Random bearing capacity evaluation of shallow foundations for Prandtl's and multi-block failure mechanisms with spatial averaging and self-weight of soil included

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ABSTRACT

An evaluation of bearing capacity probabilistic characteristics based on a kinematical approach is investigated in the study. The author utilize the Prandtl failure mechanism and a multi-block failure mechanism resulting from a kinematical method of limit analysis in conjunction with Vanmarcke's spatial averaging approach. The differences in the resulting bearing capacity for both types of failure mechanisms were investigated. Moreover, the presented approach examines the impact of asymmetrical mechanism that may occur in spatially variable soil deposit, the influence of self-weight of soil and the effect of anisotropy in random characteristics. To optimize the geometry of failure, the so-called "simulated annealing scheme" was utilized, and a numerical algorithm for the applied method was created. Finally, for reliability computations, the crude Monte Carlo method was applied. Reliability indices were evaluated for symmetrical and asymmetrical cases for two foundation widths and two soil conditions; the impact of vertical and horizontal fluctuation scales on reliability levels was investigated. The results show that for higher values of horizontal fluctuation scales, the difference between a symmetrical and asymmetrical approach becomes negligible; however, for smaller values, it can be significant.

1 Introduction

According to spatial variability in mechanical properties of homogenous soil layers and limited information about the soil deposits, the probabilistic approach is commonly used in geotechnics. In this study, as a deterministic background, the Prandtl failure mechanism (Prandtl, 1920) and multi-block failure mechanism (Michałowski, 1997) were utilized. Incorporation of the kinematical failure mechanisms requires specified soil strength parameters on each slip line. Thus, to use these mechanisms in the probabilistic analysis the adaptation of their failure geometry and corresponding bearing capacity formula is necessary. In order to include the spatial variability of soil strength parameters the Vanmarcke (1977, 1983) spatial averaging approach was utilized. Application of the multi-block failure mechanism allows to preserve the kinematical admissibility of failure mechanism and consider the inclusion of weight of soil. However, the application of the multi-block failure mechanism requires dedicated optimization procedure. This procedure allows to optimized failure geometry in order to obtain the minimum bearing capacity. Due to large number of parameters subjected to changes the simulated annealing scheme (Kirkpatrick et al. 1983; Kirkpatrick 1984) was chosen. In this study the plane strain conditions were assumed; therefore, the two dimensional failure geometries were analysed.

2 Adaptation of deterministic failure mechanisms

2.1 The Prandtl failure mechanism

Adaptation of the Prandtl failure mechanism to the probabilistic analyses was performed in earlier papers (Puła 2004, Puła & Chwała 2015). The bearing capacity formula in the case of subjecting different values of the soil strength parameters on each slip line can be expressed as in the following formula:

$$p = \frac{bp'}{\sin\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right)} \tag{1}$$

Where p' is expressed by Eq. (2)

$$p' = p'_c + p'_q + p'_\gamma \tag{2}$$

Three terms in Eq. (2) correspond to cohesion, overburden pressure and weight of soil, respectively. They can be expressed by Eqs. (3) – (5).

$$p'_c = c_1 \frac{\cos\varphi_1}{2\sin\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)} + c_3(\exp(\pi\tan\varphi_2) - 1)\frac{1}{2\sin\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)\tan\varphi_2}$$
$$+ c_3\exp(\pi\tan\varphi_2)\frac{\cos\varphi_3}{2\sin\left(\frac{\pi}{4} - \frac{\varphi_1}{2}\right)} \tag{3}$$

$$p'_q = q\exp(\pi\tan\varphi_2)\cos\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)\frac{\cos\left(\frac{\pi}{4}+\frac{\varphi_1}{2}-\varphi_3\right)}{\sin\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)} \tag{4}$$

$$\begin{aligned} p'_\gamma = {} & b\left(-\frac{1}{4}\gamma\cos\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)\right)+\frac{b\gamma}{2(1+9\tan^2\varphi_2)4\sin^2\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)}\times \\ & \times\left(\left(3\tan\varphi_2\sin\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)-\cos\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)\right)\exp\left(\frac{3}{2}\pi\tan\varphi_2\right)\right. \\ & \left.+\left(3\tan\varphi_2\sin\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)\right)\right) \\ & +b\frac{\gamma\cos\varphi_3\exp\left(\frac{3}{2}\pi\tan\varphi_2\right)\cos\left(\frac{\pi}{4}+\frac{\varphi_1}{2}-\varphi_3\right)}{8\sin^2\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)} \end{aligned} \tag{5}$$

Parameters φ_1, φ_2, φ_3, c_1, c_2 and c_3 are assigned to slip lines as shown in Fig. 1.

