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Comparison of simple stress-strain models in the moderate strain range for fine-grained soils: A review

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ABSTRACT

The prediction of stress-strain behaviour in soils is a problem that can be approached in different ways depending on the design scenario. In some cases, a multi-parameter constitutive model calibrated with non-routine soil tests may be appropriate, for example, where a model has been developed for the relevant soil at a building site subject to complex loading. However, simple characteristic parameters are desirable for examining the variability of soil behaviour especially at regional scales. This paper describes a method of assessing the suitability of simple models for simulating non-linear undrained soil stress-strain behaviour in the moderate strain range. The moderate strain range is defined by a soil strength mobilisation of 20% to 80%. Three simple stress-strain models are compared. A published database of reconstituted triaxial tests is used to evaluate the three models with selected statistical tools that quantify errors associated with the simple model approximation of the relationship between stress and strain. The paper discusses the value of computing the model error and the trade off to make between introducing a greater number of parameters (and tests) for model precision and limiting the complexity of the variability characterisation.

Keywords: Mobilisation strain framework; variability characterisation; model error; design

1. Introduction

1.1. Background

The selection of representative soil parameters for ground movement analyses requires an understanding of soil material response due to loading. Material parameters are typically measured with exploratory Ground Investigation (GI) techniques. It is not possible to replicate all changes in stress and strain within the deformation mechanism with a single test; equally it is not possible to calculate the natural variability of the soil with certainty. Reliability-based site characterisation procedures offer practicing engineers a way to identify relevant parameter ranges from available GI data.

The variability of parameters from soil tests can be characterised using reliability-based procedures by analysing large test databases. This approach is well-documented for the characterisation of undrained shear strength, c_u (e.g., Mayne 1980, Mayne 1985, Mayne & Holtz 1985, Chen & Kulhawy 1993, Mayne et al. 2009, Ching & Phoon 2013, 2014), but the variability characterisation of undrained pre-failure behaviour has been the subject of less study.

1.2. Study aims

This paper reviews the suitability of different soil models that characterise undrained stress-strain behaviour in the moderate strain range by considering their implementation within a reliability-based site characterisation framework. This study has two primary aims: (i) to review the historical development of simple models for stress-strain (in the moderate strain range) used in geotechnics (see the thesis of Beesley 2019 for further details on the review and analysis presented in this paper) and (ii) offer suggestions for the further development of these models for enhanced incorporation in geotechnical design procedures. A companion paper Beesley et al. (2023) has also been prepared which reexamines the empirical prediction of strain parameters for the use in Mobilisable Strength Design (MSD) calculation.

2. Imposed modelling limitations

2.1. Definition of "moderate strain range"

The moderate strain range for fine-grained soils was defined by Vardanega & Bolton (2011) as shear strain (γ) corresponding to a mobilised soil strength from 20% to 80%. Characterising undrained stress-strain behaviour by strength mobilisation means that the stress is normalised using the soil undrained shear strength (c_u). Strength mobilisation models can be incorporated into Mobilisable Strength Design (MSD) calculation procedures (e.g., Bolton & Powrie 1988, Bolton 1993a, Osman & Bolton 2005, Lam & Bolton 2011, Diakoumi & Powrie 2013, Bolton et al. 2014, McMahon et al. 2014, Klar & Klein 2014, Deng et al. 2021) and can also be used to select characteristic stress-strain parameters for alternative deformation analyses and further allow for the use of MSD in a reliability framework as suggested by Vardanega & Bolton (2016a).

In this paper the moderate strain range is defined only by the shear strains mobilised according to Eq. 1:

$$S = \frac{\tau_{mob} - \tau_0}{c_u - \tau_0} \qquad 0.2 \leq S \leq 0.8 \tag{1}$$

where, S = stress ratio (previously denoted as B, see Casey et al. 2016, Vardanega & Bolton 2016b, Beesley & Vardanega 2020); τ_{mob} = mobilised shear strength; τ_0 = initial shear stress; c_u = undrained shear strength.

2.2. Choice of test data

Triaxial tests are routinely used in industry to measure monotonic non-linear undrained stress-strain behaviour and shear strength. Consolidated Undrained (CU) triaxial test data were digitised from published experimental studies on reconstituted soils and compiled into a database named RFG/TXCU-278 (for further details see Beesley 2019, Beesley & Vardanega 2020). Stress-strain increments were obtained only from the undrained shear stage of the test. The digitised test data included deviator stress (q) or shear stress ($0.5q = \tau$) and axial strain (ε_a), which was used to calculate shear strain by $\gamma = 1.5\varepsilon_a$. Test modes include: isotropic consolidation triaxial compression (CIUC) and extension (CIUE); K_0-consolidation triaxial compression (CKUC) and extension (CKUE).

