Pore pressure dissipation around driven piles and cone penetration tests- analytical solutions and a simplified approach using the Gauss' divergence theorem

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ABSTRACT: Forced penetrations into the ground such as pile driving, cone penetration tests and other activities of similar nature can generate significant excess pore pressure. The generated pore pressure can take some time to dissipate if the permeability of the medium is low. The generation and dissipation of pore pressure from such types of activities may best be formulated in a cylindrical coordinate system. Although consolidation could happen both in the vertical and in the radial directions, quite often the consolidation process in the vertical direction is disregarded. This paper aims at providing solutions with due consideration for combined radial and vertical consolidation. Finite Difference discretization is provided and exact analytical solutions are derived. Then, employing the Gauss divergence theorem, a simplified and flexible approach is devised for describing the consolidation process based on predefined pore pressure profiles and average pore pressures. The simplified approach is further used for taking into account the effect of impermeable boundaries. The formulations established may also be useful for interpreting coefficients consolidation from dissipation tests performed in piezocone penetration tests without having to disregard the consolidation in the vertical direction.

Keywords: Degree of consolidation; pile driving, cone penetration tests, pore pressure dissipation around piles

1. Introduction

The process of forced penetration, for example, during pile driving and Cone Penetration Tests (CPT) into low permeable soils such as clays generates excess pore pressure in the vicinity of the penetrating structure. The generation of excess pore pressure may lead to a decrease in the effective stresses and hence a decrease in the bearing capacity of the soils. In time, the generated pore pressure dissipates. The process is termed as consolidation. The consolidation occurs in all directions from areas of high excess pore pressure to areas of low excess pore pressure as long as boundary conditions allow it. The rate of dissipation depends on the permeability of the material or more appropriately the coefficient of consolidation which combines the permeability of the medium with the resistance of the material against changes in volume. In the cases of piles and CPTs, consolidation in the vertical and radial direction can be envisaged. At and around the tip, assuming an isotropic material, the consolidation process can be assumed to happen spherically. In our next exposition we are going disregard what happens at the tip and continue to assume radial symmetry and that consolidation happens in both radial and vertical directions.

1.1. Differential equation

The consolidation process in a cylindrical co-ordinate system may be described applying the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

on the pore pressure and further assuming $\partial u/\partial \theta=0$ and radial symmetry of material properties. Accordingly, one is led to the ordinary differential equation of consolidation

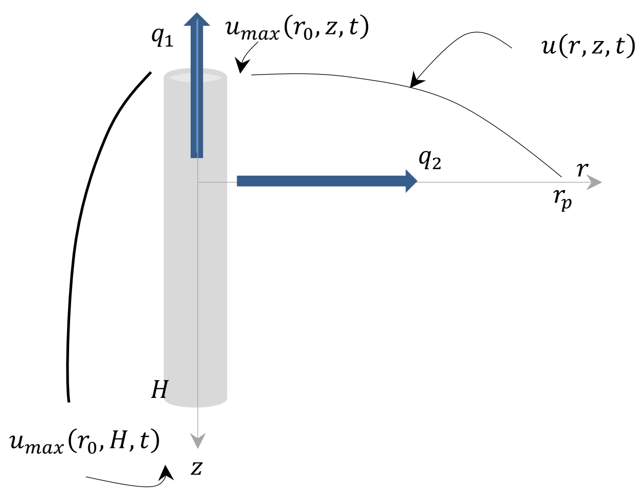

Figure 1. Schematics of flow and pore pressure distribution around driven cylindrical piles

$$\frac{\partial u}{\partial t} = c_r \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + c_z \frac{\partial^2 u}{\partial z^2}, \tag{2}$$

where c_r and c_z are coefficients of consolidation along the radial and the vertical direction. They may be given respectively as

$$c_r = \frac{k_r K}{\gamma_w},\ c_z = \frac{k_z K}{\gamma_w}. \tag{3}$$

where k_r and k_z are the permeability's of the soil in the respective directions, K is the bulk modulus of the soil and γ_w is the unit weight of water.

1.2. Boundary conditions

For solving the differential equation, boundary conditions are specified such that, vertically, the excess pore pressure vanishes at the top ground surface and, radially, the excess pore pressure vanishes at a distance r_p from the pile center. It is called the *plastified* radius. It is the radial extent at which the penetration is assumed to cause plastic deformations. For our case it is sufficient to define it as the radial extent outwardly from the periphery of the penetrating object where the pore pressure generated is insignificant. The extent of the plastified radius has in literature been related to stiffness, overconsolidation ratio and undrained strength of the soil around the penetrating object and it has been estimated from cylindrical and spherical cavity analyses [1], Fig.1.

