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Constitutive modelling of crushable sands

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ABSTRACT: When soil particles break, the particle size distribution (PSD) becomes a variable in the same way as other variables like void ratio, soil structure and anisotropy, etc. To consider particle breakage in a constitutive model, we need three key components: (i) quantification of PSD in a realistically simple manner, (ii) evolution law of PSD during particle breakage, and (iii) influences of PSD on other soil properties like strength and stiffness. This paper summarises the latest advances in the first two components, discusses new ways of quantifying PSD effects, and finally presents a new critical state model where the PSD is treated as a variable. In discussing the PSD effects, we focus on the movement of the critical state line (CSL) due to particle breakage. We introduce a new state parameter and a new evolution law of the CSL. We assume that the CSL shifts downwards in the v—ln$p$ space with increasing particle breakage under relatively low stresses, but all the CSLs for different PSDs converge to a steady state at high stresses where particle breakage eventually stops and is no longer the main mechanism for soil deformation. The proposed model is compared with other constitutive models in the literature and validated against experimental data, which demonstrates satisfactory performance.

Keywords: Granular soil; critical state; state parameter; particle breakage; constitutive modelling

1 INTRODUCTION

Particle breakage occurs in many geotechnical applications, for example, foundations built on carbonate sands, rail ballast under repeated loads, gravels produced from weak rocks like claystone and siltstone. When particles break, the particle size distribution (PSD) of the soil becomes a variable in the same way as other soil variables like pore size characteristics, soil structure and anisotropy. In classical soil mechanics, the pore size characteristics of a soil is usually quantified by the void ratio, which evolves under stress and affects other soil properties. Similarly, we need to establish three key principles to tackle the problem of particle breakage: a parameter or a model that quanitifies soil PSD, a evolution law that describes how PSD evolves with external factors like stress, strain or energy, and constitutive relations that describe the influences of PSD on other soil properies.

The aim of this paper is to propose a constitutive model that can capture the aforementioned features of granular soils. In this paper, some basic observations on particle breakage are firstly discussed. A new and simple evolution law for the critical state line with particle breakage is then proposed. The state-dependent behavior is developed by using the modified state parameter where a new reference compression line (RCL) is proposed. Finally, the proposed model is compared with other existing constitutive model and is validated against experimental triaxial test data in the literature.

2 PARTICLE BREAKAGE

2.1 Ultimate PSD

It has been widely reported that the PSD of granular soils will eventually evolve toward an ultimate steady-state even though the applied stress/strain is extremely large (Coop *et al*., 2004; Tong *et al*., 2018). An ultimate PSD implies that particles with different sizes will not break further as a result of the dynamic balance between the particle size effect and the coordination number effect (McDowell & Bolton, 1998; Tong *et al*., 2019). The ultimate PSD, however is not necessary but commonly assumed to be fractal-graded for most practical cases, which can be expressed as

$$P(d_i) = \left(\frac{d_i}{d_{\max}}\right)^{3-D_u} \tag{1}$$

where $P(d_i)$ is the mass percentage finer than d_i-sized particles, $d_{\max}$ is the maximum particle size, D_u is defined as the ultimate fractal dimension. For the sake of simplicity, D_u is taken as 2.5-2.6 for sands and 2.7 for gravel materials in several studies in the literature (Coop *et al*., 2004).

2.2 Quantification of particle breakage

An appropriate constitutive model of granular soils needs to consider the evolution of PSD during stress path, which means the PSD should be treated as a variable in a constitutive model. In that case, it is necessary to adopt a simple variable that can represent the PSD and measure the degree of particle breakage of a sample, preferably within the range of 0 to 1. A new breakage index B_λ is proposed in this paper, given by:

$$B_\lambda = \frac{\lambda_{pi} - \lambda_{pc}}{\lambda_{pi} - \lambda_{pu}}, \quad \text{with } \lambda_p = \frac{d_{63.2}}{d_{max} - d_{63.2}} \tag{2}$$

where λ_{pi}, λ_{pu}, λ_{pc} are the initial, ultimate, and current values of λ_p, respectively, and $d_{63.2}$ is the characteristic particle diameter at which 63.2% of the sample by mass is smaller, λ_p is a scale parameter controlling the extent of breakage. Tong *et al.* (2020) found a good linear relationship between λ_p and Einav's breakage index B_r^* (Einav, 2007).