3. Choice of stress-strain models

3.1. Review of published stress-strain models in the moderate strain range

Duncan & Buchignani (1976) proposed relationships between E_u/c_u and overconsolidation ratio (OCR) for ranges of clay soils categorised by plasticity index (Fig. 1). The chart was reproduced by Casey et al. (2016) who clarified that the lines shown in Fig. 1 correspond to a stress ratio of 0.5, where stress ratio is defined by Eq. 2.

$$S = \frac{\tau_{mob} - \tau_0}{c_u - \tau_0} = 0.5 \tag{2}$$

Whereas the parameter ranges recommended by Duncan & Buchignani (1976) were based on field measurements, and give no indication of non-linear behaviour, Casey et al. (2016) used a reconstituted soils database of CKUC tests to develop new empirical correlations describing the variation of secant undrained modulus (E_u) with stress ratio. To characterise non-linear

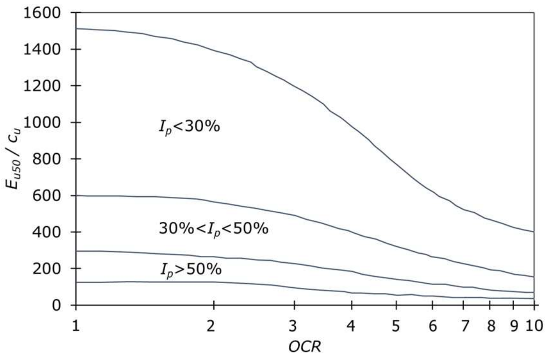

Figure 1. Chart showing relationships between E_{u50}/c_u and OCR proposed by Duncan & Buchignani (1976) for clays (digitised from the original publication)

behaviour, Casey et al. (2016) adopted equivalent elastic secant modulus values at three points on a soil's stress-strain curve which could be estimated by three different empirical equations. Casey et al. (2016) demonstrated that E_u at S = 0.25, 0.50 and 0.75, normalised by a reference effective stress (σ'_{vref} = 100kPa), varied with pre-shear vertical consolidation stress applied in the triaxial test (denoted here by σ'_{v0}) and recommended Eqs. 3a-c for normally consolidated soils:

$$\frac{E_{u25}}{\sigma'_{vref}} = 465\left(\frac{\sigma'_{v0}}{\sigma'_{vref}}\right)^{0.73} \quad \text{at } S = 0.25 \tag{3a}$$

$$\frac{E_{u50}}{\sigma'_{vref}} = 364\left(\frac{\sigma'_{v0}}{\sigma'_{vref}}\right)^{0.68} \quad \text{at } S = 0.50 \tag{3b}$$

$$\frac{E_{u75}}{\sigma'_{vref}} = 260\left(\frac{\sigma'_{v0}}{\sigma'_{vref}}\right)^{0.61} \quad \text{at } S = 0.75 \tag{3c}$$

Non-linear elasticity is a common approach used by constitutive modellers (Potts et al. 2002). A non-linear elastic stress-strain model was first implemented with finite element (FE) analysis by Duncan & Chang (1970) who explained that incremental stress-strain calculation procedures in FE analyses can be conveniently expressed by a formula describing the degradation of tangent modulus (E_t): Eq. 4 assumes:

- a hyperbolic (or modified hyperbolic) stress-strain law (recommended by Kondner 1963 and many others e.g., Hardin & Drnevich 1972, Stokoe et al. 1999, Darendeli 2001, Zhang et al. 2005, Vardanega & Bolton 2013, Vardanega & Bolton 2014, Oztoprak & Bolton 2013, Wichtmann & Triantafyllidis 2013),
- a Mohr-Coulomb failure criterion, and
- an empirical relationship to estimate initial tangent modulus (E_i) from the confining pressure applied in a triaxial test

to mathematically describe the variation of tangent modulus with any stress increment up to peak stress (Duncan & Chang 1970):

$$E_t = \left[1 - \frac{R_f(1 - sin\phi)(\sigma_1 - \sigma_3)}{2c\ cos\phi + 2\sigma_3 sin\phi}\right]^2 Kp_a\left(\frac{\sigma_3}{p_a}\right)^n \tag{4}$$

where, σ_1 = major principal stress; σ_3 = minor principal stress; R_f = the failure ratio (always less than unity) defined by the ratio of the measured value of $(\sigma_1 - \sigma_3)_{\text{failure}}$ at peak strength to the asymptotic value of $(\sigma_1 - \sigma_3)_{\text{ult}}$ as defined by the hyperbolic stress-strain equation; c and ϕ are Mohr-Coulomb strength parameters; K and n are respectively the intercept and slope coefficients obtained by fitting a straight line through a series of measurements of (E_i) at different confining stresses (σ_3) plotted in transformed axes log(E_i) and log(σ_3).

