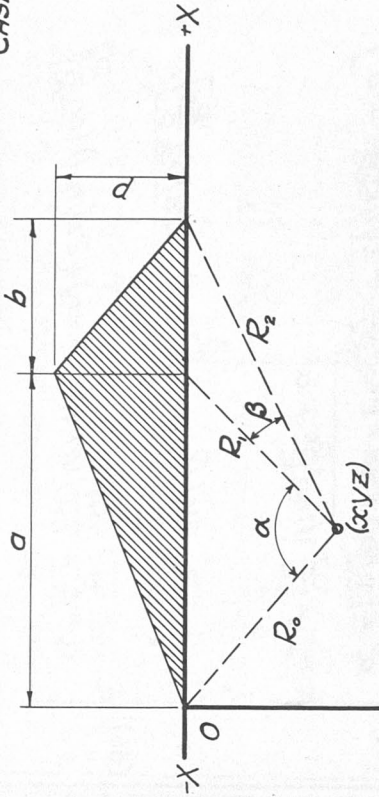
$$\begin{cases}
 \sigma_x = \frac{p}{\pi a} [\alpha\beta + x\alpha + \frac{az}{R_2^2}(x-b) + 2z \log_e \frac{R_1}{R_2}] \\
 \sigma_z = \frac{p}{\pi a} [\alpha\beta + x\alpha - \frac{az}{R_2^2}(x-b)] \\
 \sigma_y = \frac{2p}{\pi m a} [\alpha\beta + x\alpha + z \log_e \frac{R_1}{R_2}] \\
 T_{xz} = \frac{-p}{\pi a} [z\alpha - a \frac{z^2}{R_2^2}]
 \end{cases}
 \tag{7}$$

Principal Stresses:

$$\begin{cases}
 \sigma_1 = \frac{p}{\pi a} [\alpha\beta + x\alpha + z \log_e \frac{R_1}{R_2} + z \sqrt{\{\frac{a}{R_2^2}(x-b) + \log_e \frac{R_1}{R_2}\}^2 + (\alpha - \frac{az}{R_2^2})^2}] \\
 \sigma_2 = \frac{p}{\pi a} [\alpha\beta + x\alpha + z \log_e \frac{R_1}{R_2} - z \sqrt{\{\frac{a}{R_2^2}(x-b) + \log_e \frac{R_1}{R_2}\}^2 + (\alpha - \frac{az}{R_2^2})^2}] \\
 T_{max} = \frac{pz}{\pi a} \sqrt{\{\frac{a}{R_2^2}(x-b) + \log_e \frac{R_1}{R_2}\}^2 + (\alpha - \frac{az}{R_2^2})^2}
 \end{cases}
 \tag{7a}$$

$$\begin{cases}
 \delta_z = \frac{p}{\pi a E} \frac{m+1}{m} [\alpha\beta + x\alpha - \frac{az}{R_2^2}(x-b) - \frac{2}{m} \{\alpha\beta + x\alpha + z \log_e \frac{R_1}{R_2}\}] \\
 \delta_x = \frac{p}{\pi a E} \frac{m+1}{m} [\alpha\beta + x\alpha + \frac{az}{R_2^2}(x-b) + 2z \log_e \frac{R_1}{R_2} - \frac{2}{m} \{\alpha\beta + x\alpha + z \log_e \frac{R_1}{R_2}\}]
 \end{cases}
 \tag{7b}$$

CASE VIII.



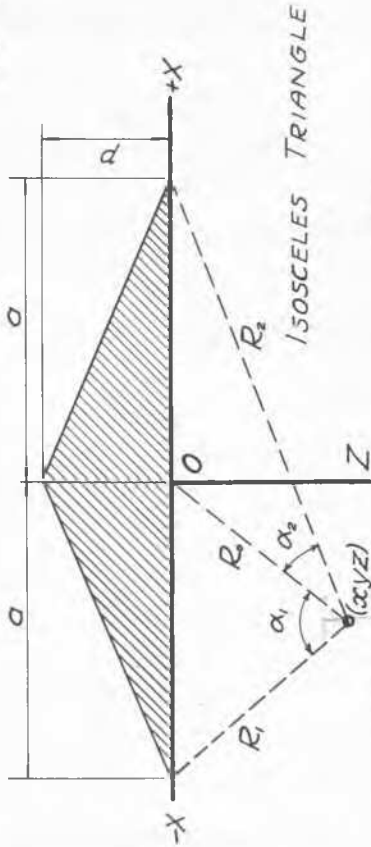
$$\begin{cases}
 \sigma_x = \frac{p}{\pi} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{2z}{a} \log_e \frac{R_1}{R_2} + \frac{2z}{b} \log_e \frac{R_1}{R_2}] \\
 \sigma_z = \frac{p}{\pi} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta] \\
 \sigma_y = \frac{2p}{\pi m} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}] \\
 T_{xz} = \frac{pz}{\pi} [\frac{\alpha}{a} - \beta]
 \end{cases}
 \tag{8}$$

