Properties of sand mixtures with identical entropy coordinates

Propriétés des mélanges de sable avec des coordonnées d'entropie identiques

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ABSTRACT: The space of the grain size distribution curves with N fractions is isomorphic to the space of the unit-sided simplex with dimension N-1. The interpolation over the simplex with dimension N-1is problematic since the number of the interpolation points may increase exponentially with N. Instead of this we interpolate a function on the entropy diagram and extend the result on each grading curve. The method is well-defined since we use such interpolation points (where measurement is made) that has not only unique inverse image but also are either mean or extreme points for the simplex. The method is applied to the dry density and the parameters of two SWCC models. Five data sets of artificial mixtures of natural sand grains were used for density and SWCC measurements. This paper presents the first results related to the functions for getting the parameters of the soil water characteristic curve (SWCC) equations of Van Genuchten and Fredlund-Xing and the dry density. Example for the practical application is given, the accuracy is analyzed preliminarily.

RÉSUMÉ: L'espace des courbes de distribution granulométrique avec N fractions est isomorphe simplex côté unité de dim N -1. L'interpolation est problématique puisque le nombre des points d'interpolation peut augmenter exponentiellement avec N. Au lieu de cela, nous interpolons une fonction sur le diagramme d'entropie et étendons sur chaque courbe de classement. La méthode est bien définie puisque nous utilisons de tels points d'interpolation (oá est effectuée la mesure) qui ont non seulement une image inverse unique mais également des points moyens ou extrámes pour une valeur d'entropie de base relative donnée (c'est-á-dire pour un mélange avec la máme structure). La méthode est appliquée á la densité sáche et aux paramátres de SWCC. Cet article présente les résultats relatifs aux fonctions d'obtention des paramátres des équations Van Genuchten et Fredlund-Xing et de la densité sáche. Exemple d'application pratique, la précision du diagramme dérivé est analysée préliminairement

KEYWORDS: sand, grading curve, grading entropy, dry density, soil water characteristic curve, interpolation in large dimensions.

1 INTRODUCTION

The interpolation of a function over the space of the grading curves needs a number of interpolation points exponentially increasing in terms of the number of the fractions N which is "too many".

To overcome this difficulty, an approximate interpolation method is suggested, based on the grading entropy concept, needing $c\,N$ to $c\,N^2$ data.

The paper presents the first application of the method to the minimum dry density moreover, to the van Genuchten and Fredlund-Xing soil water characteristic curve equations for sands through some examples.

2 GRADING ENTROPY

2.1 *Fractions*

For characterizing a grading curve, an abstract fraction system is defined. The diameter range of these are doubled with the serial number j (j=1, 2. , Table 1):

$$2^j d_0 \geq d > 2^{j-1} d_0, \tag{1}$$

where d_0 is the smallest diameter which may be equal to the height of the SiO_4 tetrahedron (d_0=2^{-22} mm).

The log diameter limits (by corrected with a constant) and the fraction serial numbers are basically the same as the grading entropy of the fraction (see Table 1).

Table 1. Definition of fractions, serial numbers and eigen-entropies

j	1	...	23	24
Limits	d_0 to 2 d_0	...	2^{22} d_0 to 2^{23} d_0	2^{23} d_0 to 2^{24} d_0
S_{0j} [-]	1	...	23	24

2.2 *Space of grading curves*

The relative frequencies of fractions x_j of grading curvesfulfil:

$$\sum_{i=1}^{N} x_i = 1, \quad x_i \geq 0, \quad N \geq 1. \tag{2}$$

where the integer variable N is the number of the fractions between the finest and coarsest non-zero fractions.

