Weighting functions for the stiffness of circular surface footings on multi-layered non-homogeneous elastic half-spaces under general loading

Fonctions de pondération pour la rigidité des semelles de surface circulaires sur des demi-espaces élastiques multicouches non homogènes sous charge générale

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ABSTRACT: The classic contact problem of a rigid, rough circular surface footing bearing on a multi-layered non-homogeneous elastic half space was studied, leading to a simplified methodology for footing stiffness under six degrees-of-freedom loading that makes use of approximate weighting functions. The proposed methodology considers the effects of non-homogeneity by computing weighted average elastic parameters. These are then used as inputs to the stiffness solutions for a homogeneous elastic half-space. A key benefit of the proposed model is that it can handle both continuously varying and multi-layered non-homogeneity. The accuracy of the new method was evaluated by comparing its predictions against three-dimensional finite element analyses. The results of the study show that the proposed methodology can provide excellent predictions of the footing stiffness under general loading for most cases, except when there are stiff layers overlying softer layers.

RÉSUMÉ: Le problème de contact classique d'un rigide, rugueux pied de surface circulaire portant sur un demi élastique multicouche non homogène espace a été étudié et une méthodologie simplifiée a été proposée afin de déterminer la rigidité de la semelle sous généraux 6 degrés de liberté de chargement avec le utilisation des fonctions approximatives de pondération. La méthodologie proposée prend en compte les effets de la non-homogénéité en calculant les paramètres élastiques moyens pondérés et en utilisant ces paramètres comme entrées dans les solutions de rigidité pour un demi-espace élastique homogène. Le principal avantage du modèle proposé est qu'il peut gérer à la fois faire varier en continu et à couches multiples non-homogénéité. La précision du modèle proposé a été évaluée en comparant ses prédictions contre les analyses par éléments finis en trois dimensions rigoureuse. Les résultats de l'étude montrent que le modèle proposé est capable de fournir des prévisions raisonnables de la rigidité de la semelle sous chargement général pour la plupart des cas, sauf quand il y a une couche rigide recouvrant des couches plus douces.

KEYWORDS: rigid, rough, circular, surface footing, stiffness, multi-layered, non-homogeneous, elastic, weighting functions, torsion

1 INTRODUCTION

The determination of the stiffness of a surface footing bearing on a non-homogeneous elastic half-space is a classic problem in shallow foundation research. Often, the non-homogeneity in the half-space is idealised as continuously varying with depth. Furthermore, most published studies focus only on the vertical loading case. Little attention has been paid to foundation stiffness under general six degrees-of-freedom loading. This paper addresses this issue by proposing a new model that provides footing stiffness solutions for general loading on a generic non-homogeneous elastic half-space, where the non-homogeneity is idealised as multi-layered or continuously varying.

1.1 *Related Work*

In previous published work, the non-homogeneity in the half-space is usually limited to the variations of shear modulus with depth. These variations are either modelled as continuously varying or multi-layered piecewise constant. For the former, the pioneering work of Gibson (1967) was concerned with a vertically loaded surface footing on an incompressible elastic half-space, with a linearly-increasing shear modulus with depth. More recently, Doherty et al. (2005) used the scaled boundary finite element method to obtain stiffness coefficients for a rigid, rough circular surface footing on a non-homogeneous elastic half-space for six degrees-of-freedom loading. This work assumed a power-law variation in the shear modulus, which can be considered as a generalisation of the Gibson soil.

For multi-layered elastic half-spaces, the existing literature focuses principally on behaviour under vertical loads. For example, Mayne and Poulos (1999) uses an exponential strain influence distribution to compute vertical settlements. There is a dearth of stiffness solutions for surface footings bearing on multi-layered elastic half-space under general loading. This paper addresses this issue by generalising the solutions of Doherty et al. (2005) to account for layering in the half-space.

2 MODEL

The hypothesis explored in the current work is that multi-layered problems can be modelled by identifying equivalent 'homogeneous' elastic parameters (e.g. shear modulus) for use with standard homogeneous half-space solutions. These equivalent parameters represent a weighted average of the non-homogeneous stiffness profile. Previous methods based on the strain influence framework (e.g. Mayne and Poulos, 1999) also fall within this family of weighted elastic parameters methods; in this case, the weighting function is determined by the stress distribution. The main goal of the current paper is to identify appropriate weighting functions for the footing stiffness problem under general loading. The proposed approach relies on certain assumptions. Firstly, it is assumed that an equivalent shear modulus, G_{eq} can be obtained by weighting the shear modulus profile as shown in Eq. 1.