The characteristic particle diameter of 63.2% was derived for mathematical convenience in reducing one of the two parameters in a Weibull function, as detailed in Tong *et al.* (2018). We assume that the PSD of a uniformly graded soil follows a two-parameter Weibull functon during particle crushing, and that PSD of any non-uniformly graded soil can be formed by particle crushing of uniformly graded soils (Zhang *et al.* 2015). λ_p is the parameter in the Weibull function that controls the extent of breakage, with κ being the other parameter that controls the type of breakage. When λ_p takes the value as defined in Equation (2), the Weibull function equals 63.2% irrespective of the κ value.

We also note that B_l follows the same idea as Einav (2007) to capture the relationship between the current, initial, and final PSDs. A clear advantage of B_λ is that it is directly measured by the parameter λ_p in the PSD function and controls the breakage extent.

2.3 *Evolution of particle breakage*

A large number of tests have indicated that particle breakage is affected by both stress and strain (e.g. Coop *et al.*, 2004; Tong *et al.*, 2020). Breakage indices are often correlated to energy quantities which are combinations of stress and strain. There is not much difference when using total input work and plastic work because the amount of elastic work is often several orders of magnitude smaller than that of plastic work in many cases, such as in ring shear test (Tong *et al.*, 2020), impact test (Xiao *et al.*, 2016), triaxial test when considerable particle breakage occurs. However, the accumulation of particle breakage during cyclic loading cannot be predicted when using total input work. In general, correlating plastic work with particle breakage indices provides a unified and flexible approach when considering particle breakage in constitutive models subjected to both monotonic and cyclic loading (Hu *et al.*, 2018). The plastic work W^p in a conventional triaxial test can be expressed as follows (Hu *et al.*, 2018):

$$W^p = \int \langle \sigma_{ij} d\varepsilon_{ij}^p \rangle = \int \langle p d\varepsilon_v^p + q d\varepsilon_s^p \rangle \tag{3}$$

where the symbol $\langle \ \rangle$ is the Macaulay's brackets (i.e., $\langle x \rangle = x$, if $x \geq 0$; $\langle x \rangle = 0$, if $x < 0$), σ_{ij} is the stress tensor, p is the mean stress, q is the deviator stress, $d\varepsilon_{ij}^p$ is the plastic strain rate tensor, $d\varepsilon_v^p$ is the plastic volumetric strain rate, and $d\varepsilon_s^p$ is the plastic deviatoric strain rate. The relationship between the plastic work and breakage index can be described by a unified hyperbolic function, regardless of the initial density or stress path. For example, Hu *et al.* (2018) showed that both B_r^* (Einav 2007) and B_u (defined as relative uniformity) could be hyperbolically related to the plastic work by extensive experimental results of different granular soils. Similarly, the relationship between B_λ and W^p can be given as

$$B_\lambda = \frac{W^p}{b \times p_r + W^p} \tag{4}$$

where b is a parameter controlling the evolution rate of PSD, p_r is the unit pressure (= 1 kPa) for ensuring the dimensionally consistency. Again, B_λ ranges from 0 (no plastic work) to 1 (infinite plastic work).

