An analysis procedure to study dynamic fully coupled hydro-mechanical behavior of unsaturated soils: theory and validation

Une procédure d'analyse pour étudier le comportement hydro-mécanique dynamique totalement couplé des sols non saturés: théorie et validation

Bo Zhang, Kanthasamy Muraleetharan
School of Civil Engineering and Environmental Science, The University of Oklahoma, USA, muralee@ou.edu

ABSTRACT: A procedure to analyze dynamic behavior of unsaturated soils is necessary to study problems such as soil liquefaction, soil-structure interaction, and deformations of unsaturated soil embankments during earthquakes. Accurate descriptions of hydro-mechanical coupling effects and hydraulic hysteresis are essential to study the dynamic behavior of unsaturated soils. In order to study the fluid flow and elastoplastic behavior in unsaturated soils subjected to dynamic loading, a fully coupled dynamic analysis procedure is developed. The fully coupled equations governing the dynamic behavior of unsaturated soils are presented with solid skeleton displacement, pore water pressure, and pore air pressure as primary unknown variables. The stress-strain behavior of soils is described by a fully coupled hydro-mechanical elastoplastic constitutive model for unsaturated sands and silts. This model includes the hysteretic soil water characteristic curves (SWCCs) based on the bounding surface plasticity concept and coupling mechanisms between SWCCs and the stress-strain behavior of the solid skeleton. As a preliminary validation of the analysis procedure, the numerical simulation results are compared with a drying-wetting experiment.

RÉSUMÉ : Une procédure d'analyse du comportement dynamique des sols non saturés est nécessaire pour étudier des problèmes tels que la liquéfaction des sols, l'interaction sol-structure et les déformations des remblais de sols non saturés durant les tremblements de terre. Des descriptions précises des effets de couplage hydro-mécaniques et de l'hystérésis hydraulique sont essentielles pour étudier le comportement dynamique des sols non saturés. Afin d'étudier le mouvement de fluide et le comportement élastoplastique dans des sols non saturés soumis à une charge dynamique, une procédure d'analyse dynamique entièrement couplée est développée. Les principales variables inconnues des équations totalement couplées régissant le comportement dynamique des sols non saturés sont présentées comme étant le déplacement du squelette solide, la pression de l'eau interstitielle et la pressio d'air interstitielle. Le comportement contrainte-déformation des sols est décrit par un modèle constitutif élastoplastique hydro-mécanique entièrement couplé pour les sables non saturés et les limons. Ce modèle comprend les courbes caractéristiques de l'eau du sol hystérétique (SWCC) basées sur le concept de plasticité de la surface limite et les mécanismes de couplage entre les SWCC et le comportement contrainte-déformation du squelette solide. Comme validation préliminaire de la procédure d'analyse, les résultats de la simulation numérique sont comparés à une expérience de séchage-mouillage.

KEYWORDS: unsaturated soils, dynamics, fully coupled, finite element method

1 INTRODUCTION

Unsaturated soils are three-phase porous media consisting of a solid skeleton (s), pore water (w), and pore air (a). Dynamic behavior of unsaturated soils has received increasing attention within the geotechnical earthquake engineering community in the last few decades for a wide variety of problems, such as soil-structure interaction (Ravichandran *et al.* 2015), liquefaction evaluation (Okamura and Soga 2006, Unno *et al.* 2008, Zhang *et al.* 2016), wave propagation (Chen *et al.* 2011, Steeb *et al.* 2014), and earthquake resistance of structures (Khoei and Mohammadnejad 2011, Matsumaru and Uzuoka 2016). Analysis procedures to study dynamic behavior of unsaturated soils have been developed beginning in 1990's (Zienkiewicz *et al.* 1990, Schrefler and Xiaoyong 1993). Although progress has been made in improving these analysis procedures, many simplifying assumptions such as constant air pressure (usually atmospheric pressure) (Sheng *et al.* 2008, Callari and Abati 2009, Pedroso 2015) and quasi-static condition (Schrefler and Xiaoyong 1993, Laloui *et al.* 2003, Oettl *et al.* 2004) are still made. Furthermore, important effects such as elastoplasticity, hydro-mechanical coupling, and hydraulic hysteresis are rarely taken into account in the constitutive models used in these analysis procedures (Ravichandran and Muraleetharan 2009, Uzuoka and Borja 2012).