Furthermore, the boundary conditions

- alongz: at $z = 0$; $u = 0$; at $z = H$, $\partial u/\partial z = 0$

and

- along r: at $r = 0$, $\partial u/\partial r = 0$; at $r = r_p$, $u = 0$

are considered.

2. Finite Difference discretization

The differential equation, Eq.2, can be solved using numerical methods in its generality. For instance, it can be approximated by a Finite Difference discretization as:

$$\frac{u_{ij}(t+\Delta t)-u_{ij}(t)}{\Delta t} = c_r\frac{1}{r_{ij}}\frac{u_{ij+1}(t)-u_{ij-1}}{2\Delta r} + \frac{c_r}{\Delta r^2}\{u_{ij+1}(t)-2u_{ij}(t)+u_{ij-1}(t)\} + \frac{c_z}{\Delta z^2}\{u_{i+1j}(t)-2u_{ij}(t)+u_{i-1j}(t)\}. \tag{4}$$

Further rearranging, the pore pressure at time $t+\Delta t$ at node i,j can be determined as:

$$u_{i,j}(t+\Delta t) = u_{i,j}(t) + \frac{c_r\Delta t}{\Delta r}\frac{u_{i,j+1}(t)-u_{i,j-1}(t)}{2r_{ij}} + \frac{c_r\Delta t}{\Delta r^2}\{u_{i,j+1}(t)-2u_{i,j}(t)+u_{i,j-1}(t)\} + \frac{c_z\Delta t}{\Delta z^2}\{u_{i+1,j}(t)-2u_{ij}(t)+u_{i-1,j}(t)\}, \tag{5}$$

in which i is a node counter in the z direction, j is a node counter in the radial direction, and Δt, Δr and Δz are increments in time, in vertical distance and in radial distance respectively, Fig. 2.

The scheme together with appropriate boundary conditions and steps handles a consolidation process in a cylindrical coordinate system and it is conditionally stable and begins to be unstable when $c_i\Delta t/\Delta x^2 \geq 0.5$.

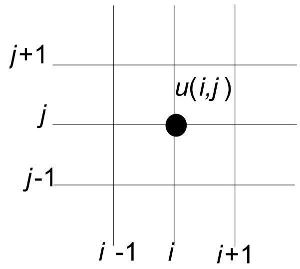

Figure 2. Illustration of the Finite Difference discretization

3. Analytical solution using separation of variables

Next, we proceed with separation of variables to solve the differential equation, Eq.2, by considering a solution of the form

$$u(r,z,t) = u(r)u(z)u(t). \tag{6}$$

Well-known solutions exist when the consolidation is only radial or when the consolidation is only vertical. When consolidation occurs only vertically, i.e., the solution of the differential equation is that of Fourier's solution for the one-dimensional heat flow problem which was adopted by Terzaghi [2] in formulating his one-dimensional consolidation theory. Say,

$$u(z) = c_j f(z) = c_j\,cos\left\{(2j-1)\pi\frac{H-z}{2H}\right\}, j = 1,2,3,\ldots, \tag{7}$$

in which c_j are some constants, which depend on the initial conditions.

When the consolidation occurs only in radial direction, again the solution is the same as that of the solution laid down for the propagation of heat in a cylindrical coordinate system of similar boundary conditions [3,4]. In fact, the solution is a combination of Bessel functions of the first and the second kind. Say for instance,

$$u(r) = c_i f(r) = c_i J_0(\lambda_i r), \tag{8}$$

in which J_0 is Bessel function of the first kind and order zero and c_i are constants dependent on the initial condition.

After some manipulations, a general solution is obtained as

$$u(r,z,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij} J_0(\lambda_i r)\cos\left\{(2j-1)\pi\frac{H-z}{2H}\right\} f_{ij}(t) \quad (9)$$

$$f_{ij}(t) = exp\left\{-c_r\lambda_i^2 t - c_z\frac{(2j-1)^2}{4H^2}\pi^2 t\right\}$$

in which the coefficients are found such that the initial condition is satisfied as

$$c_{ij} = \frac{4\int_{r_0}^{r_p}\int_0^H u(r,z,0) r J_0(\lambda_i r)\cos\left\{(2j-1)\pi\frac{H-z}{2H}\right\} drdz}{(r_p^2 J_1^2(\lambda_i r_p) - r_0^2 J_0^2(\lambda_i r_0))H} \quad (10)$$