Jardine et al. (1986) presented a periodic logarithmic function (Eq. 5) that was shown to model the non-linear relationship between E_u/c_u and axial strain (ε_a) of a low plasticity, lightly overconsolidated clay over four logarithmic cycles of strain:

$$\frac{E_u}{c_u} = A + Bcos\left\{\propto \left[log_{10}\left(\frac{\varepsilon_a}{C}\right)\right]^{\theta}\right\} \tag{5}$$

where, A, B, C, α and θ are constants derived by fitting Eq. 5 to stiffness-strain measurements of a single CU triaxial test.

Equation 4 was developed by Duncan & Chang (1970) for routine triaxial tests, whereas Jardine et al. (1986) introduced Eq. 5 to better fit small-strain non-linearity observed by use of local-strain measuring techniques. While Eq. 4 and Eq. 5 in principle can simulate stress-strain behaviour incrementally at any stress ratio, Jardine et al. (1986) pointed out that lower and upper strain limits must be selected for Eq. 5. Each model has five model parameters; however, at least three triaxial tests are needed to calibrate Eq. 4 for a soil. In contrast, adopting the approach by Casey et al. (2016) involves fewer model parameters (Eq. 3) and offers the advantage that it has been calibrated by a large database of reconstituted soils (73 CKUC tests); however, their method provides only 3 points on the stress-strain curve.

An alternative representation of stress-strain behaviour was described by Bolton (1993a) as plastic strength (c_u) mobilisation with the development of plastic shear strains (γ), suggesting that the relationship may be described by a power law (Bolton 1993b):

$$S = \frac{\Delta\tau}{\Delta\tau_p} = \left(\frac{\Delta\gamma}{\gamma_p}\right)^b \qquad (6)$$

where, in undrained soils, $\Delta\tau_p$ = applied shear stress to mobilise peak strength (c_u- τ_0); γ_p = shear strain mobilised at peak strength (c_u); $\Delta\tau$ = an increment of shear stress (τ_{mob}- τ_o); $\Delta\gamma$ = an increment of shear strain; and b = a fitted exponent.

Power-law functions were proposed in earlier studies to describe stress-strain relationships observed in laboratory soil tests (Brinch Hansen 1965) and to define p-y curves for offshore structures (Matlock 1970; Zhang & Andersen 2017). Whereas the model by Matlock (1970) assumed a set 'b' value (of 0.33), the variation of b was recognised by Brinch Hansen (1965) and Bolton (1993b) and later formalised by Vardanega & Bolton (2011) who proposed the mobilisation strain framework (MSF) framework (an equation of the form shown in this paper as Eq. 7) for reliability-based design calculations (Vardanega & Bolton 2016a):

$$S = \frac{\tau_{mob}}{c_u} = 0.5\left(\frac{\gamma}{\gamma_{50\,CIU}}\right)^{b_{CIU}} \qquad 0.2 \le \tau_{mob}/c_u \le 0.8 \quad (7)$$

where, τ_{mob} = the mobilised shear strength; γ = shear strain; $\gamma_{50\ CIU}$ = shear strain to mobilise $0.5c_u$ under isotropically-consolidated undrained conditions (denoted in the authors' study as $\gamma_{M=2}$); and b_{CIU} is an exponent to describe non-linearity. Since τ_0 = 0, $\tau_{mob}/c_u = S$. The subscript CIU has been added to the model parameters of Eq. 7 in this work to acknowledge isotropic (K_0 = 1) consolidation stresses prior to undrained shear.

Eq. 7 makes use of a reference mobilisation strain ($\gamma_{50\ CIU}$) at S = 0.5 which may be measured in routine triaxial tests. If soil stress-strain may be accurately represented by power-law curves, a varying exponent is indicative of varying stress-strain non-linearity. A similar approach using a power-law and hardening exponent was suggested by Hollomon (1945) to describe ranges of plastic flow (ductility) in metals. Beesley & Vardanega (2021) and Vardanega et al. (2021) also present recent studies on the variability of the b parameter from the power law models when applied to soils.