In order to study the fluid flow and elastoplastic behavior in unsaturated soils subjected to dynamic loading, a fully coupled dynamic analysis procedure is developed in this paper. The fully coupled equations governing the dynamic behavior of unsaturated soils are presented with solid skeleton displacement, pore water pressure, and pore air pressure as primary unknown variables. The basic equations include mass balance equations and linear momentum balance equations for the mixture, pore air, and pore water. The stress-strain behavior of the soils is described by a fully coupled hydro-mechanical elastoplastic constitutive model for unsaturated sands and silts (Liu and Muraleetharan 2012a, b). This model includes a hysteretic model for SWCCs based on the bounding surface plasticity concept and coupling mechanisms between SWCCs and the stress-strain behavior of the solid skeleton. As a preliminary validation of the analysis procedure, the numerical simulation results are compared with a drying-wetting experiment.

2 GOVERNING EQUATIONS

2.1 *Basic definitions*

The pore space of the solid skeleton is completely filled with the fluid phases β ($\beta = w, a$). Considering a representative elementary volume (REV) in unsaturated soils, each phase has a mass δm_α and a volume δV_α , $\alpha = s, w, a$, the overall volume δV of REV is obtained from the sum of the volume of $\alpha-$phase, $\delta V = \sum \delta V_\alpha$ and the corresponding mass δm is given by $\delta m = \sum \delta m_\alpha$. The volume fraction n^α of $\alpha-$phase is defined as $n^\alpha = \delta V_\alpha / \delta V$; hence the volume fractions of the mixture satisfy $n^s + n^w + n^a = 1$ and the degree of saturation $S_{r\beta}$ of fluid $\beta-$phase is expressed as

$S_{r\beta} = n^\beta / \sum n^\beta$. The relationships between volume fractions and the parameters typically used in soil mechanics are given by: $n = n^w + n^a$, $n^s = 1 - n$, $n_w = nS_{rw}$, $n^a = n(1 - S_{rw})$, $S_{rw} + S_{ra} = 1$, where n is porosity of the soil and n_w is the volumetric water content. The intrinsic mass density of $\alpha -$ phase is denoted as $\rho_a = \delta m_a / \delta V_a$, whereas the macroscopic partial density of $\alpha -$ phase is written as $\rho^a = \delta m_a / \delta V$. Thus the macroscopic partial density of $\alpha -$ phase can be expressed by

$$\rho^s = (1-n)\rho_s\,,\quad \rho^w = nS_{rw}\rho_w\,,\quad \rho^a = n(1-S_{rw})\rho_a \tag{1}$$

The overall density ρ of the three-phase mixture is given by

$$\rho = (1-n)\rho_s + nS_{rw}\rho_w + n(1-S_{rw})\rho_a \tag{2}$$

The velocities for flowing fluid constituents (water and air) relative to the moving solid $\tilde{\mathbf{v}}^{\beta s}$ are defined as $\tilde{\mathbf{v}}^{\beta s} = n^\beta \left(\mathbf{v}^\beta - \mathbf{v}^s\right)$, where $\mathbf{v}^\beta$ and $\mathbf{v}^s$ are the intrinsic or absolute velocities for $\beta -$ phase and solid skeleton.