The relative frequencies x_i can be identified with the barycentre coordinates of the points of an N-1 dimensional, closed simplex (which is the N-1 dimensional analogy of the triangle or tetrahedron, the 2 and 3 dimensional instances) and, the space of the grading curves with N fractions can be identified with the N-1 dimensional, closed simplex. The vertices of the simplex represent the fractions, and the 2 dimensional edges are related to the two-mixtures etc. The sub-simplexes of a simplex surface are partly continuous, partly gap-graded. The continuous sub-simplexes have a lattice structure, as is indicated in Fig. 1(a).

2.3 Entropy parameters of grading curves

The grading entropy S can be separated into the sum of two parts (Lőrincz et al. 2005):

$$S = S_0 + \Delta S \quad (3)$$

where S_0 is base entropy, ΔS is entropy increment. The base entropy is the weighed mean of the 2 base log of the diameters

$$S_0 = \sum x_i S_{0i} = \sum x_i i \quad (4)$$

where S_{0k} is the k-th fraction entropy (Table 1). Its normalised form, the relative base entropy A is independent of d_0:

$$A = \frac{(S_o - S_{o\min})}{S_{o\max} - S_{o\min}} \quad (5)$$

where S_{0max} and S_{0min} are the entropies of the largest and the smallest fractions, respectively. The entropy increment ΔS:

$$\Delta S = -\frac{1}{\ln 2} \sum_{x_i \neq 0} x_i \ln x_i. \quad (6)$$

The normalized entropy increment B:

$$B = \frac{\Delta S}{\ln N} \quad (7)$$

2.3 Entropy diagram, inverse image

Any grading curve can be represented as a point in terms of the entropy coordinates (Figs 1(b), 2(b)). Four maps can be defined between the N-1 dimensional simplex (fixed N) and the two dimensional real Euclidean space of the entropy coordinates, the non-normalized, normalized, and two partly normalized.

The inverse image of the maximum line of the diagram (critical values of the map) is the optimal line, visualised in Figs 2(c) and 3(a). The optimal point or grading curve has finite fractal distribution with the following relative frequencies:

$$x_1 = \frac{1}{\sum_{j=1}^{N} a^{j-1}} = \frac{1-a}{1-a^N}, \quad x_j = x_1\, a^{j-1} \quad ()$$

where parameter a is the root of the following equation (Imre et al, 2009):

$$y = \sum_{j=1}^{N} a^{j-1}[j-1-A(N-1)] = 0. \quad (9)$$

The inverse image of a regular value of the diagram is an N-3 dimensional sphere (Figs 2(d), 3(b)). The related grading curves for a given A have the same sub-graph area, the deviation from the optimal grading curve depends on B (Fig 3).

The optimal grading curve is the mean grading curve for the grading curves for given A value, and within this, separately for those that have a fixed B independently of the dimension.

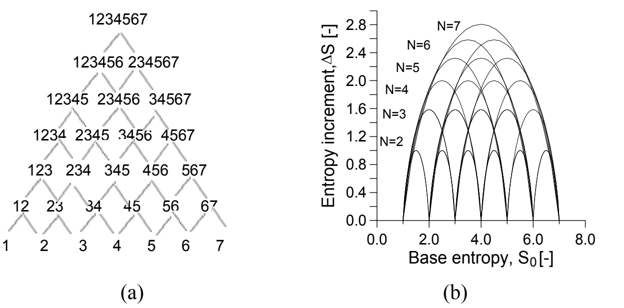


Figure 1. (a) The lattice of the continuous sub- simplexes of the simplex (the one unit integers are the serial number of fractions, the k-unit integers denote k-component mixtures: k≤N). (b) Entropy diagrams, the image of the optimal lines of the continuous sub- simplexes.