3 EVOLUTION OF CSL

3.1 *CSL and ICL*

The isotropic compression line (ICL) of a granular soil is not unique, and highly depends on its initial void ratio. Those ICLs will eventually converge into a unique line referred to as the Limit Compression Line (LCL) (Pestana & Whittle, 1995), which can be expressed as a straight line in the space of logarithm of void ratio versus logarithm of mean effective stress:

$$\ln(e_{lcl}) = \ln(N) - \lambda \ln(p/p_r) \tag{5}$$

where e_{lcl} is the void ratio on the LCL, N is the void ratio on the LCL when $p = p_r$, λ is the slope of LCL in the ln(e) - ln(p) space. To model the nonlinear CSL and ICLs as observed by many laboratory studies (e.g., Yamamuro & Lade, 1996), and to avoid the negative void ratio at high stresses, a family of the Isotropic Compression Lines (ICLs) can be given by adding one parameter in the Equation (6) (Sheng *et al.*, 2008):

$$\ln(e_{icl}) = \ln(N) - \lambda \ln[(p + p_{icl})/p_r] \tag{6}$$

where e_{icl} is the void ratio on the ICLs, p_{icl} is defined as a shifting stress controlling the curvature of the ICL, which depends on the initial void ratio of sample, i.e., a smaller initial void ratio leads to a larger p_{icl} value. It is also found that the ICLs are considered to be parallel to the CSL at high stresses (e.g. Sheng *et al.*, 2008). A similar form of CSL with Equation (7) is also defined by Sheng *et al.* (2008), which takes the form:

$$\ln(e_{csl}) = \ln(\Gamma) - \lambda \ln[(p + p_{csl})/p_r] \tag{7}$$

where e_{csl} is the void ratio on the CSL, Γ is the void ratio on the CSL when $p + p_{csl}$ = 1kPa, p_{csl} is defined as a shifting stress controlling the curvature of the CSL.

3.2 *Evolution of CSL*

The critical state line (CSL) in the *p* - *q* space can be represented by a straight line and is assumed to be independent of particle breakage in this paper, which is consistent with most studies as discussed previously. The CSL in the *e* - log(*p*) space, however, will significantly be affected by particle breakage. The constitutive framework developed based on the assumption of *parallel shifting* of CSL in the *e* - log(*p*) space (Muir Wood & Maeda, 2008) is considered as an effective approach to model particle breakage of granular soils.

Typical experimental and numerical results show that the CSL shifts downwards with increasing extent of particle breakage under a relatively low stress level (Bandini & Coop, 2011). It should be noted that it is not possible to explore the effect of PSD on the CSL at a high stress level because the PSD at such high stresses is continuously changing and it is not easy to measuring the PSD during the test. Realising that experimental results tend to show a steady-ultimate state of particle breakage, a new evolution law of CSL will be adopted in this paper according to the assumption that the CSLs of samples with various degrees of particle breakage will eventually converge to steady-state at a high stress level where particle breakage completes and is no longer the main deformation mechanism. As shown in Figure 1, the proposed evolution of CSL moves downwards with decreasing slope of CSL at the same mean effective stress as particle breakage progresses. This assumption is reasonable and in consistent with the experimental results by Bandini & Coop (2011), who found that particle breakage will not only result in a downward shift, but also a rotation of the CSL in the *e* - log(*p*) space.

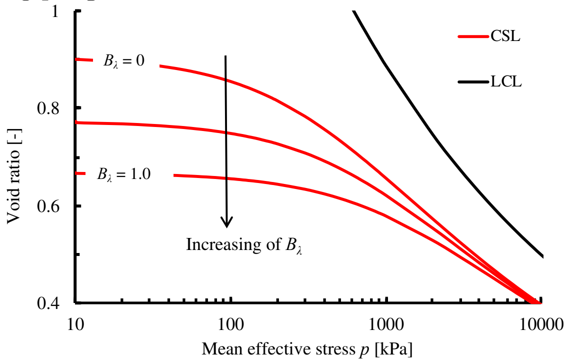

Figure 1. Evolution of CSL with particle breakage

The LCL as shown in Figure 1 will not change as it is an asymptotic line of the ICLs at high stresses, which is similar with the observations by McDowell, who found the compression index (C_c = $(-\Delta e)/\Delta\log(\sigma'_v)$) is independent of the initial PSD (McDowell, 2002) and the slope of LCL in the log (*e*)-log (*p*) space depends on the ultimate fractal dimension, but is independent of the initial PSD. In this case, the parameter *N* and *λ* of the LCL as shown in the Equation (6) are then independent of breakage index B_λ.