Principal Stresses:

$$\begin{cases}
 \sigma_1 = \frac{p}{\pi} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}] + \frac{pz}{\pi} \sqrt{(\frac{1}{a} \log_e \frac{R_1}{R_2} + \frac{1}{b} \log_e \frac{R_1}{R_2})^2 + (\frac{\alpha}{a} - \frac{\beta}{b})^2} \\
 \sigma_2 = \frac{p}{\pi} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}] - \frac{pz}{\pi} \sqrt{(\frac{1}{a} \log_e \frac{R_1}{R_2} + \frac{1}{b} \log_e \frac{R_1}{R_2})^2 + (\frac{\alpha}{a} - \frac{\beta}{b})^2} \\
 T_{max} = \frac{pz}{\pi} \sqrt{(\frac{1}{a} \log_e \frac{R_1}{R_2} + \frac{1}{b} \log_e \frac{R_1}{R_2})^2 + (\frac{\alpha}{a} - \frac{\beta}{b})^2}
 \end{cases}
 \tag{8a}$$

$$\begin{cases}
 \delta_z = \frac{p}{\pi E} \frac{m+1}{m} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta - \frac{2}{m} \{\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}\}] \\
 \delta_x = \frac{p}{\pi E} \frac{m+1}{m} [\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}] \\
 \quad - \frac{2}{m} \{\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a} \log_e \frac{R_1}{R_2} + \frac{z}{b} \log_e \frac{R_1}{R_2}\}
 \end{cases}
 \tag{8b}$$

CASE IX.



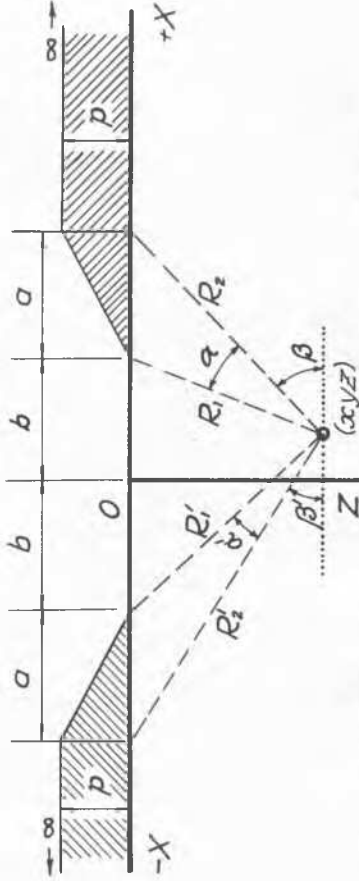
$$\left. \begin{aligned} \sigma_x &= \frac{P}{\pi a} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - 2z \log_e \frac{R_1 R_2}{R_2^2}] \\ \sigma_z &= \frac{P}{\pi a} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2)] \\ \sigma_y &= \frac{2P}{\pi a m} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - z \log_e \frac{R_1 R_2}{R_2^2}] \\ T_{xz} &= -\frac{Pz}{\pi a} [\alpha_1 - \alpha_2] \end{aligned} \right\} (9)$$

Principal Stresses:

$$\left. \begin{aligned} \sigma_1 &= \frac{P}{\pi a} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - z \log_e \frac{R_1 R_2}{R_2^2}] + \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_1 R_2}{R_2^2} + (\alpha_1 - \alpha_2)^2} \\ \sigma_2 &= \frac{P}{\pi a} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - z \log_e \frac{R_1 R_2}{R_2^2}] - \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_1 R_2}{R_2^2} + (\alpha_1 - \alpha_2)^2} \\ T_{max} &= \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_1 R_2}{R_2^2} + (\alpha_1 - \alpha_2)^2} \end{aligned} \right\} (9a)$$

$$\left. \begin{aligned} \delta_z &= \frac{P}{\pi a E} \cdot \frac{m+1}{m} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - \frac{2}{m} \{a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - z \log_e \frac{R_1 R_2}{R_2^2}\}] \\ \delta_x &= \frac{P}{\pi a E} \cdot \frac{m+1}{m} [a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - 2z \log_e \frac{R_1 R_2}{R_2^2} \\ &\quad - \frac{2}{m} \{a(\alpha_1 + \alpha_2) + x(\alpha_1 - \alpha_2) - z \log_e \frac{R_1 R_2}{R_2^2}\}] \end{aligned} \right\} (9b)$$

CASE X.