Vardanega & Bolton (2011) compiled a large database of 92 CIU shear tests on intact soils and demonstrated that b_{CIU} = 0.608 (mean) ± 0.158 (standard deviation). Vardanega et al. (2021) examined (in part) the effect of shear mode on the b parameter for collected London clay test data including pressuremeter measurements. Beesley & Vardanega (2021) presented a detailed analysis of the variation of b using RFG/TXCU-278 and a series of triaxial tests on intact Bothkennar Clay presented in the report of SERC (1989), concluding that b is insensitive to OCR but is affected by shear mode and sample state: lower b values may be expected in reconstituted and SHANSEP-consolidated samples.

Casey (2016), in response to the discussion of Vardanega & Bolton (2016b), observed that a large difference in mobilised reference strain at S = 0.5 measured in triaxial compression (TC) may occur due to the application of an isotropic or K_0-consolidation stress path. Vardanega & Bolton (2016b) agreed that Eq. 8 is needed to describe K_0-consolidated triaxial tests:

$$S = \frac{\tau_{mob} - \tau_0}{c_u - \tau_0} = 0.5\left(\frac{\gamma}{\gamma_{50\,CKU}}\right)^{b_{CKU}} \qquad 0.2 \le S \le 0.8 \quad (8)$$

where, τ_0 = initial shear stress; $\gamma_{50\ CKU}$ (denoted by Vardanega & Bolton (2016b) as γ_{ref}) = shear strain to mobilise $0.5(c_u - \tau_0)$; and b_{CKU} is an exponent describing soil non-linearity.

Klar & Klein (2014) pointed out that expressing stress-strain behaviour with a power-law leads to infinitely high initial stiffness and instead proposed an exponential function (Eq. 9) to model CIU test results:

$$S = \frac{\tau_{mob}}{c_u} = \left(1 - e^{-0.693\frac{\varepsilon_a}{\varepsilon_{50\,CIU}}}\right) \qquad (9)$$

where, ε_a = axial strain; $\varepsilon_{50\ CIU}$ = axial strain mobilised at $0.5c_u$ in a CIU triaxial test. The issue of the infinite stiffness at low strain was also pointed out in Bateman et al. (2022) with a linear-power law function proposed.

Puzrin & Burland (1996) proposed a logarithmic function, Eqs. 10(a-c), to characterise measurements outside the small strain region by defining a lower strain limit as $x_1 = \varepsilon_e$+e-1, where ε_e = the elastic region limit and e = 2.718. As for Eq. 6, the stress-strain curve is normalized with respect to both a limiting strain and stress, taken as measured at the point of peak stress:

$$\frac{q}{q_p} = \frac{\varepsilon}{\varepsilon_r} - \alpha\frac{\varepsilon}{\varepsilon_r}\left[\ln\left(1 + \frac{\varepsilon}{\varepsilon_r}\right)\right]^R \qquad (10a)$$

where,

$$R = \frac{\left(1+\frac{\varepsilon_p}{\varepsilon_r}\right)ln\left(1+\frac{\varepsilon_p}{\varepsilon_r}\right)}{\frac{\varepsilon_p}{\varepsilon_r}\left(\frac{\varepsilon_p}{\varepsilon_r}-1\right)}\ , \qquad (10b)$$

$$\alpha = \frac{\left(\frac{\varepsilon_p}{\varepsilon_r}-1\right)}{\frac{\varepsilon_p}{\varepsilon_r}\left[ln\left(1+\frac{\varepsilon_p}{\varepsilon_r}\right)\right]^R}\ , \qquad (10c)$$

and, q = deviator stress; q_p = peak deviator stress; ε_p = strain at peak deviator stress; ε_r = a reference strain defined by q_p/E_{max}; and E_{max} = initial stiffness modulus.

Puzrin & Burland (1996) also demonstrated a separate function for small-strain data. The two adjoined functions, known as L4, was incorporated into a constitutive model to predict pre-failure deformations caused by tunnelling in London Clay; see Addenbrooke et al. (1997).

3.2. Evaluation of published stress-strain models in the moderate strain range

To select an appropriate stress-strain model for geotechnical variability characterisation analyses, Beesley (2019) suggested that such a model requires:

1. that the model is sufficiently representative of a soil's physical behaviour (e.g., Klar & Klein 2014);
2. the lowest possible number of model parameters while maintaining reasonable accuracy (Puzrin & Burland 1996);
3. that the model parameters should be representative of physical properties (Puzrin & Burland 1996);
4. that the parameters should be simple to derive (Puzrin & Burland 1996);
5. that the parameters can be evaluated by reliability-based procedures (e.g., as suggested by Vardanega & Bolton 2011, 2016a).