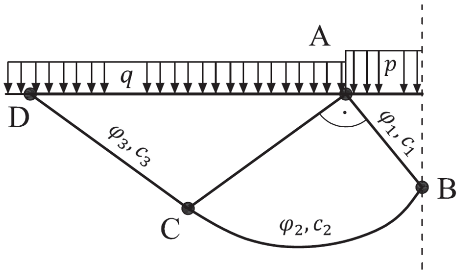

Fig. 1: Assignment of the soil strength parameters to the slip lines. Due to the symmetry only left side of failure mechanism is shown.

The kinematical admissibility of the solution cited above is no longer preserved. This is caused by the fact, that the geometry of the Prandtl failure mechanism is strictly determined according to weightless soil assumption; therefore, there is no possibility in adjusting its shape. However, despite the presented adaptation does not meet the upper bound condition it can be utilized in the probabilistic evaluation of bearing capacity. The received response (bearing capacity) should be interpreted as an average resulting from soil spatial variability.

2.2 The multi-block failure mechanism

Incorporation of the multi-block failure mechanism allows to preserve the kinematical admissibility of failure mechanism. The bearing capacity formula after the adaptation to the probabilistic analyses can be expressed as in Eq. (6).

$$p = 2\left(\sum_{i=1}^{n-1} c_i l_i \cos\varphi_i\, v_{ii+1} + \sum_{i=2}^{n-1} c_{i+n-1} l_{ii+1}\cos\varphi_{i+n-1}\, v_{i+1}\right) + 2l_n v_{n|} q + 2\sum_{i=1}^{n} g_i v_{i|} \tag{6}$$

Where n denotes the number of rigid block on the one side of the failure mechanism, c_i and φ_i are soil strength parameters, l is the length of specified slip line, v is the velocity discontinuity (index $v_|$ denotes the vertical component) and g is the gravitational force acting on each block. The three terms presented in Eq. (6) are dedicated to cohesion, overburden pressure and weight of soil, respectively. The above formula is derived for symmetrical failure mechanism; thus, the factor 2 was added. In the case of asymmetrical failure mechanism the summation has to be performed for all slip lines (there is no multiplying by 2). In Fig. 2 the asymmetrical failure geometry corresponding to $n = 6$ is shown.

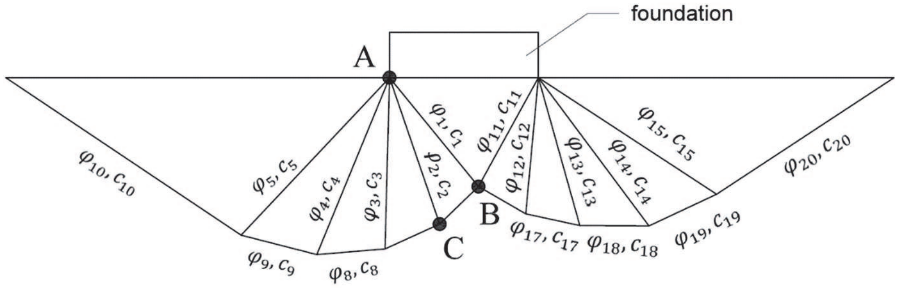

Fig. 2: Assignment of the soil strength parameters to the slip lines in the case of asymmetrical failure mechanism.

2.2.1 Optimization procedure

According to Introduction, the usage of multi-block failure mechanism requires the optimisation procedure. Due to multi-parameter issue, the simulated annealing scheme (Kirkpatrick et al. 1983; Kirkpatrick 1984) was chosen and applied to the considered problem. The detailed description of the proposed optimization procedure is given in the paper by Puła & Chwała (2018). The method has a stochastic background and is based on iterative process which is controlled by the probability of acceptance of a worse solution.