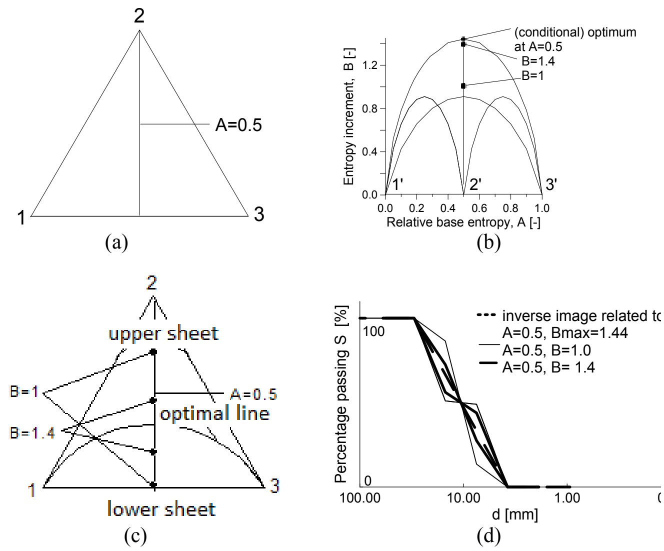


Figure 2. Entropy map, N=3. (a) The simplex and its A=0.5 hyper-plane section. (b) The entropy diagram with 3 points on coordinate line A=0.5. **(c)** and (c) The inverse image in the simplex and in the space of the possible grading curves (optimal line, N-3=0 dimensional circles).

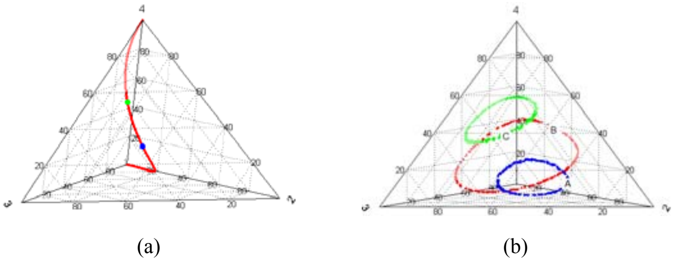


Figure 3. Inverse image for an N=4 simplex. (a) The optimal line. (b) inverse image (N-3=1 dimensional sphere) of three regular entropy diagram points (curve C: A=0.66 B=1.2 (set 9), curve B: A=0.5 B=1.2 (set 21), curve A: A=0.3 B=1.2).

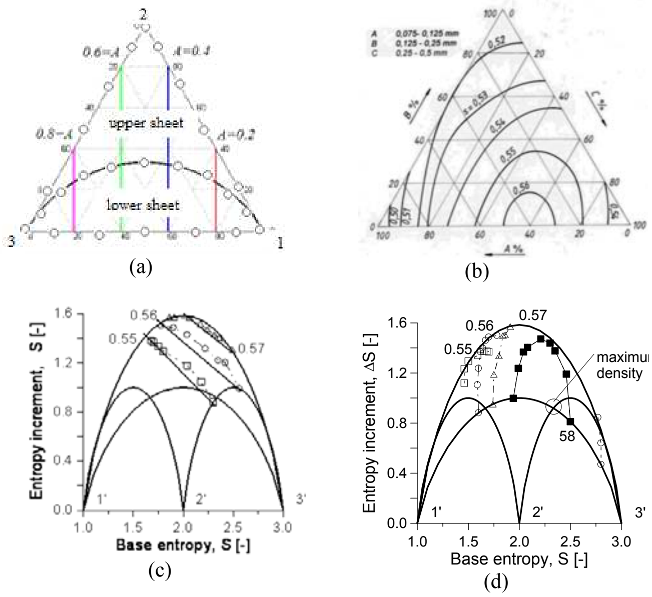
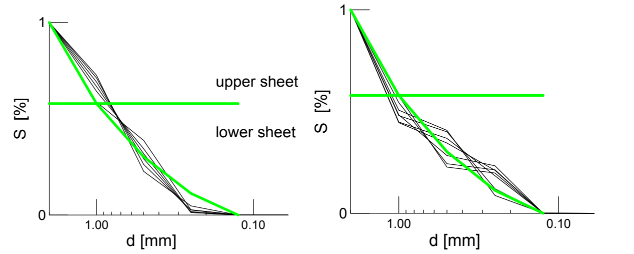


Figure 4. The interpolation of s_{min}. (a) The sections (sheets) and the interpolation points. (b) The precise interpolation of the iso-lines. (c) and (d): The s_{min} lines interpolated on the diagram using the forwarded interpolation points.