Aside from Eq. 5, all models presented by the cited studies use a form of strength mobilisation to describe soil stress-strain behaviour. Only Eq.'s 6 to 10 use a form of strain mobilisation. If Eq. 1 is used to describe strength mobilisation, it seems plausible that a reasonable description of the normalised stress-strain data may be achieved by hyperbolic, power, exponential, or logarithmic functions, or three elastic secant moduli. Casey et al. (2016), Vardanega & Bolton (2011) and Puzrin & Burland (1996) provided evidence of model calibrations with large datasets.

Power and hyperbolic laws are unlikely to represent realistic soil behaviour close to peak stresses: Vardanega & Bolton (2011) recommended an upper limit of $S = 0.8$; Duncan & Chang (1970) stated that R_f varies between 0.75 and 1.0. Lower stress limits are also relevant: Klar & Klein (2014) selected an exponential law (Eq. 9) to limit the initial stiffness to a finite value rather than use the power law approximation that leads to infinitely high stiffness. The stress-strain models proposed by Puzrin & Burland (1996) and Klar & Klein (2014) are defined by a collapse limit but without lower limits for stress ratio.

To simplify the modelling process, a single model that can simulate stress-strain behaviour over the relevant engineering range for the design calculation is desirable. A model that simulates a continuous curve offers more information to the engineer, but it comes at a cost: whereas only a single parameter is needed to predict three secant moduli using Eqs. 3(a-c), at least two parameters are required to use Eqs. 4-10. This raises the question of how one might quantify the variability of multiple parameters for a single soil model, and, moreover, whether additional information about the stress-strain behaviour is worthy of the extra computation effort.

R_f is a parameter that has no real physical meaning and can be understood as a model correction factor and additional variable that is difficult to predict (Duncan & Chang 1970). Parameters E_i and E_{max} (Duncan & Chang 1970, Puzrin & Burland 1996) are not straightforward to measure consistently in routine triaxial tests (without bender elements or local high-resolution displacement gauges) (cf. Atkinson 2000). The non-linearity parameter (b_{CIU} or b_{CKU}) of the power-law model (Vardanega & Bolton 2011) is arguably representative of soil ductility or more nuanced differences in yielding behaviour; this requires further study. The exponential function is inflexible in shape and hence a constant soil non-linearity is inherently assumed (Klar & Klein 2014).

3.3. Models proposed for further investigation

Based on the aforementioned selection criteria, three mathematical functions were selected to model non-linear stress-strain behaviour in the moderate strain range, defined as $0.2 \leq S \leq 0.8$ according to Eq. 1, using the tests on reconstituted fine-grained soils in database RFG/TXCU-278: exponential (Eqs. 11a and 11b, Klar & Klein 2014), power (Eqs. 12a and 12b, Vardanega & Bolton 2011), and logarithmic (Eqs. 13a and 13b, Beesley 2019):

$$S = \left(1 - e^{-0.693\frac{\gamma}{\gamma_{50\,CIU}}}\right) \tag{11a}$$

$$S = \left(1 - e^{-0.693\frac{\gamma}{\gamma_{50\,CKU}}}\right) \tag{11b}$$

$$S = 0.5\left(\frac{\gamma}{\gamma_{50\,CIU}}\right)^{b_{CIU}} \tag{12a}$$

$$S = 0.5\left(\frac{\gamma}{\gamma_{50\,CKU}}\right)^{b_{CKU}} \tag{12b}$$

$$S = \beta_{CIU} \cdot \log_{10}\left(\frac{\gamma_{50\,CIU}}{\gamma}\right) + 0.5 \tag{13a}$$

$$S = \beta_{CKU} \cdot \log_{10}\left(\frac{\gamma_{50\,CKU}}{\gamma}\right) + 0.5 \tag{13b}$$

where, γ = shear strain ($= 1.5\varepsilon_a$); ε_a = axial strain; CIU = isotropically-consolidated undrained conditions; CKU = K_0-consolidated undrained conditions; γ_{50} = a reference shear strain to mobilise $0.5(c_u - \tau_0)$ according to each function; b = an exponent to describe non-linearity (power-law function); β = a factor to describe non-linearity (logarithmic function); the exponential function is inflexible in shape.