2.2 *Stress and strain variables*

The effective stress principle plays a significant role in constitutive modeling of saturated and unsaturated soils. For unsaturated soils, the effective stress (in standard soil mechanics sign convention) can be described as follows (Bishop 1959):

$$\boldsymbol{\sigma}' = \left(\boldsymbol{\sigma} - p_a \mathbf{I}\right) + \chi s_c \mathbf{I} \tag{3}$$

where, $\boldsymbol{\sigma}'$ is the effective stress tensor, $\boldsymbol{\sigma}$ is the total stress tensor, p_a is pore air pressure, $\mathbf{I}$ is the second-order unit tensor, χ is Bishop's parameter, $s_c = p_a - p_w$ is matric suction and p_w is pore water pressure. Following Wei and Muraleetharan (2002a, 2002b), χ is defined as the volumetric water content n_w. In this paper, $\boldsymbol{\sigma}'$ is referred to as the intergranular stress tensor, which is conjugated with the strain tensor $\boldsymbol{\varepsilon}$ of the solid skeleton.

2.3 *Fully coupled governing equations*

The mathematical description of the behavior of unsaturated soils presented in this research is similar to Muraleetharan and Wei (1999) and Wei (2001) and is derived by extending Biot's formulation (Biot 1941). Without loss of generality, the following assumptions are made: a) all the physical processes are assumed to be under isothermal condition; b) the solid grains are incompressible; c) exchange among the solid, water and air phases are neglected.

The balance of mass for $\alpha -$ phase is written as

$$D^\alpha \left(n^\alpha \rho_\alpha\right) / Dt + n^\alpha \rho_\alpha \,\mathrm{div}\, \mathbf{v}^\alpha = 0 \tag{4}$$

where $\mathbf{v}^\alpha$ is the velocity of $\alpha -$ phase, D^α / Dt denotes the material time derivative with respect to the motion of α - phase. The linear momentum balance equation for the mixture is obtained by adding the linear momentum balance equations for three phases and disregarding the fluid relative accelerations with respect to the solid phase and convective terms

$$\rho\ddot{\mathbf{u}} - \mathrm{div}\,\boldsymbol{\sigma} - \rho\mathbf{b} = 0 \tag{5}$$

where $\mathbf{u}$ is the displacement vector for the solid skeleton, $\boldsymbol{\sigma}$ is the total Cauchy stress tensor, $\mathbf{b}$ is the body force tensor. The linear momentum balance equations for pore water and pore air phases result in the generalized Darcy's law

$$\tilde{\mathbf{v}}^{\beta s} = -\frac{\mathbf{k}k_{r\beta}}{\mu_\beta}\left(\nabla p_\beta - \rho_\beta \mathbf{b} + \rho_\beta \ddot{\mathbf{u}}\right) \tag{6}$$

where $\mathbf{k}$ is the intrinsic permeability tensor, $k_{r\beta}$ is the relative permeability of $\beta -$ phase , which is dependent on the degree of relative saturation ($0 \le k_{r\beta} \le 1$), μ_β is the dynamic viscosity. In this paper, the following equations based on the van Genuchten-Mualem (VGM) model (van Genuchten 1980) are utilized

$$k_{rw} = S_e^{0.5}\left[1 - \left(1 - S_e^{1/m}\right)^m\right]^2 \tag{7}$$

$$k_{ra} = \left(1 - S_e\right)^{0.5}\left(1 - S_e^{1/m}\right)^{2m} \tag{8}$$

where m is the material parameter, S_e is the effective degree of water saturation which is given by

$$S_e = \left(S_{rw} - S_{rw}^r\right) / \left(S_{rw}^s - S_{rw}^r\right) \tag{9}$$

where S_{rw}^r and S_{rw}^s are the residual (minimum) and saturated (maximum) degree of water saturation.

Following Wei and Muraleetharan (2002a, 2002b), the volumetric water content is assumed to be a function of matric suction and volumetric strain of the solid skeleton, which is given as

$$n_w = f\left(\varepsilon_v, s_c\right) \tag{10}$$

Applying the chain rule, the material time derivative of volumetric water content $\dot{n}_w$ can be given in terms of matric suction s_c and volumetric strain of the solid skeleton, ε_v

$$\dot{n}_w = \frac{\partial n_w}{\partial \varepsilon_v}\dot{\varepsilon}_v + \frac{\partial n_w}{\partial s_c}\dot{s}_c \tag{11}$$