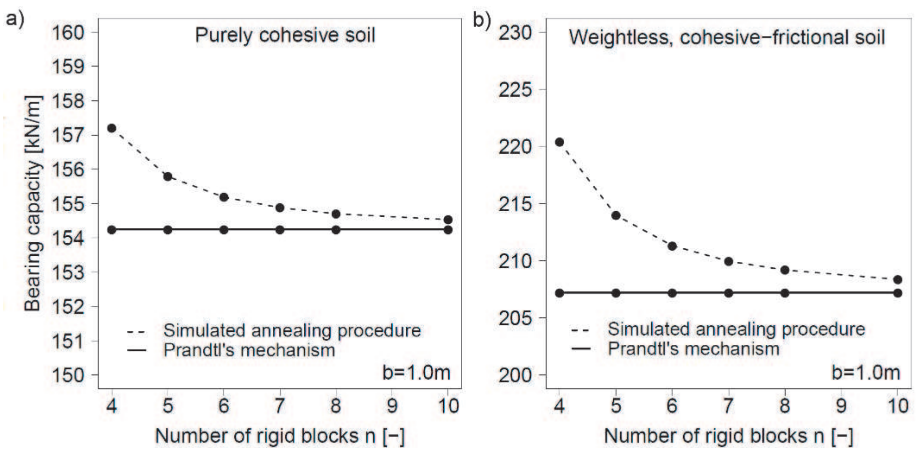

Fig. 3: Comparison of the exact Prandtl solution and multi-block mechanism versus a varying number of blocks: a) purely cohesive soil c = 30kPa, b) cohesive frictional soil φ = 25°, c = 10kPa. Analyses were performed for foundation width b = 1.0 m. note that the vertical axes started at 150 and 200 kN/m.

In this study, the minimum value of the formula for bearing capacity is searched; the result of the optimization is the corresponding geometry of failure. In order to test the correctness of the optimization procedure, the well-known solutions were compared with those obtained by the optimization procedure. The resulting bearing capacities are shown in Fig. 3. Fig 3a concerns bearing capacity for purely cohesive soil as a function of rigid block number. In Fig 3b the analogous results for weightless soil are shown. According to Fig. 3 the convergence of the result obtained for the Prandtl failure mechanism and for the proposed optimization procedure with an increase in rigid block number is clearly visible. However, the reasonable compromise between accuracy and computational time has to be accepted. Therefore, for further calculations six block failure mechanism was chosen.

3 Spatial averaging

The soil strength parameters, i.e. angle of internal friction and cohesion were modelled as lognormal random fields. Such description is necessary when spatial variability of soil parameters is considered and when relatively large volume of soil participate in the failure mechanism. Advantages of such description were indicated in e.g. Fenton & Griffiths (2008), ISO 2394 (2015) or Rackwitz (2000). According to procedure proposed by Vanmarcke (1977, 1983) the averaged random field of property X (in this study, it is the random field of the angle of internal friction φ or cohesion c) can be expressed as follows:

$$X_V = \frac{1}{|V|}\iiint_V X(x,y,z)dxdydz \tag{7}$$

Where V is the averaging domain and $|V|$ denotes its measure. If a kinematical failure mechanism is considered, the natural areas of averaging are slip lines. Therefore, the random fields of strength parameters are averaged along slip lines; consequently, V denotes a segment of a straight line (or logarithmic spiral for the Prandtl mechanism), and $|V|$ denotes its length. In the present study due to the assumption of the stationary random fields, the mean value of a new random field (after averaging) is preserved. However, its variance are reduced. The level of reduction depends on the choice of V. The variance of the new random field X_V is given by Eq. (8).

$$\mathrm{Var}(X_V) = \sigma_V^2 = \gamma(V)\sigma_X^2 \tag{8}$$

In each field, the correlation structure is assumed to be characterized by the Gaussian covariance function:

$$\mathrm{R}(\Delta x, \Delta z) = \sigma_X^2 \exp\left\{-\left[\left(\frac{\Delta x}{\omega_2}\right)^2 + \left(\frac{\Delta y}{\omega_1}\right)^2\right]\right\} \tag{9}$$

where Δx and Δy are the distances along the x and y axis. Distinguishing between the parameters ω_1 and ω_2 leads to anisotropy in the correlation structure. Both parameters are related to the fluctuation scale (Fenton & Griffiths, 2008) in the following way:

$$\omega_1 = \frac{\theta_v}{\sqrt{\pi}}, \quad \omega_2 = \frac{\theta_h}{\sqrt{\pi}} \tag{10}$$