4. Assessment of three alternative simple non-linear stress-strain models

4.1. Visualisation of modelled stress-strain data

Before any observations can be made from model parameters about the variability of stress-strain behaviour, the model error first needs to be evaluated. Empirical curves developed using a poorly fitted model will introduce uncertainty into the corresponding model parameters. Although it is possible to quantify curve fitting error, a serious error in fit between a model and the test data should be avoided.

In Fig. 2, the error data (in total n = 5199 data points) demonstrate that none of the functions perfectly fit the stress-strain data of RFG/TXCU-278 as would be expected. The error varies with S: the exponential model (Eq. 11) is significantly biased and overestimates γ between $0.2 \leq S \leq 0.65$; the power-law model (Eq. 12) and logarithmic model (Eq. 13) better fit the test data within the range of $0.3 \leq S \leq 0.7$. Fig. 2 demonstrates that the power-law cannot simulate the full curvature of the stress-strain curve (particularly for CKUE tests) while the logarithmic function exacerbates it; most measurements of γ have values between the two approximations of behaviour.

4.2. Quantifying model error

In addition to visual inspection of the error data in Fig. 2, the bulk of the error data can be used to quantify model bias. 10th and 90th percentiles of the ratio (Measured shear strain/Modelled shear strain) are used to identify the central 80% of the error data, as illustrated to the right of each plot. These percentiles demonstrate the bias and spread of error of the exponential model (corresponding to a factor error as high as 3.91 at the 10th percentile) and on this basis the model is discounted from further consideration in the variability characterisation framework. The 10th and 90th percentiles of the error data are symmetrical about the zero-error line for both power-law and logarithmic models, and they have about the same spread of error. The variation of error with S indicates that for most CU triaxial tests, apart from CKUE, the power-law model is closer to the shape of the stress-strain curve over a larger range of S than the logarithmic model.

Alternatively, the model error evaluation can be undertaken by examining the regression statistics per triaxial test: the coefficient of determination (R^2) and the standard error (SE) (see e.g. Kvalseth 1983 for full definitions of these statistical terms). The method of least squares was used to estimate a "best fit" linear regression line using one or two logarithmically transformed axes as appropriate to the model.

SE may be used as a relative measure of accuracy when comparing models. When comparing different models fitted to a single triaxial test, *SE* should be calculated using non-transformed axes. In this case the parameters do not represent accuracy of the regressions but, rather, quantify the average residual error of predicted stress-strain (here in terms of strength mobilisation and strain).

Applying Eq. 12 (power-law) to the database RFG/TXCU-278 (n = 271 tests) gave a range of *SE* values of 0.002 to 0.092 with an associated range of R^2 of 0.80 to 1.00. Applying Eq. 13 (logarithmic) to the database RFG/TXCU-278 (n = 271 tests) gave a range of *SE* values of 0.0002 to 0.087 with an associated range of R^2 of 0.73 to 1.00. Notably, for 56% of the tests Eq. 13 outperformed Eq. 12 on both the *SE* and R^2 metric. While the power-law may be outperformed by the logarithmic function in 56% of cases (for this database), the *SE* and R^2 metrics provide no information about the distribution of error at different points on the stress-strain curve.

Based on Fig. 2, three stress ratios at S = 0.3, 0.5 and 0.7 were chosen to quantify the effect of model error on strain γ; measurements of γ were normalised by the reference strains predicted by Eq. 12 and Eq. 13 with the reference strains named γ_{30}, γ_{50} and γ_{70} respectively. The normalised shear strains γ/γ_{50} and γ/γ_{70} are plotted against measured S in Fig. 3; as an approximation, the error distribution of normalised shear strains was characterised within 0.01 of the stress ratio (the bounds of ±0.005 are indicated by red lines in Fig. 3). Fig. 3 shows only a subset of the database but it is sufficiently representative to demonstrate that, at S = 0.5 and 0.7, the power-law model (Eq. 12) produces lower strain error than the logarithmic model (Eq. 13) and the distributed errors are generally skewed to a lesser extent. Normalised shear strains predicted by Eq. 12 are also more closely distributed within the 10th and 90th percentiles than those predicted by Eq. 13. Beesley (2019) and Beesley et al. (2023) report further details on the model error analysis of γ/γ_{30}, γ/γ_{50} and γ/γ_{70}, carried out using RFG/TXCU-278.

4.3. Implications for variability characterisation procedures

Fig. 3 further shows that γ_{70} may be a superior normaliser of γ than the γ_{50} parameter for the database RFC/TXCU-278; with less model error affecting γ_{70} than γ_{50}, intuitively this means that, should a relationship exist between mobilised shear strain and other material parameters such as *OCR*, γ_{70} is a more reliable candidate parameter to investigate such relationships. A full statistical analysis of the model strain parameters is shown in Beesley et al. (2023).