4 STATE VARIABLES

4.1 *State variable of Been & Jefferies (1985)*

The commonly-used state parameter proposed by Been & Jefferies (1985) provides a normalised description of granular soils at various mean effective stresses and densities, and has been successfully used in modelling the state-dependent behaviour of many granunlar soils in the literature. However, such a state parameter is not able to capture the effect of particle breakage, see Tong *et al.* (2022).

4.2 *A new reference compression line*

To describe the state-dependent behavior of the crushable soils, new reference lines have been proposed to define the state parameter instead of the traditional CSL. A Reference Compression Line (RCL) that intersects with the CSL at point of p = 0, $e = e_{cs0}$ and converges to the LCL at high stresses is adopted to further modify the state parameter in this paper. Substituting the point of $p = 0$, $e = e_{cs0}$ into (6) gives

$$p_{icl} = \left(\frac{N}{e_{cs0}}\right)^{1/\lambda} p_r \tag{8}$$

The new RCL can be obtained by substituting Equation (8) into (6), which is expressed as

$$\ln(e_{rcl}) = \ln(N) - \lambda \ln\left[p + \left(\frac{N}{e_{cs0}}\right)^{1/\lambda}\right] \tag{9}$$

where e_{rcl} is the void ratio on the RCL. According to Equation (8), the new RCL needs three parameters: the void ratio on the LCL at p = 1kPa, the slope of LCL in the ln(*e*)-ln(*p*) space, and the void ratio on the CSL at p = 0. The third parameter can be adopted as the void ratio on the CSL at a low mean effective stress if the critical state void ratio at p = 0 is not available. For example, the RCL obtained from Equation (8) with $e_{cs0} = e_{csl}$ at p = 10 kPa almost coincides with the CSL, with only some minor differences at low stresses.

4.3 *Modified state parameter considering particle breakage*

The modified state parameter Ψ is then defined as the difference between the current void ratio and the void ratio on the RCL at the same mean effective stress, which can be written as

$$\Psi = e - e_{rcl} = e - \frac{N}{\left[p + \left(\frac{N}{e_{cs0}}\right)^{1/\lambda}\right]^{\lambda}} \tag{10}$$

Since the RCL and CSL intersects at a very low mean effective stress (ideally at $p = 0$), the RCL of a soil sample will shift downwards subsequently at a low stress level with development of particle breakage. Again, all the RCLs of samples with various degrees of particle breakage will eventually converge to the LCL at a high stress level.

5 CONSTITUTIVE MODELLING

5.1 *Elastic strain increment*

The elastic volumetric strain increment and the elastic deviatoric strain increment can be calculated as:

$$\begin{cases} d\varepsilon_{\mathrm{v}}^{\mathrm{e}} = \dfrac{dp}{K} \\ d\varepsilon_{\mathrm{s}}^{\mathrm{e}} = \dfrac{dq}{3G} \end{cases} \tag{11}$$

where the subscripts v and s represent volumetric and deviatoric component, respectively; K and G are the elastic bulk modulus and the elastic shear modulus, respectively, and are dependent on mean effective stress and void ratio. The nonlinear hypoelastic relation proposed by Richart *et al.* (1970) is adopted for calculating the elastic shear modulus

$$G = G_0 \frac{(2.97-e)^2}{(1+e)} \sqrt{p \times p_{\mathrm{r}}} \tag{12}$$

where G_0 is a material constant, and e is the void ratio.