Assuming that $u(r,z,0) = u_0 f(r) f(z)$, c_{ij} can also be multiplicatively split as

$$c_{ij} = u_0 c_i c_j \quad (11)$$

where

$$c_i = \frac{2}{r_p^2 J_1^2(\lambda_i r_p) - r_0^2 J_0^2(\lambda_i r_0)}\int_{r_0}^{r_p} r J_0(\lambda_i r) f(r) dr, \quad (12)$$

$$c_j = \frac{2}{H}\int_0^H \cos\left\{(2j-1)\pi\frac{H-z}{2H}\right\} f(z) dz. \quad (13)$$

Similarly, the pore pressure at a given time t at the coordinate (r,z) can be written as:

$$u_w(r,z,t) = u_0 f(r,t) f(z,t), \quad (14)$$

where

$$f(r,t) = \sum_{i=1}^{\infty} c_i J_0(\lambda_i r)\, exp\{-c_r\lambda_i^2 t\}$$

and

$$f(z,t) = \sum_{j=1}^{\infty} c_j \cos\left\{(2j-1)\pi\frac{H-z}{2H}\right\}\omega_j;$$

$$\omega_j = exp\left\{-c_z\frac{(2j-1)^2}{4H^2}\pi^2 t\right\}$$

In Fig.3 an example normalized pore pressure contour generated according to Eq.9 for an initial uniformly distributed pore pressure is shown.

The stage of the consolidation process is often inferred from a single variable called the degree of consolidation. In this case, the degree of consolidation may be defined as

$$U := 1 - \frac{\int_0^H\int_{r_0}^{r_p} u(r,z,t) drdz}{\int_0^H\int_{r_0}^{r_p} u(r,z,0) drdz} =$$

$$1 - \frac{\int_{r_0}^{r_p} f(r,t) dr \int_0^H f(z,t) dz}{\int_{r_0}^{r_p} f(r,0) dr \int_0^H f(z,0) dz} \quad (15)$$

Specifying the initial pore pressure, particular solutions that describe the pore pressure evolution with time and the corresponding degree of consolidation can be worked out.

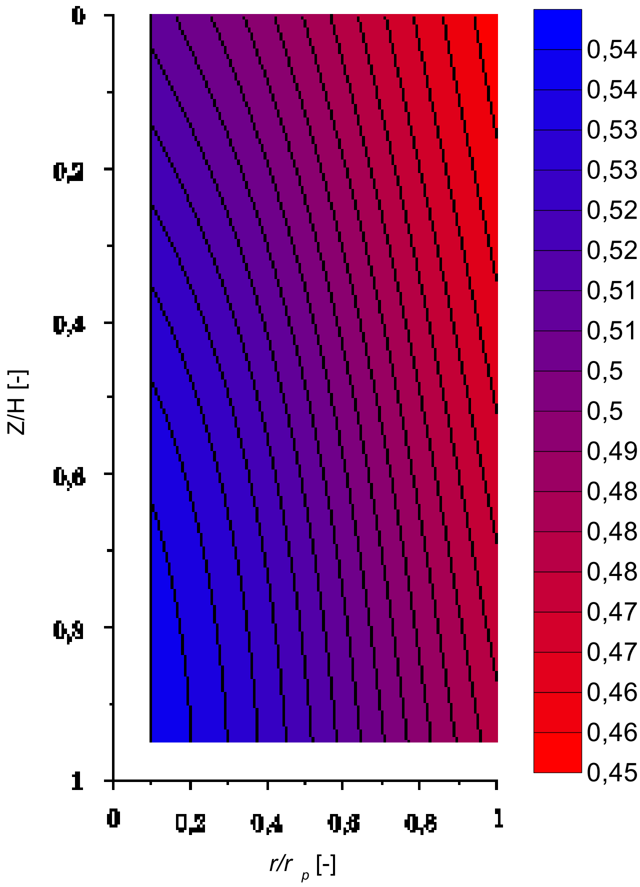

Figure 3. : An example normailized pore pressure contour generated according to Eq.9 for an initial uniformly distributed pore pressure.

4. Simplified approach applying the Gauss divergence theorem

4.1. Single pile, unobstructed radial and vertical seepage

Consider a volume element, Ω with a boundary of area, S, Fig.4. In the volume element, enclosed by the boundary surface we will describe the shape of the pore pressure distribution and assume that the shape will remain the same throughout the consolidation process. Once the pore pressure distribution is set, the average changes of the pore pressure within the specified volume enclosed in the boundary can be established by quantifying the seepage through the specified boundary. This is facilitated by the Gauss divergence theorem as follows.