$$\frac{1}{G_{eq}} = \frac{\int_0^\infty w(z) \cdot \frac{1}{G(z)}\, dz}{\int_0^\infty w(z)\, dz} \tag{1}$$

where $w(z)$ is the weighting function and $G(z)$ is the non-homogenous shear modulus profile with depth. For practical purpose, the integration in Eq. 1 can be truncated at a finite depth below which $w(z)$ is negligible. This weighting process assumes that each non-homogeneous layer acts like a set of springs in series. This contrasts with other researchers who assume that each layer acts like a set of springs in parallel (e.g. Huajian et al., 1992). Secondly, it is assumed that the appropriate weighting functions are largely invariant to the nature of the stiffness non-homogeneity.

This paper does not attempt a rigorous analysis of the above assumptions. Instead, it assesses the robustness of the proposed approach by comparing the *a posteriori* predictions resulting from these *a priori* assumptions. Moreover, it is assumed that the *w(z)* is exponential in nature; not dissimilar to the function proposed by Mayne and Poulos (1999) for vertical loading. In the current approach, the two-parameter Weibull distribution function (Eq. 2) is adopted as the weighting function,

$$w\left(\frac{z}{D},\lambda,\beta\right)=\frac{\beta}{\lambda}\left(\frac{z}{D}\cdot\frac{1}{\lambda}\right)^{\beta-1}\exp\left(-\left(\frac{z}{D}\cdot\frac{1}{\lambda}\right)^{\beta}\right) \quad (2)$$

where β and λ are the shape and scale parameters of the Weibull distribution. This weighting function has the advantages of flexibility and parsimony. Despite only being a two-parameter model, the Weibull function is able to reproduce a wide range of radially decreasing distribution shapes. To complete the model, β and λ are identified for each loading type (vertical, horizontal, moment, torsional). For this purpose, the solutions of Doherty et al. (2005) are used as the training data.

Finally, to calculate the footing stiffness, the equivalent shear modulus computed using Eqs. 1 and 2 is used as input to the stiffness solutions for a homogeneous elastic half-space listed in Table 1. These solutions correspond to the vertical stiffness, lateral stiffness, rotational stiffness, torsional stiffness and a cross-coupling stiffness between horizontal and moment loading. They were obtained from Doherty et al. (2005) for the case where the shear modulus is constant with depth.

Table 1. Stiffness solutions for a rigid, rough circular surface footing on a homogeneous elastic half-space for ν = 0.2 and 0.499. (Nomenclature: G_{eq} = equivalent shear modulus, D = footing diameter, V, H, M, T = vertical, horizontal, moment, torsional load respectively. w, u, θ, ω = vertical, horizontal displacement, rotation, twist respectively)

	Equation	$\nu = 0.2$	$\nu = 0.499$
Vertical	$V/(wG_{eq}D)$ =	2.65	4.04
Lateral	$H/(uG_{eq}D)$ =	2.286	2.705
Rotational	$M/(\theta G_{eq}D^3)$ =	0.465	0.676
Torsional	$T/(\omega G_{eq}D^3)$ =	0.675	0.675
Coupling	$H/(\theta G_{eq}D^2)$ = or $M/(uG_{eq}D^2)$ =	-0.142	-0.0106

2.1 *Calibration*

The Doherty et al. (2005) solutions provide dimensionless stiffness coefficients corresponding to compressible and incompressible material (ν = 0.2, 0.499) and five shear modulus profiles that vary according to Eq. 3.

$$G(z) = G_R\left(\frac{2z}{D}\right)^{\alpha} \quad (3)$$

where z = depth below ground surface, D = footing diameter, G_R = reference shear modulus and α = dimensionless factor that controls the rate of increase of the shear modulus with depth. The five shear modulus profiles correspond to α values of 0, 0.2, 0.4, 0.6, 0.8 and 1. The unknown parameters (β and λ) in the Weibull weighting function are the values that best reproduced the outputs (stiffness coefficients) for each set of inputs (ν and α). Using ordinary least squares (OLS) regression, the best-fit values of the parameters for each loading type were obtained and are listed in Table 2.