$$\left. \begin{aligned} \sigma_x &= \frac{P}{\pi a} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + 2z \log_e \frac{R_2 R_2'}{R_1 R_1'}] \\ \sigma_z &= \frac{P}{\pi a} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha')] \\ \sigma_y &= \frac{2P}{\pi a m} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + z \log_e \frac{R_2 R_2'}{R_1 R_1'}] \\ T_{xz} &= -\frac{Pz}{\pi a} [\alpha - \alpha'] \end{aligned} \right\} (10)$$

Principal Stresses:

$$\left. \begin{aligned} \sigma_1 &= \frac{P}{\pi a} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + z \log_e \frac{R_2 R_2'}{R_1 R_1'}] + \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_2 R_2'}{R_1 R_1'} + (\alpha - \alpha')^2} \\ \sigma_2 &= \frac{P}{\pi a} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + z \log_e \frac{R_2 R_2'}{R_1 R_1'}] - \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_2 R_2'}{R_1 R_1'} + (\alpha - \alpha')^2} \\ T_{max} &= \frac{Pz}{\pi a} \sqrt{\log_e^2 \frac{R_2 R_2'}{R_1 R_1'} + (\alpha - \alpha')^2} \end{aligned} \right\} (10a)$$

$$\left. \begin{aligned} \delta_z &= \frac{P}{\pi a E} \cdot \frac{m+1}{m} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') - \frac{2}{m} \{a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') \\ &\quad + z \log_e \frac{R_2 R_2'}{R_1 R_1'}\}] \\ \delta_x &= \frac{P}{\pi a E} \cdot \frac{m+1}{m} [a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + 2z \log_e \frac{R_2 R_2'}{R_1 R_1'} \\ &\quad - \frac{2}{m} \{a(\beta + \beta') - b(a + \alpha') + x(\alpha - \alpha') + z \log_e \frac{R_2 R_2'}{R_1 R_1'}\}] \end{aligned} \right\} (10b)$$

CASE XI

INFINITE LINE LOAD,  $\bar{p}$ , (SEE FIG. 3)

$$R^2 = k^2 x^2 + z^2$$

(12)

$$\sigma_x = \frac{2\bar{p}}{\pi} \cdot \frac{kx^2 z}{R^2 R'^2} \quad \sigma_z = \frac{2\bar{p}}{\pi} \cdot \frac{kz^3}{R^2 R'^2}$$

$$\sigma_y = \frac{4\bar{p}}{\pi m} \cdot k \cdot \frac{x^2 + z^2}{R^2 R'^2} \quad \tau_{xz} = \frac{2\bar{p}}{\pi} \cdot \frac{kxz^2}{R^2 R'^2}$$

$$\sigma_z = \left( \frac{1}{E_V} - \frac{1}{m^2 E_H} \right) \sigma_z - \frac{m+1}{m^2 E_H} \cdot \sigma_x$$

$$\sigma'_x = \left( \frac{1}{E_H} - \frac{1}{m^2 E_V} \right) \sigma_x - \frac{m+1}{m^2 E_V} \cdot \sigma_z$$

INFINITE STRIP LOAD

$R^2 = k^2 x^2 + z^2$

(13)

$$\sigma_x = \frac{\bar{p}}{\pi(1-k)} [(\phi_1 - \phi_2) - k(\phi_1' - \phi_2')] \quad \sigma_z = \frac{\bar{p}}{\pi(1-k)} [(\phi_1' - \phi_2') - (\phi_1 - \phi_2)]$$

$$\sigma_y = \frac{2\bar{p}}{\pi m} (\phi_1' - \phi_2') \quad \tau_{xz} = \frac{2\bar{p}}{\pi(1-k)} \left[ \log_e \frac{R_2}{R_1} - \log_e \frac{R_2'}{R_1'} \right]$$

Discontinuities and Aeolotropy

Expressions for stresses in discontinuous or in aeolotropic (anisotropic) solids are for the most part involved and comparatively difficult to apply. Several examples may be found in the following: [6], [8], [14], [15], [16], [19]. The analysis for such cases has not been advanced as broadly as that for the homogeneous, isotropic body. The practical need of such solutions is great because of the inevitable stratification and heterogeneity of natural soil beds, but at present the available solutions are not adequate for securing more than a qualitative idea of the influence of these complicating factors on the stress distribution.

Formulae (11), (12), and (13) give the stresses in a semi-infinite elastically aeolotropic, homogeneous solid possessing the following properties:

- Modulus of elasticity in directions normal to the z-axis,  $E_H$
- " " " " " parallel to the z-axis,  $E_V$

Poisson's number, m, the same in all directions.

$$k^2 = \frac{E_V}{E_H}$$