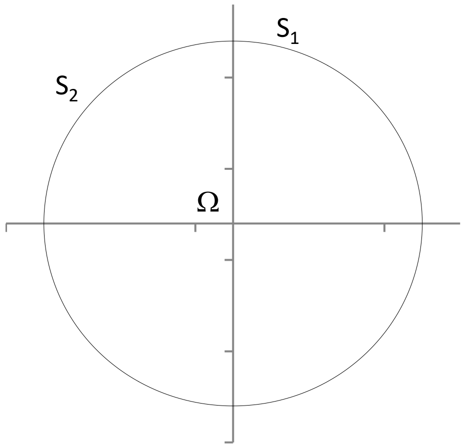

Figure 4. Flow through boundary S from an enclosed volume Ω (where S_1 is an seepage surface and S_2 is a closed surface)

We start with the volume integral of the differential equation in Eq.2,

$$\int_{\Omega}\frac{\partial u}{\partial t}d\Omega = c_r\int_{\Omega}\frac{1}{r}\nabla_r u d\Omega + c_r\int_{\Omega}\nabla_r^2 u d\Omega + c_z\int_{\Omega}\nabla_z^2 u d\Omega. \tag{16}$$

Note that $\frac{1}{\Omega}\int_V\ u\mathrm{d}\Omega$ is the average pore pressure, say $\overline{u}$. Therefore

$$\int_{\Omega}\frac{\partial u}{\partial t}d\Omega = \Omega\frac{\partial\bar{u}}{\partial t}. \tag{17}$$

The integral at the left-hand side of Eq.16 can be written as

$$\int_{\Omega}\frac{\partial u}{\partial t}d\Omega = \Omega\frac{\partial}{\partial t}\left(\frac{1}{\Omega}\int_{\Omega}ud\Omega\right). \tag{18}$$

Next, we consider each volume integral on the right side of the equal sign and change them into surface integral making use of the Gauss divergence theorem. Let us begin with the first term at the right side in Eq.16 and write

$$\int_{\Omega}\frac{1}{r}\nabla_r u d\Omega = \int_S\frac{1}{r}undS, \tag{19}$$

where n is a unit normal at the boundary. This integral vanishes where u vanishes. Such a boundary condition is to be considered. Hence, the integral in Eq.19 is disregarded. In the same manner, we continue to employ the Gauss divergence theorem on the other integrals:

$$\int_{\Omega}\nabla_r^2 u d\Omega = -\frac{\gamma_w}{k_r}\int_{\Omega}\nabla_r q_r d\Omega = -\frac{\gamma_w}{k_r}\int_S q_r dS_{\perp}, \tag{20}$$

where

$$q_r = -\frac{k_r}{\gamma_w}\nabla_r u \tag{21}$$

and $dS_{\perp}$ is the infinitesimal surface area perpendicular to flow. In the same manner we have

$$\int_{\Omega}\nabla_z^2 u d\Omega = -\frac{\gamma_w}{k_z}\int_{\Omega}\nabla_z q_z d\Omega = -\frac{\gamma_w}{k_z}\int_S q_z dS_{\perp}. \tag{22}$$

Combining Eqs.17, 20 and 22 we have

$$\Omega\frac{\partial\bar{u}}{\partial t} = c_r\frac{\gamma_w}{k_r}\int_S q_r dS_z + c_z\frac{\gamma_w}{k_z}\int_S q_z dS_r. \tag{23}$$

Considering

$$dS_z = 2\pi r_p dz,\ dS_r = 2\pi r dr \tag{24}$$

we have

$$\Omega\frac{\partial\bar{u}}{\partial t} = 2\pi K r_p\int_0^H q_r dz + 2\pi K\int_{r_0}^{r_p} q_z r dr. \tag{25}$$

Let us assume a pore pressure distribution in which separation of variables according to

$$u(r,z,t) = u(r_0,H,0)f(r)f(z)f(t) \tag{26}$$

is possible in the same manner as we did earlier while formulating the analytical solutions. Only this time, the shape of the pore pressure distribution is assumed to remain the same throughout the consolidation process. Then, we obtain

$$q_r = -\frac{k_r}{\gamma_w}\frac{\partial u(r,z,t)}{\partial r} = -k_r u(r_0,H,0)f(z)f(t)\frac{\partial f(r)}{\partial r}, \tag{27}$$

$$q_z = -\frac{k_z}{\gamma_w}\frac{\partial u(r,z,t)}{\partial r} = -k_z u(r_0,H,0)f(r)f(t)\frac{\partial f(z)}{\partial z} \tag{28}$$

and

$$\bar{u} = \frac{1}{\Omega}2\pi u(r_0,H,0)f(t)\int_{r_0}^{r_p} rf(r)dr\int_0^H f(z)dz. \tag{29}$$