where θ_v and θ_h denote the vertical and horizontal fluctuation scales, respectively. The averaging procedure leads to discretization of the random field such that after averaging along the line l_i, a single random variable X_{l_i} is obtained. The all covariances between each two variables X_{l_i} and X_{l_j} in the case of two dimensional failure mechanisms was derived. As an example, the formula for the variance of the random variable X_{l_1} (slip line AB, see Fig. (2)) is given below:

$$\mathrm{Var}(X_{AB}) = \sigma_X^2 \int_0^1 \int_0^1 \exp\left[-\left(\frac{(x_B - x_A)(t_1 - t_2)}{\omega_2}\right)^2\right] \exp\left[-\left(\frac{(y_B - y_A)(t_1 - t_2)}{\omega_1}\right)^2\right] dt_1 dt_2 \tag{11}$$

where x_A, y_A, x_B and y_B are the Cartesian coordinates of points A and B. To use the above procedure, the variances and covariances for all considered slip lines have to be derived. In the case of the Prandtl failure mechanism the covariance matrix consists of 9 components (there are three slip lines); however, for symmetrical failure mechanism there are 100 components (10 slip lines) and for asymmetrical one there are 400 components (20 slip lines, see Fig. 2). The covariance matrix is symmetrical and positive definite.

5 Numerical algorithm

For the purpose of numerical analysis the dedicated numerical algorithm was proposed. Note, that in the case of the Prandtl failure mechanism some steps (in the procedure presented below) can be abandoned. The algorithm was implemented in Mathematica software (Wolfram Research, 2010). Detailed description of the numerical procedure in the case of multi-block mechanism can be found in recent paper by Puła & Chwała (2018). The main steps of the algorithm are shown below:

Step 1. Generate independent soil strength parameters $\varphi_1, \dots, \varphi_n$ and $c_1, \dots, c_n$ on each slip line (based on initial probabilistic characteristics).

Step 2. Find the optimal geometry of the failure mechanism for the soil parameters from Step 1.

Step 3. Determine the covariance matrix for the failure geometry obtained in step 2, then compute the new set of averaged soil strength parameters bounded by the covariance matrix ($\overline{\varphi_1}, \dots, \overline{\varphi_n}$ and $\overline{c_1}, \dots, \overline{c_n}$).

Step 4. Find the optimal geometry of the failure mechanism for the soil parameters from Step 3. Compute the resulting bearing capacity value.

Step 5. Repeat steps 1, 2, 3 and 4 *N* times. *N* should be sufficient to evaluate reliability measures or approximate the probability distribution to a histogram of bearing capacity.

By repeating numerical procedure *N* times (as mentioned in step 5), the bearing capacity histogram is obtained. The number of simulation *N* was set individually to each considered issue. The lognormal theoretical probability distribution function was chosen based on the shape of the bearing capacity histograms. For each of the histogram the approximation of theoretical probability distribution was carried out. Therefore, based on the fitted probability distribution of bearing capacity (p_{fit}), the probability of failure (P_f) can be defined as:

$$P_f = \mathrm{P}\left\{p_{fit} < \frac{p_{exp}}{F}\right\} \tag{12}$$

Where p_{exp} is the value of bearing capacity calculated for expected values of soil strength parameters, that are specific to the given issue. The reliability index β can be established by the well-known formula, Eq. (13):

$$\beta = -\Phi^{-1}[P_f] \tag{13}$$

where Φ^{-1} is the inverse function of the cumulative distribution function of the standard normal distribution. Fitted probability distributions were tested by Kolmogorov-Smirnov test.

6 Results

The considered soil properties are shown in Tab. 1 and Tab 2.

Tab. 1: Properties concerning undrained conditions.

Parameter	Mean value	Standard deviation	Coefficient of variation	Probability distribution
Unit weight, γ	18.2 kN/m^3	1.092 kN/m^3	0.06	Normal random variable
Undrained shear strength, c_u	57.7 kPa	28.65 kPa	0.497	Lognormal random field
Overburden pressure, q	14.4 kPa	-	-	Non-random

Tab. 2: Properties concerning cohesive-frictional soil.