Table 2. Best-fit values for the parameters of the Weibull (Eq. 2) weighting function for each loading type

	$\nu = 0.2$		$\nu = 0.499$	
	β	λ	β	λ
Vertical	1.23	0.58	1.66	1.02
Lateral	1.33	0.23	1.48	0.20
Rotational	1.46	0.16	2.00	0.33
Torsional	1.60	0.081	1.60	0.081
Coupling	1.11	0.11	1.04	11.3

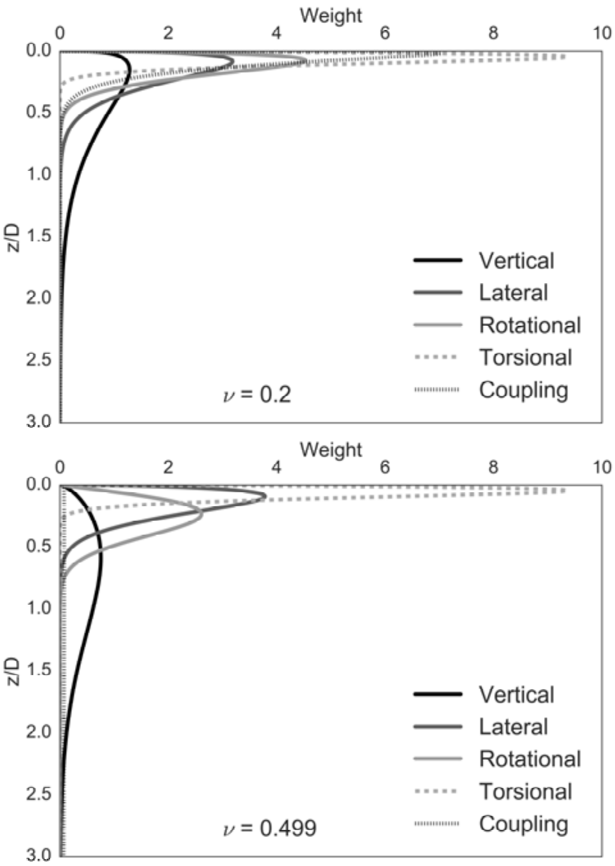


Figure 1. Weighting functions for each loading type

Fig 1 shows the weighting function obtained for each loading type. It is observed that the zone of influence is relatively shallow for non-vertical loading, with the exception of the coupling stiffness in incompressible material where the weight is low throughout the depths. While the zone of influence for vertical loading can extend to a depth of three diameters, the zone of influence for non-vertical loading stops at a depth of one diameter. For clarity, the zone of influence is defined as the depth below which the weighting is negligible. Interestingly, the shape of the weighting function obtained for vertical loading is similar to the strain influence distribution of Mayne and Poulos (1999); even though they were obtained using different approaches. To test if the assumed weighting approach is robust, the predictions using the calibrated Weibull weighting functions were compared against the original Doherty et al. (2005) solutions.

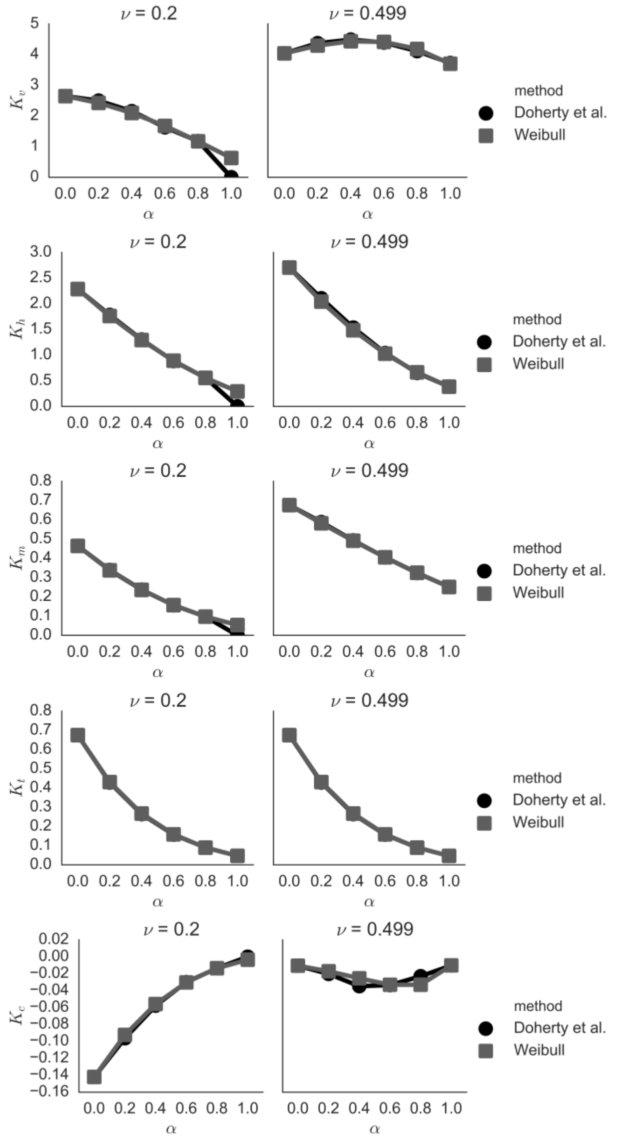


Figure 2. Comparison of dimensionless stiffness predictions by the proposed model using the calibrated Weibull weighting functions against the solutions of Doherty et al. (2005).