Combining Eqs.25, 27, 28 and 29 we have

$$\Omega\frac{\partial\bar{u}}{\partial t} = 2\pi u(r_0,H,0)\int_{r_0}^{r_p} rf(r)dr\int_0^H f(z)dz\frac{\partial f(t)}{\partial t} \tag{30}$$

$$\int_0^H q_r dz = -\frac{k_r}{\gamma_w}u(r_0,H,0)f(t)\left.\frac{\partial f(r)}{\partial r}\right|_{r=r_p}\int_0^H f(z)dz \tag{31}$$

$$\int_{r_0}^{r_p} q_z r\,d\,r = -\frac{k_z}{\gamma_w}u(r_0,H,0)f(t)\left.\frac{\partial f(z)}{\partial z}\right|_{z=H}\int_{r_0}^{r_p} rf(r)dr \tag{32}$$

Eq.30 becomes

$$\int_{r_0}^{r_p} rf(r)dr \int_0^H f(z)dz \frac{\partial f(t)}{\partial t} =$$
$$-c_r r_p \frac{\partial f(r)}{\partial r}\Big|_{r=r_p} \int_0^H f(z)dz\, f(t) -$$
$$c_z \frac{\partial f(z)}{\partial z}\Big|_{z=H} \int_{r_0}^{r_p} rf(r)dr\, f(t). \quad (33)$$

Let

$$A = \int_{r_0}^{r_p} rf(r)dr \int_0^H f(z)dz, \quad (34)$$

$$B = \frac{\partial f(r)}{\partial r}\Big|_{r=r_p} \int_0^H f(z)dz, \quad (35)$$

and

$$C = \frac{\partial f(z)}{\partial z}\Big|_{z=0} \int_{r_0}^{r_p} rf(r)dr. \quad (36)$$

Rearranging Eq.34 and making use of Eqs.34-36, we obtain

$$A\frac{\partial f(t)}{\partial t} = -\{c_r r_p B + c_z C\} f(t). \quad (37)$$

This gives us the time dependence function

$$f(t) = exp\left\{-\left(\frac{c_r r_p B + c_z C}{A}\right)t\right\}. \quad (38)$$

We have now obtained the desired result, *i.e*., the function that describes how the pore pressure filed changes with time as follows

$$u_w(r,z,t) =$$
$$u(r_0, H, 0) f(r) f(z)\, exp\left\{-\left(\frac{c_r r_p B + c_z C}{A}\right)t\right\}. \quad (39)$$

This result is simple to use and is also flexible to accommodate different pore pressure profiles. A particular formula is obtained when $f(r)$ and $f(z)$ are specified.

The degree of consolidation may then be defined as in before as

$$U = 1 - \frac{u(r,z,t)}{u(r,z,0)} = 1 - exp\left\{-\left(\frac{c_r r_p B + c_z C}{A}\right)t\right\}. \quad (40)$$

The desired properties $\lim_{t\to 0} U = 1$ and $\lim_{t\to\infty} U = 0$ are satisfied. We can also solve for the consolidation time as

$$t = \frac{\bar{T} r_0^2}{c_r}, \quad (41)$$

where $\overline{T}$ is a time factor and given as

$$\bar{T} = \frac{A\, ln(1-U)}{r_0^2\left(r_p B + \frac{c_z}{c_r} C\right)}. \quad (42)$$

A, B and C are determined once $f(r)$ and $f(z)$ are specified. Before proceeding to a particular solution, let us show how A, B and C are related to each other.

$$BC = \frac{\partial f(r)}{\partial r}\Big|_{r=r_p} \frac{\partial f(z)}{\partial z}\Big|_{z=0} \int_0^H f(z)dz \int_{r_0}^{r_p} rf(r)dr \quad (43)$$

$$BC = \frac{\partial f(r)}{\partial r}\Big|_{r=r_p} \frac{\partial f(z)}{\partial z}\Big|_{z=0} A \quad (44)$$

$$\frac{BC}{A} = \frac{\partial f(r)}{\partial r}\Big|_{r=r_p} \frac{\partial f(z)}{\partial z}\Big|_{z=0} \quad (45)$$

Note that the righthand terms are the hydraulic gradients at the boundaries, $z = 0$ and $r = r_p$. We can thus alternatively write

$$\frac{BC}{A} = i_{r_p} i_{z=0} \quad (46)$$

where i_{r_p} is the hydraulic gradient at the boundary $r = r_p$ and i_H is the hydraulic gradient at $z = 0$.