For the finite element discretization of the fully coupled governing equations for unsaturated porous media with solid skeleton displacement $\mathbf{u}$, pore water pressure p_w , and pore air pressure p_a as the primary unknown variables, the shape functions $\mathbf{N_u}$, $\mathbf{N_{p_w}}$ and $\mathbf{N_{p_a}}$ are used for the approximation of the displacement field, water pressure and air pressure fields, respectively

$$\mathbf{u} = \mathbf{N_u U}\,,\quad p_w = \mathbf{N_{p_w} P_w}\,,\quad p_a = \mathbf{N_{p_a} P_a} \tag{12}$$

where $\mathbf{U}$, $\mathbf{P_w}$ and $\mathbf{P_a}$ are the global vectors of the nodal values for the displacement of the solid skeleton and for pore water pressure and pore air pressure. After discretizing the weak form of the non-linear partial differential equations using a standard finite element procedure, the following fully coupled algebraic equations can be derived

$$\begin{aligned}
&\mathbf{M_s\ddot{U}} + \int_\Omega \mathbf{B}^\mathrm{T}\boldsymbol{\sigma}' d\Omega + \mathbf{H_{sw}P_w} + \mathbf{H_{sa}P_a} = \mathbf{F_s}\\
&\mathbf{M_w\ddot{U}} + \mathbf{C_{ws}\dot{U}} + \mathbf{C_{ww}\dot{P}_w} + \mathbf{C_{wa}\dot{P}_a} + \mathbf{H_{ww}P_w} = \mathbf{F_w}\\
&\mathbf{M_a\ddot{U}} + \mathbf{C_{as}\dot{U}} + \mathbf{C_{aw}\dot{P}_w} + \mathbf{C_{aa}\dot{P}_a} + \mathbf{H_{aa}P_a} = \mathbf{F_a}
\end{aligned} \tag{13}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{B}$ is the strain-displacement matrix, $\mathbf{H}$ is the flow matrix, $\mathbf{C}$ is the damping matrix, and $\mathbf{F}$ is the load vector.

3 ELASTOPLASTIC CONSTITUTIVE MODEL

A fully coupled hydro-mechanical elastoplastic constitutive model for unsaturated sands and silts (CM4USS) in the general stress space (Liu and Muraleetharan 2012a, b) is utilized to describe the stress-strain behavior of the soils. The formulation of this model is based on the critical state soil mechanics framework and bounding surface plasticity. The bounding surface plasticity concept was incorporated into the hydro-mechanical behaviors in order to simulate the unsaturated sand behavior under both monotonic and cyclic loading conditions. The most distinct features of this constitutive model are the hysteretic model for SWCCs and coupling mechanisms between SWCCs and the stress-strain behavior of the solid skeleton. The capability of CM4USS for predicting the cyclic behavior of unsaturated sands has been demonstrated (Zhang *et al.* 2016).

CM4USS is developed in an incremental form. An additive decomposition of the total strain and the volumetric water content is assumed:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p , \quad \dot{n}_w = \dot{n}_w^e + \dot{n}_w^p \tag{14}$$

where $\boldsymbol{\varepsilon}$ is the total strain of the solid skeleton, n_w is the volumetric water content, superscripts e and p indicate the elastic and plastic components, and a dot over the variable represents the time derivative.

3.1 *Elastic behavior*

The elastic behavior is described as

$$\dot{\varepsilon}_v^e = \frac{\dot{I}}{K} , \quad \dot{\boldsymbol{\varepsilon}}_q^e = \frac{\dot{\mathbf{s}}}{2G} , \quad \dot{n}_w^e = \frac{\dot{s}_c}{\Gamma^e} \tag{15}$$

where ε_v^e and $\boldsymbol{\varepsilon}_q^e$ are the elastic components of volumetric strain and deviatoric strain of the solid skeleton, which are given by $\varepsilon_v = tr(\varepsilon_{ij})$ and $\boldsymbol{\varepsilon}_q = dev(\varepsilon_{ij})$, n_w^e is the elastic component of volumetric water content, I is the hydrostatic intergranular stress given by $I = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3$, $\mathbf{s}$ is the deviatoric stress tensor defined as $\mathbf{s} = dev(\sigma'_{ij})$, Γ^e is capillary elastic modulus, K and G are bulk modulus and shear modulus, which are the functions of I.