Parameter	Mean value	Standard deviation	Coefficient of variation	Probability distribution
Unit weight, γ	18.2 kN/m^3	1.092 kN/m^3	0.06	Normal random variable
Angle of internal friction, φ	30.0°	4.5°	0.15	Lognormal random field
Cohesion, c	5.0 kPa	1.0 kPa	0.20	Lognormal random field
Overburden pressure, q	14.4 kPa	-	-	Non-random

Based on the performed analyses the following observations can be drawn:

1) In Fig. 3 the impact of horizontal fluctuation scale on the shape of the bearing capacity histogram is shown. The biggest differences are observed in transition from isotropic case ($\theta_h = \theta_v$) to anisotropic one for $\theta_h = 5\theta_v$. The strong dependence of bearing capacity histogram shape on the fluctuation scales is clearly visible.

2) The observed differences in results obtained for the Prandtl failure mechanism and 6-block symmetrical failure mechanism are shown in Fig. 4. Note, that in the case of the Prandtl mechanism large values of bearing capacity are more frequent than those obtained for symmetrical 6-block failure mechanism. Similar, but inverse behaviour can be observed for lower values of bearing capacity. Therefore, the bearing capacity mean values are lower for 6-block mechanism, however the standard deviations are greater for the Prandtl mechanism. This can be explained by the consideration of weight of soil during the optimization process for 6-block mechanism.

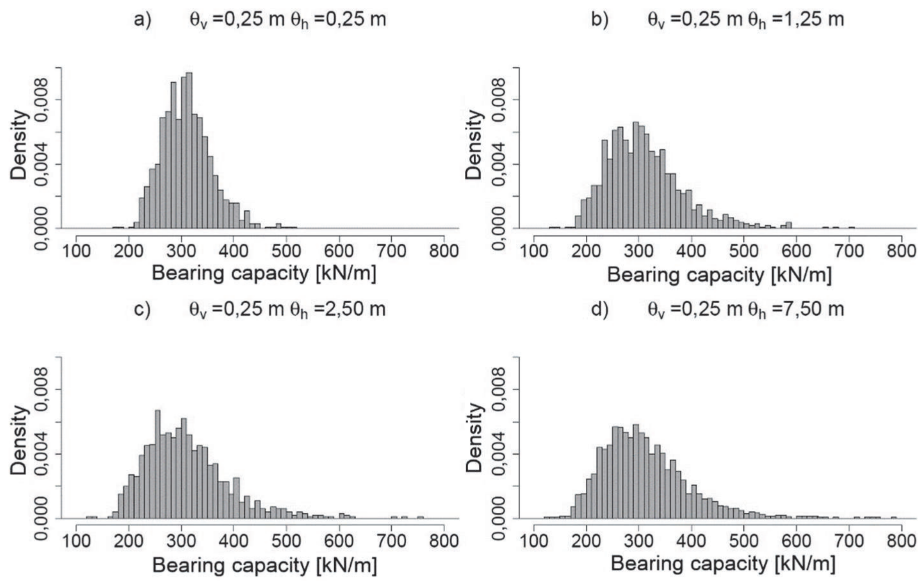

Fig. 3: Example bearing capacity histograms obtained for the Prandtl failure mechanism, undrained conditions, b=1,0 m, vertical fluctuation scale 0.25 m.

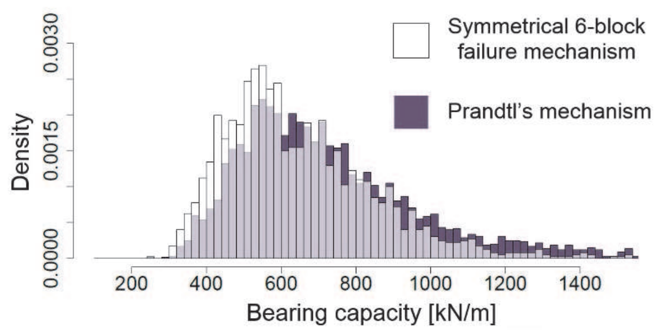

Fig. 4: Comparison of the bearing capacity histograms obtained for cohesive-frictional soil for the Prandlt failure mechanism and symmetrical 6-block failure mechanism, $\theta_v = 0.50$ m and $\theta_h = 30\theta_v$.