Now, let us consider functions for $f(r)$ and $f(z)$ according to parabolic functions given as:

$$f(r) = 1 - \left(\frac{r-r_0}{r_p-r_0}\right)^2; f(z) = 1 - \left(\frac{z-H}{H}\right)^2. \quad (47)$$

It is further assumed that the pore pressure follows the same distribution throughout the consolidation process. This may not be entirely true but we hold it to be accurate enough when we are interested in ranges of higher degree of consolidation. If for instance Terzaghi's [2] solution for the differential equation that describes the one-dimensional consolidation is considered, for small time, t from the start of consolidation, the curve of $f(z,t)$ is more accurately described by an error function. The curves become more and more parabolic towards the steady state condition as $t \to \infty$. For the degree of consolidation greater than 25%, the difference between the exact analytical solution and the approximation with a parabolic pore pressure profile is practically small and may hence be considered accurate enough.

For parabolic functions $f(r)$ and $f(z)$ given in Eqs.47 we obtain

$$i_{z=0} = -\frac{2}{H} \text{ and } i_{r_p} = -\frac{2}{r_p - r_0} \quad (48)$$

and therefore

$$A = \int_{r_0}^{r_p} r\left[1 - \left(\frac{r-r_0}{r_p-r_0}\right)^2\right]dr \int_0^H \left[1 - \left(\frac{z-H}{H}\right)^2\right]dz$$
$$= \frac{-4r_p^3 r_0 + 3r_p^4 + 12 r_p r_0^3 - 5r_0^4 - 6r_p^2 r_0^2}{12(r_p - r_0)^2} . \frac{2H}{3}$$
$$\text{for } r_p >> r_0, A \approx \frac{1}{6} H r_p^2, \quad (49)$$

$$B = r_p i_{r_p} \int_0^H \left[1 - \left(\frac{z-H}{H}\right)^2\right] dz = \frac{2}{3} r_p i_{r_p} H, \tag{50}$$

$$C = \left.\frac{\partial f(z)}{\partial z}\right|_{z=0} \int_{r_0}^{r_p} r f(r) dr = i_{r_p} i_H \frac{A}{B}. \tag{51}$$

Accordingly, $u(r,z,t)$, the degree of consolidation, U, and the time factor $\overline{T}$ can be determined making use of Eqs.39 and 41 respectively. Example plots of the normalized *plastified* radius versus the time factor is shown in Fig.5. The solid lines are from the approximate solutions and the points are from measurements [5].

The pore pressure field equation can now be summarized as

$$u(r,z,t) = u(r_0,H,0) f(r,t) f(z,t) \tag{52}$$

in which,

$$f(r,t) = f(r)\, exp\left\{-c_r r_p \frac{B}{A} t\right\} \tag{53}$$

and

$$f(z,t) = f(z)\, exp\left\{-c_z \frac{C}{A} t\right\}. \tag{52}$$

The solution can be easily adapted for other types of pore pressure profiles.

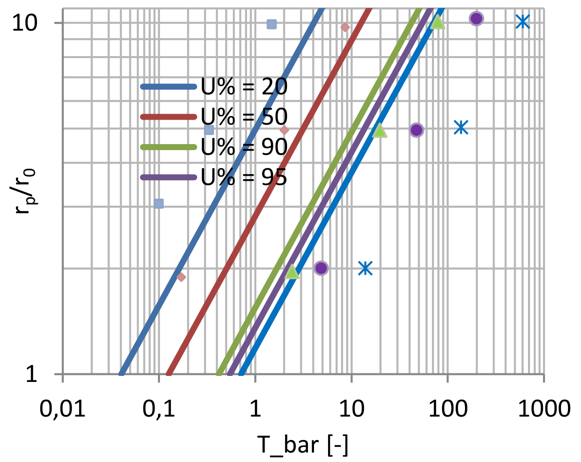

Figure 5. Time factor with normalized plastified radius for different degrees of consolidation: Theory (lines) and Experiment (points)

4.2. Reduced seepage surface due to multiple pile neighbors and other impermeable boundary conditions

There are many conditions in practice where it may be appropriate to consider portions of the seepage boundaries as impermeable. For instance, the radial seepage boundary may get reduced in the case of multiple piles, if the piles are placed at less than a distance of $2r_p$ from each other. See Fig.6 for the illustration for the case of two neighboring piles.