Fig. 2 clearly shows that the proposed weighting approach is capable of excellent reproduction of the original solutions; except for K_v, K_h and K_m when ν = 0.2 and α = 1. In this figure, K_v, K_h, K_m, K_t and K_c are dimensionless stiffness coefficients computed as per Table 3, where G_R is the reference shear modulus in Eq. 3. Thus far, the work demonstrates that the proposed weighting function (Eq. 2) coupled with the calibrated parameters (Table 2) and homogeneous stiffness solutions (Table 1) can provide accurate stiffness solutions for the case where the shear modulus is increasing continuously. However, this is no different to the solutions of Doherty et al. (2005). The achievement of the current work is the generalisation to a shear modulus that varies discontinuously in the elastic half-space in the form of piecewise constant steps.

Table 3. Dimensionless stiffness coefficients definitions

Vertical	$K_v = V/(wG_RD)$
Lateral	$K_h = H/(wG_RD)$
Rotational	$K_m = M/(\theta G_RD^3)$
Torsional	$K_t = T/(\omega G_RD^3)$
Coupling	$K_c = H/(\theta G_RD^2)$ or $M/(uG_RD^2)$

3 MULTI-LAYERED CASE STUDIES

A numerical study consisting of a 1 m diameter (*D*) rigid, rough circular surface footing, bearing on six multi-layered Young's modulus, *E*, profiles was carried out. These profiles are similar to those explored by Poulos (1978). The shear modulus is related to the Young's modulus via Eq. 4.

$$G = \frac{E}{2(1+\nu)} \quad (4)$$

Details of the Young's modulus profiles are presented in both Table 4 and Fig. 3. Additionally, Fig. 3 also shows continuous approximations of the multi-layered profiles based on Eq. 3, which were obtained using OLS regression to a depth of 2D below the surface (as Fig. 1 shows that there is negligible influence below that depth). The process of approximating a multi-layered profile with a continuously varying one is common in practice and highlights one of the key difficulties in using existing elastic solutions for real engineering problems; which is how to best approximate a multi-layered profile with a continuously varying one. Different practitioners end up with different approximations, depending on their judgment of how deep the zone of influence should be.

Table 4. Young's modulus variation with depth for the six cases tested

	Young's Modulus, E (MPa)		
Case	z < 0.5D	0.5D ≤ z ≤ D	z > 0.5D
1	100	200	400
2	100	400	200
3	200	100	400
4	200	400	100
5	400	100	200
6	400	200	100

For the numerical study, the predictions of the proposed weighting approach, the Doherty et al. (2005) solutions (based on the continuous modulus approximation) and the Mayne and Poulos (1999) solutions (for K_v only) are benchmarked against the results from rigorous, three-dimensional finite element (3D FE) analyses using the 3D FE software, Abaqus v6.13.