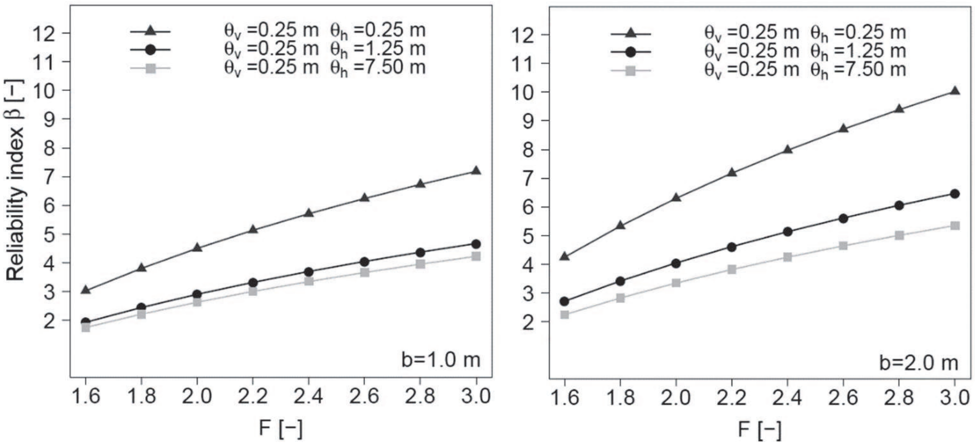

Fig. 5: Reliability indices as a function of global safety factor F obtained for undrained conditions, Prandtl failure mechanism and two foundation widths.

3) The reliability indices as a function of the global safety factor are shown in Fig. 5. Presented results were obtained for the undrained conditions by utilizing Prandtl failure mechanism. The presented results indicate crucial rule of horizontal fluctuation scale in reliability approach. Thus, the analysis performed for isotropic case highly underestimate the failure probability. This indicates the importance of research on horizontal fluctuation scales in the natural soils.

4) Analogous result to those presented in section 3), but obtained for cohesive-frictional soil are shown in Fig. 6. The conducted analyses concerns two types of the multi-block failure mechanism; namely: symmetrical and asymmetrical. Dashed lines are related to reliability indices obtained in the case of asymmetrical failure mechanism. Similarly like for the Prandtl mechanism the strong influence of horizontal fluctuation scale can be observed. Moreover, the differences in reliability indices between symmetrical and asymmetrical mechanism are significant for isotropic cases and becomes negligible in the case of strong anisotropy in spatial correlation structure (see Fig. 6).

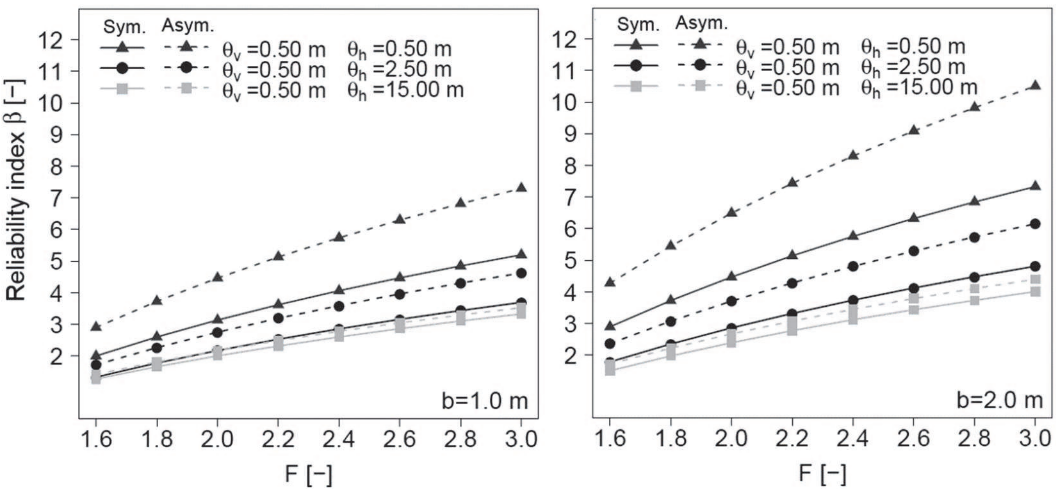

Fig. 6: Reliability indices as a function of global safety factor F obtained for cohesive frictional soil for symmetrical and asymmetrical failure mechanism.

7 Concluding remarks

To summarize, the kinematical failure mechanism can be successfully incorporated in the probabilistic analyses of bearing capacity. Obtained results can be used for reliability assessment. In the case of undrained condition the Prandtl failure mechanism can be utilized; however for soils with friction application of multi-block failure mechanisms result in better evaluation of bearing capacity. The horizontal fluctuation scale is significant for reliability analyses; however, the determination of its value for natural soil is still a matter of concern.

8 Literature

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