Next, we will devise a simple approach for the consideration of the impermeable portions of the seepage boundaries. Recall that

$$\Omega \frac{\partial \bar{u}}{\partial t} = c_r \frac{\gamma_w}{k_r} \int_S q_r dS_z + c_z \frac{\gamma_w}{k_z} \int_S q_z dS_r. \tag{53}$$

Let β_i be the central angle subtended by the impermeable portion of the radial seepage surface of radius r_p. Similarly, let θ_i be the central angle subtended by the impermeable portion of the vertical seepage boundary. The infinitesimal area of the seepage surfaces in the respective flow directions are then written as

$$dS_z = (2\pi - \Sigma\beta_i) r_p dz,\; dS_r = (2\pi - \Sigma\theta_i) r dr. \tag{54}$$

The summation implies that all β_i's and θ_i's are to be summed in their respective directions. Accordingly, Equation (53) is written as

$$\Omega \frac{\partial \bar{u}}{\partial t} = (2\pi - \Sigma\beta_i) K r_p \int_0^H q_r dz + (2\pi - \Sigma\theta_i) K \int_{r_0}^{r_p} q_z r dr \tag{55}$$

which leads us to

$$2\pi A \frac{\partial f(t)}{\partial t} = -\left\{(2\pi - \Sigma\beta_i) c_r r_p B + (2\pi - \Sigma\theta_i) c_z C\right\} f(t) \tag{56}$$

and thereby

$$f(t) = exp\left\{-\left(\frac{(2\pi - \Sigma\tilde{\alpha}_i) c_r r_p B + (2\pi - \Sigma\theta_i) c_z C}{2\pi A}\right) t\right\}. \tag{57}$$

The pore pressure field is then given by

$$u_w(r,z,t) = u(r_0,H,0) f(r) f(z).\, exp\left\{-\left(\frac{(2\pi - \Sigma\beta_i) c_r r_p B + (2\pi - \Sigma\theta_i) c_z C}{2\pi A}\right) t\right\} \tag{58}$$

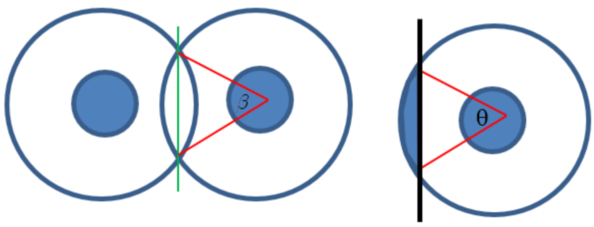

Figure 6. Possible seepage surface reductions due to impermeable obstructions

where A, B and C are as defined elsewhere. Following this result, the degree of consolidation may be defined as

$$U = 1 - \frac{u_w(r,z,t)}{u_w(r,z,0)} = 1 - exp\left\{-\left(\frac{(2\pi-\Sigma\beta_i)c_r r_p B+(2\pi-\Sigma\theta_i)c_z C}{2\pi A}\right)t\right\} \tag{59}$$

We can also solve for the consolidation time as

$$t = \frac{\bar{T} r_0^2}{c_r} \tag{60}$$

where,

$$\bar{T} = \frac{2\pi A \, ln(1-U)}{r_0^2\left((2\pi-\Sigma\beta_i)r_p B+(2\pi-\Sigma\theta_i)\frac{c_z}{c_r}C\right)}. \tag{61}$$

The higher $\sum \beta_i$ and/or $\sum \theta_i$, the higher is the consolidation time required for achieving a given degree of consolidation. The pore pressure field can also be written as

$$u_w(r,z,t) = u(r_0,H,0)f(r,t)f(z,t), \tag{62}$$

in which

$$f(r,t) = f(r)\, exp\left\{-(1-\frac{1}{2\pi}\Sigma\beta_i)c_r r_p \frac{B}{A} t\right\} \tag{63}$$

and

$$f(z,t) = f(z)\, exp\left\{-(1-\frac{1}{2\pi}\Sigma\theta_i)c_z \frac{C}{A} t\right\}. \tag{64}$$

For long piles, practically, the predominant consolidation is radial. Thence,

$$u(r,z,t) \approx u(r_0,H,0)f(r,t) \tag{65}$$

may be considered. In that case the time factor will be

$$\bar{T}_r = \frac{2\pi\, ln(1-U)}{r_0^2(2\pi-\Sigma\beta_i)r_p}\frac{A}{B} \tag{66}$$

Let us denote the timefactor, $T_{\beta,r}$ for the pile whose seepage surface in radial direction is defined by $2\pi - \Sigma\beta_i$ ($\Sigma\beta_i$ being the sum of all central angles subtended by the seepage obstractions in the radial directuion). The ratio of the time factor $T_{\beta,r}$ to the time factor T is then given by

$$\frac{\bar{T}_{\beta,r}}{\bar{T}_r} = f_\alpha = \frac{2\pi}{2\pi-\Sigma\beta_i} \tag{67}$$

Fig.7 shows the plot of values of f_α gainst $\frac{\Sigma\beta_i}{\pi}$.