(a) (b)

Figure 5. Halving the inverse image grading curve set 9 into (a) upper sheet and (b) lower sheet, separated by x_{4opt} of the optimal grading curve.

2.3 Interpolation

Two 2-dimensional surfaces ('sections') bounded by mean and some extreme lines of the simplex are used for the approximate interpolation, independently of the simplex dimension.

In the approximate interpolation the iso-lines are interpolated using forwarded interpolation points. These are selected from the optimal line and from the continuous 2-edges (edge series 1 to 2, 2 to 3,…, N-1 to N) in case of the upper sheet. Alternatively, for the lower sheet interpolation, these are selected on the optimal line and on the edge 1 to N.

If one sheet is used then the interpolated value is constant in the inverse image, which is an N-3 dimensional sphere. If two sheets are used then the inverse image is halved into upper and lower parts along the optimal line with the conditions: $x_N <= x_{Nopt}$ and $x_N > x_{Nopt}$, resp., in relation to the two functions.

In case of N=3, the interpolation is visualized in Fig. 4. The upper simplex sheet is bounded by the optimal line and the continuous 2-edges 1-2, 2-3, the lower sheet is bounded by the optimal line and the gap-graded edge 1-3.

3. TRANSFER FUNCTION RESULTS

3.1 *SWCC model parameter transfer function determination*

The measurements were performed on some sand fractions and artificial, optimal mixtures composed of sand grains with A=2/3 in the optimal points indicated in Figures 5 to 7, with the methods of the Soil Science Institute (SSI) (see Várallyay, 1973; Rajkai 1993, Imre et al 2003 and 2013). In the lower suction range of u_a-$u_w \leq$ 50 kPa sand boxes are used and, the suction is applied by water pressure decrease. In the higher suction range (u_a-u_w > 50 kPa) pressure membrane extractor is used, the suction is applied by the axis translation technique.

The following functions were fitted to the measured data.

The Fredlund – Xing (1994) water retention curve equation:

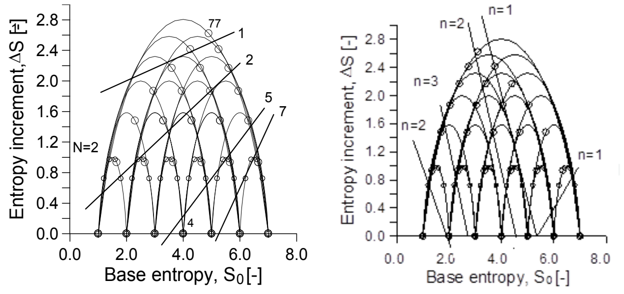
$$w = w_r + \frac{w_s - w_r}{\left\{ \ln\left[e + \left(\frac{u_a - u_w}{a} \right)^n \right] \right\}^m} \tag{10}$$

The van Genuchten (1980) water retention curve equation:

$$w = w_r + \frac{w_s - w_r}{\left(1 + \left[a\left(u_a - u_w\right)\right]^n\right)^m} \tag{11}$$

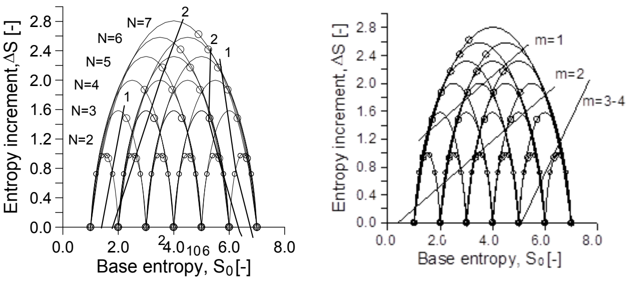
where w is water content (w_r residual, w_s saturated), $u_a - u_w$ is suction, and model parameters are a, n, m.