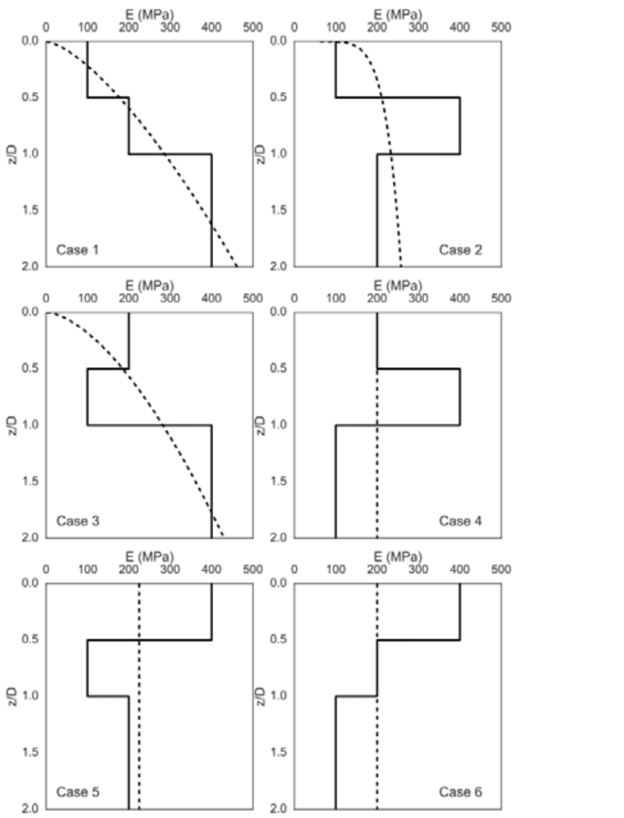
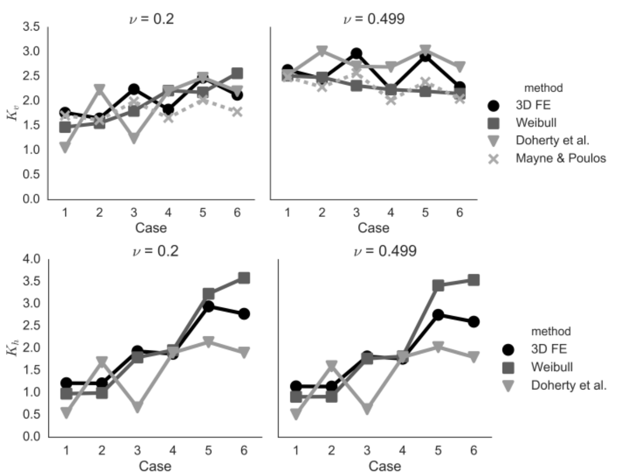


Figure 3. Young's Modulus profiles for each case study (solid lines) and their respective best-fit continuous approximations (dashed lines)

Fig. 4 shows the comparison of the dimensionless stiffness predictions from all the different models. They are computed as per Table 3, where G_R = 100/(2(1 + ν)) MPa. For K_v, the proposed Weibull function has similar predictive capabilities to the Mayne and Poulos (1999) solution. This is not surprising given the similarity in shape between the calibrated weighting function and the strain influence distribution. For K_h and K_m, it was observed that the proposed model overpredicts the footing stiffness for Cases 5 and 6 where there is a stiff layer overlying softer layers. For K_t, the predictions of the proposed model are excellent in all cases. Interestingly, the proposed model provides excellent predictions for K_c for the case where ν = 0.2 while the predictions for the incompressible case were less satisfactory. It is apparent from Fig. 4 that the proposed model overwhelmingly outperforms the predictions of Doherty et al. (2005) for the multi-layered cases.



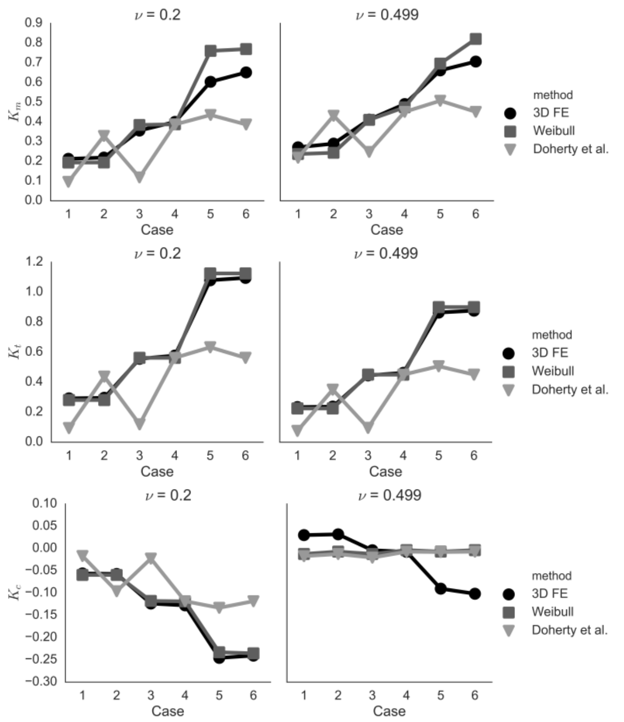


Figure 4. Comparison of dimensionless stiffness predictions by the proposed model and the solutions of Doherty et al. (2005) against the benchmark 3D FE results.

4 CONCLUSION

A new approach for computing footing stiffness values is presented. The model works on the premise that effects of non-homogeneity can be incorporated within the stiffness solutions for a homogeneous elastic half-space by adopting equivalent weighted average homogeneous elastic parameters. The proposed model provides excellent footing stiffness predictions for most cases. The signficance of the work is demonstrated in the application of the model to multi-layered cases. The overestimation for cases with stiff layers over softer layers indicates that further work is still needed, though the work represents a significant improvement on current practice.

5 ACKNOWLEDGEMENTS

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