3.2 *Yield, critical, bounding and dilatancy surfaces*

The CM4USS formulation is developed by introducing the surfaces in the general stress space, the bounding surface, critical surface, and dilatancy surface, which correspond to the bounding (or peak), dilatancy (or phase transfer), and critical stress ratios, respectively. The yield surface has a closed cap-like shape at the tip, which is given in the multiaxial stress space as follows

$$f(\sigma', s_c; \boldsymbol{\alpha}, m) = \sqrt{(\mathbf{s} - I\boldsymbol{\alpha}) : (\mathbf{s} - I\boldsymbol{\alpha})} - \sqrt{2/3}\, mI\sqrt{1 - (I/I_0)^{\beta}} = 0 \tag{16}$$

where $\boldsymbol{\alpha}$ is kinematic hardening parameter, which is the center of the yield surface, m is the isotropic hardening parameter that defines the size of the yield surface, I_0 is the initial hydrostatic intergranular stress, β is a model parameter.

For unsaturated sands and silts, the bounding curves accounting for the hysteresis in SWCCs during wetting- drying cycles are described by

$$\text{Wetting:} \quad n_w = \frac{n_{ws} + n_{wr}(s_{c0w}/b_2)^{d_2}}{1 + (s_{c0w}/b_2)^{d_2}} \tag{17}$$

$$\text{Drying:} \quad n_w = \frac{n_{ws} + n_{wr}(s_{c0d}/b_3)^{d_3}}{1 + (s_{c0d}/b_3)^{d_3}} \tag{18}$$

where: n_{ws} is the volumetric water content at zero suction, n_{wr} is the residual volumetric water content at very high suction, b_2, d_2, b_3 and d_3 are material parameters , s_{c0w} and s_{c0d} are the suction on the wetting and drying bounds, respectively.

3.3 *Plastic behavior*

The plastic strain tensor of the solid skeleton and plastic volumetric water content are given as follows:

$$\dot{\boldsymbol{\varepsilon}}^p = \langle \Lambda \rangle \left(\mathbf{n} + \frac{1}{3} D\mathbf{I} \right) \tag{19}$$

$$\dot{n}_w^p = \dot{s}_c / \Gamma^p \tag{20}$$

where Λ is loading index given by considering the coupling effects between the intergranular stress and matric suction, $\mathbf{n}$ is the unit deviatoric stress ratio tensor, D is the dilatancy coefficient which is related to the distance from the dilatancy surface, Γ^p is the capillary plastic modulus. In this model, the evolution equations for the kinematic hardening parameter $\boldsymbol{\alpha}$ and isotropic hardening parameter are given as

$$\dot{\boldsymbol{\alpha}} = \langle \Lambda \rangle h\left(\boldsymbol{\alpha}_\theta^b - \boldsymbol{\alpha}\right) \tag{21}$$

$$\dot{m} = c_m(1 + e_0)\dot{\varepsilon}_v^p + c_v\left(\frac{s_c n_w}{p_{ref}}\right)^{\varpi} \dot{n}_w^p \tag{22}$$

where $\boldsymbol{\alpha}_\theta^b$ is the bounding image stress ratio, h is a positive scalar-valued function, e_0 is the initial void ratio. c_v, c_m, ϖ are model parameters, p_{ref} is the reference pressure.