5.2 *Plastic strain increment*

A simple yield surface that plastic deformation occurs whenever there is a change in stress ratio η (= q/p) proposed by Li & Dafalias (2000) for the triaxial compression is adopted:

$$f = q - \eta p = 0 \tag{13}$$

The loading direction (n_{fv}, n_{fs}) is defined as

$$\begin{cases} n_{\mathrm{fv}} = \dfrac{-\eta}{\sqrt{1+\eta^2}} \\ n_{\mathrm{fs}} = \dfrac{1}{\sqrt{1+\eta^2}} \end{cases} \tag{14}$$

The plastic flow direction (n_{gv},n_{gs}) is defined as

$$\begin{cases} n_{\mathrm{gv}} = \dfrac{d_{\mathrm{g}}}{\sqrt{1+d_{\mathrm{g}}^2}} \\ n_{\mathrm{gs}} = \dfrac{1}{\sqrt{1+d_{\mathrm{g}}^2}} \end{cases} \tag{15}$$

where d_{g} is the state-dependent dilatancy equation, and can be written as

$$d_{\mathrm{g}} = \frac{\partial g}{\partial p} \Big/ \frac{\partial g}{\partial q} = d_0 \left(e^{m\Psi} - \frac{\eta}{M} \right) \tag{16}$$

In equation (16), g is the plastic potential function, d_0 and m are two positive material constants. A non-associated flow rule is adopted in this paper. The plastic strain increment can be written as

$$\begin{cases} d\varepsilon_{\mathrm{v}}^{\mathrm{p}} = \dfrac{n_{\mathrm{fv}} n_{\mathrm{gv}}}{H_{\mathrm{p}}} dp + \dfrac{n_{\mathrm{fs}} n_{\mathrm{gv}}}{H_{\mathrm{p}}} dq \\ d\varepsilon_{\mathrm{s}}^{\mathrm{p}} = \dfrac{n_{\mathrm{fv}} n_{\mathrm{gs}}}{H_{\mathrm{p}}} dp + \dfrac{n_{\mathrm{fs}} n_{\mathrm{gs}}}{H_{\mathrm{p}}} dq \end{cases} \tag{17}$$

where H_{p} is the plastic modulus. The expression of H_{p} should satisfy the three conditions as suggested by Li & Dafalias (2000), i.e., (1) $H_{\mathrm{p}} = +\infty$ at $\eta = 0$, (2) $H_{\mathrm{p}} = 0$ at the critical state, and (3) $H_{\mathrm{p}} = 0$ at the drained peak stress ratio. A simplified form of H_{p} as suggested by Liu & Gao (2016) is adopted in this paper

$$H_p = H_0 G \frac{M_{\mathrm{p}}^2 - \eta^2}{\eta} \tag{18}$$

where H_0 is a model constant; M_{p} is the virtual peak stress ratio, which is given as

$$M_{\mathrm{p}} = M e^{[n\langle -\Psi \rangle]} \tag{19}$$

where n is a material constant. As indicated by Equation (19), when the sample is at a loose state ($-\Psi < 0$), we have $M_{\mathrm{p}} = M$ (i.e., hardening); when the sample is at a dense state ($-\Psi > 0$), we have $M_{\mathrm{p}} > M$ (i.e., softening).

5.3 *Stress-strain relationship*

As can be obtained from Equation (11) and Equation (17), the stress-strain relations in the p-q space can be finally written as

$$\begin{pmatrix} d\varepsilon_{\mathrm{v}} \\ d\varepsilon_{\mathrm{s}} \end{pmatrix} = \begin{bmatrix} \dfrac{1}{K} + \dfrac{n_{\mathrm{fv}} n_{\mathrm{gv}}}{H_{\mathrm{p}}} & \dfrac{n_{\mathrm{fs}} n_{\mathrm{gv}}}{H_{\mathrm{p}}} \\ \dfrac{n_{\mathrm{fv}} n_{\mathrm{gs}}}{H_{\mathrm{p}}} & \dfrac{1}{3G} + \dfrac{n_{\mathrm{fs}} n_{\mathrm{gs}}}{H_{\mathrm{p}}} \end{bmatrix} \begin{pmatrix} dp \\ dq \end{pmatrix} \tag{20}$$

6 MODEL VALIDATION

The proposed model has 12 model parameters, which can be obtained by isotropic compression tests and conventional triaxial tests as described in Tong *et al.* (2022).