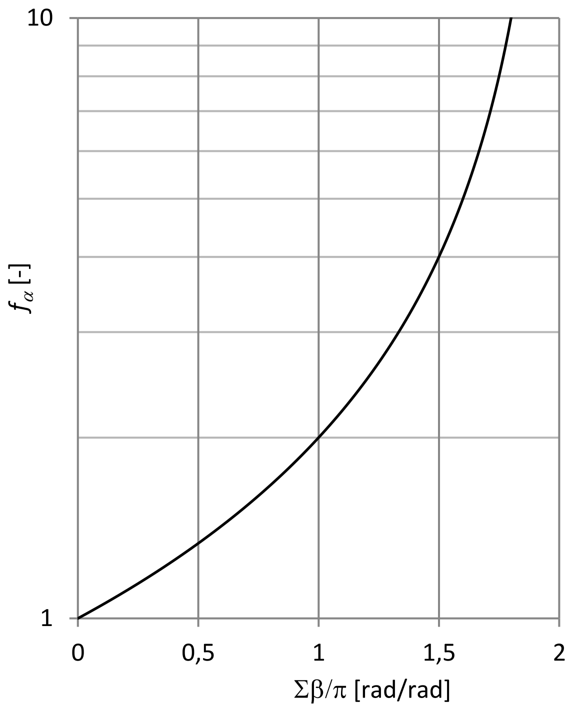

Figure 7. : Modified hydrodynamic period for a pile group with interfering angle $\Sigma\beta_i$ normalized by a hydrodynamic period for unobstructed consolidation. Only radial consolidation is considered.

5. Conclusions

We have presented solutions for describing a consolidation processes in a cylindrical coordinate system. We began with Finite Difference discretization of the differential equation of consolidation which can be used for solving the problem numerically for more general conditions when it is not possible to assume a homogeneous material and simple boundary conditions. We then looked for possible exact analytical solutions and finally, we presented simplified approximations by fixing the shape of the pore pressure profile and the boundary conditions in time and employing the Gauss divergence theorem for describing the average consolidation processes. The solutions so obtained may differ from the exact analytical solutions at the beginning of the consolidation process. The propagation of the pore pressure front in radial direction is also not considered. However, such solutions can be employed for simple approximations and elucidating qualitative aspects of the consolidation process. The approximate solutions can also be valuable for sharpening engineering judgement of the process. The approach can be employed when problems of similar nature are encountered. For instance, they can be for approximating effect of drains on the consolidation processes.

In this paper, we have treated the problem as a decoupled problem in such a way it is accessible to analytical solutions. The problem can be investigated in a more general setup using advanced numerical methods, such as FEM, where coupled flow and deformation phenomena can be considered.

A tribute

…to my loving grandmother, the late **Emahoy Truwork Asfaw** who devoted her life to raising me with aboundance of unconditional love whom I miss dearly.

Symbols

Variables

$i_{z=0}$: Hydraulic gradient at z =0
i_{r_p}: Hydraulic gradient at $r = r_p$
n: Unit normal to the surface area, S
q: Seepage (with sucscrpt r-> radial direction, with subscript z-in the vertical direction)
r: Radial distance
r_0: Pile radius
r_p: Plastified radius
S: Surface area
u: pore pressure
$\bar{u}$: Average pore pressure (over a volume)
U: Degree of consolidation
t: Time
$\bar{T}$: Time factor
z: Vertical distance
Ω: Volume element

Parameters/coefficients/constants
β_i: Central angle subtended by obstructions in the radial seepage direction
c_r: Radial consolidation coefficient
c_z: Vertical consolidation coefficient
γ_w: Unit weight of water
H: Pile height (length) or half the pile length in the case of two way drainage
k_r: Permeability in the radial direction
k_z: Permeability in the vertical direction
c_i, c_j: coefficients that depend on the initial pore pressure profile
λ_i: Eigen values of the Bessel function, J_0
θ_i: Central angle subtended by obstructions in the vertical seepage direction
Functions
J_0: Bessel function of the first kind and order zero
$f(r)$: A function that describes the pore pressure profile in the radial direction
$f(z)$: A function that describes the pore pressure profile in the vertical (z) direction
Operators
d: Infinitesimal increment
Δ: Finite increment
$\frac{\partial}{\partial x}$: Partial differential wrt the variable x
$\nabla = \frac{\partial}{\partial x}$: Spatial variation (with subscript r-in the r direction, with subscript z, in the z (vertical) direction)
$\nabla^2 = \frac{\partial^2}{\partial x^2}$: Laplacian (with subscript r-in the r direction, with subscript z, in the z (vertical) direction)

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