The parameters were identified for measured water retention curves. On the basis of these results, a preliminary transfer function was determined on the continuous sheet for the non-linearly dependent parameters a, n and m. These are given by different iso-lines as shown in Figures 6 to 8.



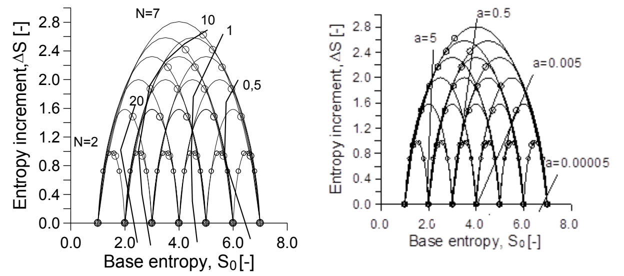
(a) (b)

Figure 6. Transfer function for parameter n, continuous sheet. (a) Fredlund-Xing model (b) van Genuchten model (d_{min}= 0.03 mm).



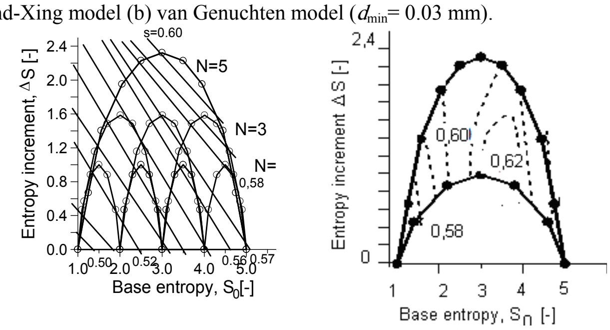
(a) (b)

Figure 7. Transfer function for parameter m, continuous sheet. (a) Fredlund-Xing model (b) van Genuchten model (d_{min}= 0.03 mm).



(a) (b)

Figure 8. Transfer function for parameter a, continuous sheet. (a) Fredlund-Xing model (b) van Genuchten model (d_{min}= 0.03 mm).



(a) (b)

Figure 9. The s_{min} transfer function, from optimal mixtures constructed from data of Lőrincz, (a) upper sheet, (b) lower sheet (d_{min}= 0.125 mm).

3.2 *Dry density data and transfer functions*

The two functions were inferred using the data of Lőrincz, the results are shown in Figure 8 (Imre et al, 2013, Fig 8).

Table 2. Entropy coordinates of two diagram points denoted by 9 and 21

	S_0 [-]	ΔS [-]	d_{min}	A [-]	B [-]
21	4.5	1.66	0.25	0.5	1.2
9	4	1.66	0.125	2.3	1.2

Table 3. Results of the statistical evaluation for set 9 (Bochum data).

9	s_{max} [-]	s_{min} [-]	s ratio [-]
Mean	0,679	0,556	0,819
Upper part mean	0,671	0,552	0,823
Lower part mean	0,682	0,560	0,821

Table 4. Results of the statistical evaluation for set 21 (Bochum data).

21	s_{max} [-]	s_{min} [-]	s ratio [-]
Mean	0,673	0,578	0,859
Upper part mean	0,667	0,564	0,846
Lower part mean	0,681	0,593	0,873

Table 5. Diagram values for s_{min} [-] interpolated at points 9 and 21.