In order to investigate the coupling effects between the mechanical and hydraulic behavior of unsaturated sands or silts, the evolution of the bounding suctions on the drying bound and wetting bound (s_{c0d} and s_{c0w}) are given below

$$\text{Wetting:} \quad \dot{s}_{c0w} = s_{c0w}\zeta v\dot{\varepsilon}_v^p + \Gamma_{0w}^p \dot{n}_w^p \tag{23}$$

$$\text{Drying:} \quad \dot{s}_{c0d} = s_{c0d}\zeta v\dot{\varepsilon}_v^p + \Gamma_{0d}^p \dot{n}_w^p \tag{24}$$

where ζ is a material parameter, v is the specific volume, which is defined in terms of the porosity n as: $v = 1/(1-n)$, Γ_{0w}^p and Γ_{0d}^p are the capillary plastic moduli on the wetting and drying bounds, respectively.

4 NUMERICAL EXAMPLE

As a preliminary evaluation of the capabilities and performance of the proposed analysis procedure, an experiment involving a saturated sand column with a height of 60 cm and drying and wetting reported by Gillham *et al.* (1979) is simulated. The time history of the pore water pressure applied at the bottom of the sand column to simulate drying and wetting is shown in Figure 1. The sand column is simulated as a one-dimensional problem using a column of 120 four-node quadrilateral elements. The top and lateral boundaries were impervious for the water flow. All the boundaries were prescribed with zero pore air pressure. The simulation was run with a constant time step of 15 s. The predicted water pressure heads and volumetric water contents are compared with the measured data at various elevations in Figure 1 and Figure 2, respectively. The predicted water content profiles at different times are compared with measured profiles in Figure 3. The numerical predictions compare very well with the measured results.

5 CONCLUSION

A fully coupled dynamic analysis procedure is presented for the fluid flow and elastoplastic behavior analysis in unsaturated soils subjected to dynamic loading. The fully coupled equations governing the dynamic behavior of unsaturated soils are developed with solid skeleton displacement, pore water pressure, and pore water pressure as the primary unknown variables. The elastoplastic stress-strain behavior of the unsaturated soils is captured by a fully coupled hydro-mechanical elastoplastic constitutive model for unsaturated sands and silts that includes hydraulic hysteresis and hydro-mechanical coupling. The numerical simulation results of a one-dimensional column of sand subjected to drying and wetting compares well with the measured results. Additional validations of the analysis procedure are currently underway using dynamic centrifuge model test results of unsaturated soils.

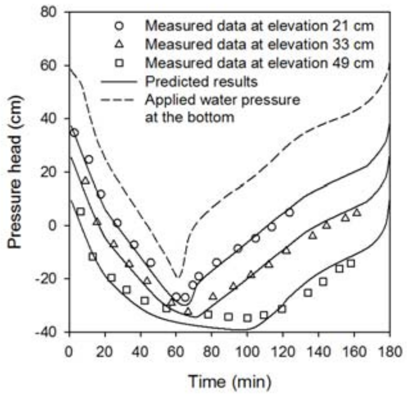

Figure 1. Comparison between measured and predicted pressure heads at different elevations

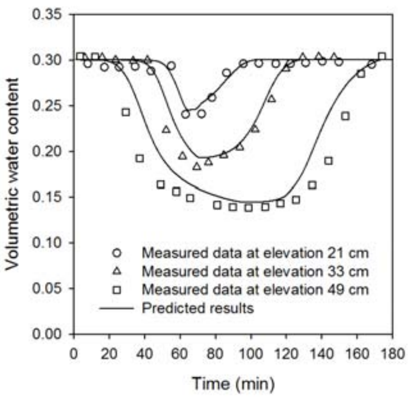

Figure 2. Comparison between measured and predicted water contents at different elevations

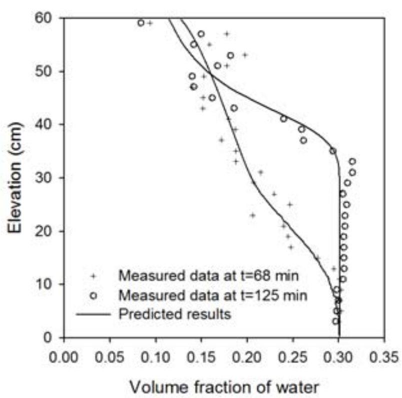

Figure 3. Comparison of measured and predicted water content profiles at different times

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