5. Concluding remarks

5.1. Summary

This paper has reviewed various simple stress-strain models used for the 'moderate strain range'. The following concluding remarks are made:

1. Such models have a long history in geotechnical applications but only recently have been formalised for use in reliability frameworks, calibrated by large geotechnical databases such as RFG/TXCU-278.
2. While the effect of shear mode has been shown for c_u only recently has this been comprehensively studied for shear strain parameters e.g., γ_{50} or γ_{70}.
3. With the simulation of a continuous non-linear stress-strain curve in a limited engineering range as the modelling objective, it was demonstrated that a non-linearity variable (e.g., b) is required to predict CU soil behaviour without bias in the range $0.2 \leq S \leq 0.8$ in addition to a limiting stress (c_u) and a reference shear strain (e.g., γ_{50}).
4. As expected, none of the investigated models perfectly replicate the stress-strain measurements in the test database. The power-law model cannot simulate the full curvature of the stress-strain curve while the logarithmic model exacerbates it; most measurements of γ have values between the two approximations of behaviour.
5. However, by quantifying model error using the normalisation procedures demonstrated in this

paper, it was shown that the power-law model (Eq. 12) is a rational choice to investigate further for implementation in reliability frameworks.

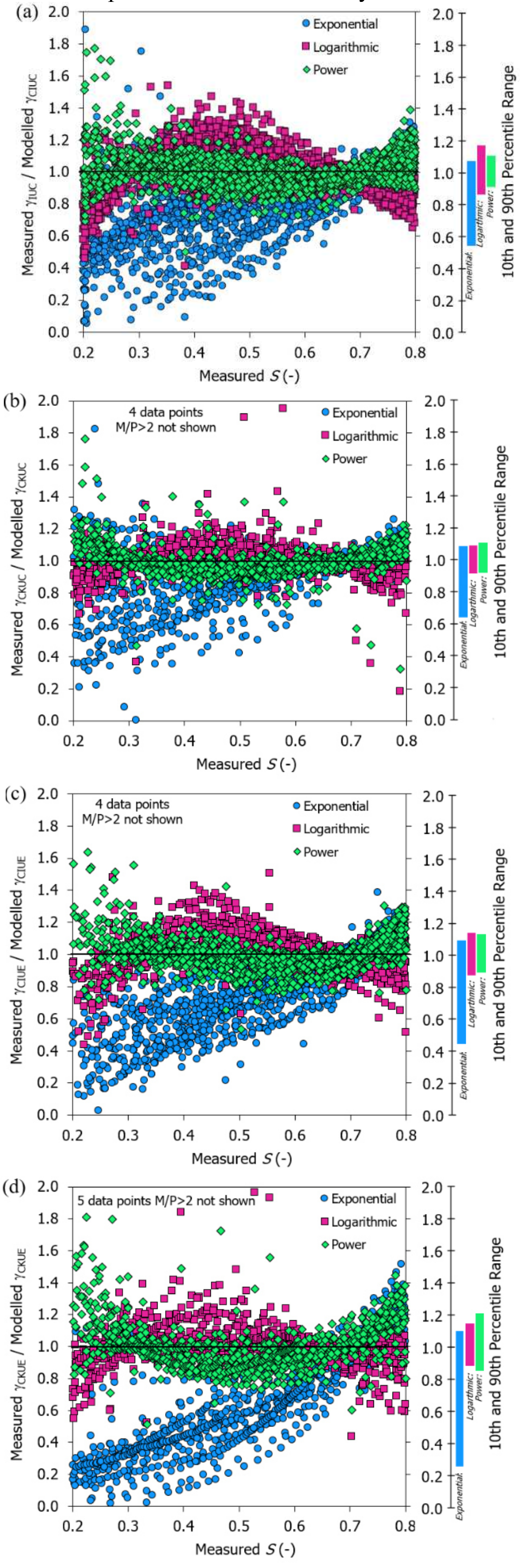

Figure 2. Measured γ / Modelled γ of all stress-strain data between 20% ≤ S ≤ 80% of database RFG/TXCU-278 using Eq. 10, Eq. 11 and Eq. 12: (a) CIUC, 114 tests, n = 2069 data points, (plot also shown in Beesley et al. 2023); (b) CKUC, 67 tests, n = 1049 data points; (c) CIUE, 55 tests, n = 1217 data points; (d) CKUE, 30 tests, n = 864 data points.