Lőrincz data	Lőrincz data	Bochum data
$x_4 > x_{4opt}$ (lower)	$x_4 < x_{4opt}$ (upper)	x $x_4 < x_{4opt}$ (upper)
0.61	0.598	0.565
0.61	0.595	0.56

Table 6. Grading curve data and entropy coordinate computation.

d [mm]	S_{oi}	Finer [%]	x_i	$x_i \ln(x_i)$	$x_i S_{oi}$
2,0	7	100			
1,0	6	80,00	0,17352	-0,30	1,04110
0,5	5	46,60	0,36044	-0,37	1,80219
0,25	4	27,71	0,18894	-0,31	0,75574
0,125	3	13,81	0,13896	-0,27	0,41689
0,062	2	3,88	0,09931	-0,23	0,19863
0.03	1	0,00	0,03883	-0,13	0,03883

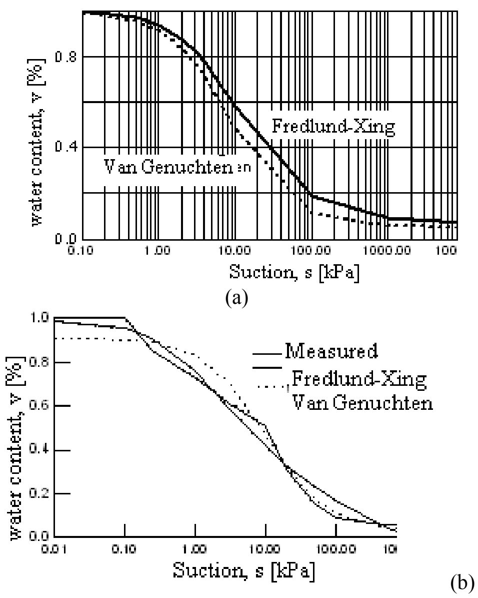


Figure 10. Example of SWCC estimation for the grading curve shown in Table 6, using Figs 6 to 9. (a) The computed functions. (b) As a comparison, a measured SWCC for *N*=7, A=0.66 (Imre et al 2009).

4. EXAMPLES

4 .1 *Inverse image - accuracy issues*

To test the variability of the dry density related to two entropy diagram points, two sets of grading curves were generated as the inverse image of two entropy diagram points (sets 9, 21 see Table 2, Fig. 5). The sand mixtures were tested for dry density according to the DIN 18126, results are shown in Tables 3 to 4.

The value interpolated (Table 5) was greater than the measured values due to some measuring problems (Imre et al, 2013). However, the upper mean was smaller than the lower mean, as qualitatively expected (Fig. 9, Tables 3 and 4).

4.2 *Interpolation example*

Using a known grading curve (Table 6), assuming minimum dry density state, SWCC was estimated as follows. The entropy parameters (S_0=4.25; ΔS =2.33; A=0.65, B=1.3) were computed. Then the parameters of SWCC were selected as follows: a=9, *n=1, m=2* for the Fredlund-Xing model and *a*=0.1, *n=1.1, m=0.95* for the van Genuchten model, respectively. The minimum solid volume ratio was estimated as 0.59 on the basis of the transfer function. The saturated gravimetric water content was 0.41 from this. The residual water content was estimated as 0.05 to 0.08 on the basis of previous experience. The computed SWCC is comparable with the measurements (Fig 10).

5. DISCUSSION, CONCLUSION

Comparing Figs 4 and 9, it can be seen that the two kinds of minimum dry density transfer functions are qualitatively the same independently of the fraction number *N*. The SWCC transfer functions have near-parallel lines in the upper sheet, the lower sheet transfer function contains reversal in the iso-lines.

The true value measured at the inverse image of the two entropy diagram points reflected, if the simplex was split into two parts, qualitatively good values (smaller mean at the upper, greater mean at the lower part, see Tables 3 to 4).

The SWCC of a well-graded sand mixture was successfully estimated using the transfer functions of two water retention curve equations (upper sheet, Figs 6 to 8). The results were similar to some measured data. The saturated gravimetric water content was estimated from the minimum dry density chart.

Further research is suggested on the accuracy of the interpolation, the grading entropy-based dry density model and, the dry density measurement methods. The transfer functions of the same SWCC equations (on the gap-graded sheet) are also suggested to be elaborated in further research.

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