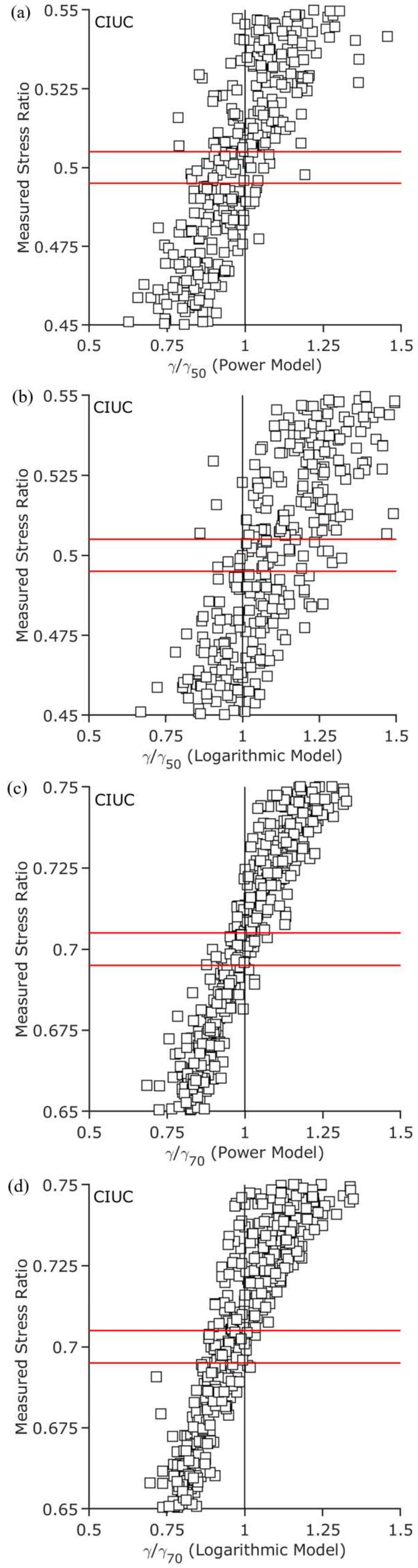

Figure 3. Normalised shear strains within 0.01 of the reference stress ratio, S, indicated by red lines, including CIUC test mode data only from RFG/TXCU-278 with reference strain at: (a) S = 0.5 using Eq. 11a; (b) S = 0.5 using Eq. 12a; (c) S = 0.7 using Eq. 11a; (d) S = 0.7 using Eq. 12a

5.2. Future directions

By reviewing the history of simple stress-strain models (relevant to the moderate strain range), it was identified that defining limited stress ranges is a useful way of approaching the problem of characterising the variability of soil stress-strain behaviour. Using c_u as the maximum stress, the power-law model strain parameters have been shown to be influenced by shear mode and *OCR* (Beesley & Vardanega 2020) as well as void ratio and liquid limit (Beesley et al. 2023) for reconstituted soils; this is evidence that there is merit in further developing the reliability framework for natural soils. The stress ratio range of 0.2 to 0.8 was selected for model calibration in this work, but different stress ranges of engineering interest may be equally valid for future research on soil variability characterisation procedures. It is likely that less model error (associated with the approximation of a measured CU stress-strain curve) would be observed with the tested models if a reduced stress ratio range were to be adopted with the inevitable trade-off that a smaller range of potential geo-structural performance outcomes are able to be studied with a narrower stress ratio range.

Following the methodology presented in this paper and in the companion paper (Beesley et al. 2023), new models could be developed specific to various design scenarios using large well-managed geotechnical test databases. The relative importance of different test modes and procedures needs to be considered carefully when characterising the variability of stress-strain behaviour relevant to the in-situ stress path and with respect to the settlement limits specified by the design contract. For implementation in reliability frameworks, the ideal "simple" stress-strain model would have the minimum number of parameters that provide sufficiently accurate information about the design calculation to make informed decisions about design risk. In certain cases, this could mean that a single linear elastic modulus is perfectly adequate for characterising soil variability, and in other scenarios more information about the non-linear behaviour would be needed. Further work is needed to identify the "simple" modelling requirements for different design scenarios and correspondingly the reliability-based procedures required to select multiple parameters (such as c_u, γ_{50} and b) for a single soil model. Field tests, construction field trials, and asset performance databases should be used collaboratively to improve the predictions of ground movements based on the characterised variability of GI databases.

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