To validate the proposed model, the experimental data of drained triaxial tests on granular soils in the literature were adopted, i.e., the Cambria sand (Yamamuro & Lade, 1996). Furthermore, a constitutive model named as LB model is adopted in this paper for comparing with the proposed model. The only difference between the proposed model and LB model is the evolution of CSL with the development of particle breakage, where a parallel shifting of CSL is adopted in LB model as proposed in the literature (Muir Wood & Maeda, 2008; Hu *et al.*, 2018). All model parameters are calibrated as discussed in Tong *et al.* (2022) and list in Table 1.

A series of drained and undrained triaxial tests on the Cambria sand were conducted by Lade and Yamamuro (Yamamuro & Lade, 1996). The sand tested, which was composed of two main mineral constituents (i.e., 54%

quartz, and 39% lithic) was uniformly graded with particle sizes between 0.83 and 2 mm. All the samples were prepared with an initial void ratio of 0.52 before isotropic compression. The predicted CSL and ICL using Equations (6)-(7) are compared with the measured CSL and ICL as shown in Figure 2. The RCL is obtained by Equation (9) with known values of N, λ, and $e_{cs, ref}$ (e_{cs0} at $B_\lambda = 0$). It is shown that the proposed functions for CSL and ICL fit well with the measured results. Figure 3 shows the calibration of breakage parameters for the Cambria sand. The ultimate fractal dimension of the Cambria sand for calculating the relative PSD index B_λ is adopted as 2.6. A good agreement is obtained by using Equation (2) with b = 5408 and a = 1.30.

Table 1. Model parameters of the three granular soils

Soil name	Elastic parameters		CSL & ICL related parameters				Breakage parameters		Dilatancy parameters		Hardening parameters	
	G_0	μ	λ	M	N	$e_{CS,ref}$	a	b	m	d_0	n	H_0
Cambria sand	350	0.25	1.12	1.35	96800	0.58	1.30	5408	0.50	2.50	0.60	0.45

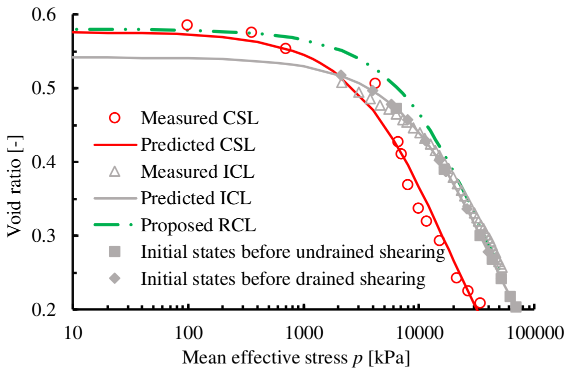

Figure 2. Measured and predicted CSL, ICL and proposed RCL of the Cambria sand.

Figure 4 show the comparison between the measured and the predicted results of drained shearing tests with confining pressure varying between 15.0 MPa and 52.0 MPa. The initial void ratios after isotropic compression at different confining pressures can be determined by the ICL. The volumetric strain, however, decreases with increasing confining pressure when it is larger than 17.2 MPa, as shown in the experimental data in Figure 4. Such behaviour is expected for crushable materials because more input work for the samples will be obtained after isotropic compression at a larger confining pressure, which will lead to a larger breakage index. A larger breakage index will also lead to a lower initial position of CSL and RCL in the e - ln(p) space, which means samples after isotropic compression at the larger confining pressure behaves like a soil in a 'loose' state, while samples after isotropic compression at the lower confining pressure behaves like a soil in a 'dense' state. The present model can describe such behaviour as shown in Figure 4 that less volumetric contraction during shearing for the sample after isotropic compression at 52 MPa is observed than that of 40 MPa. However, the LB model fails to capture such behavior as indicated in Figure 4. It is clear that the proposed model performs significantly better than that of LB model in describing the main response of drained tests within a wide range of confining pressures.

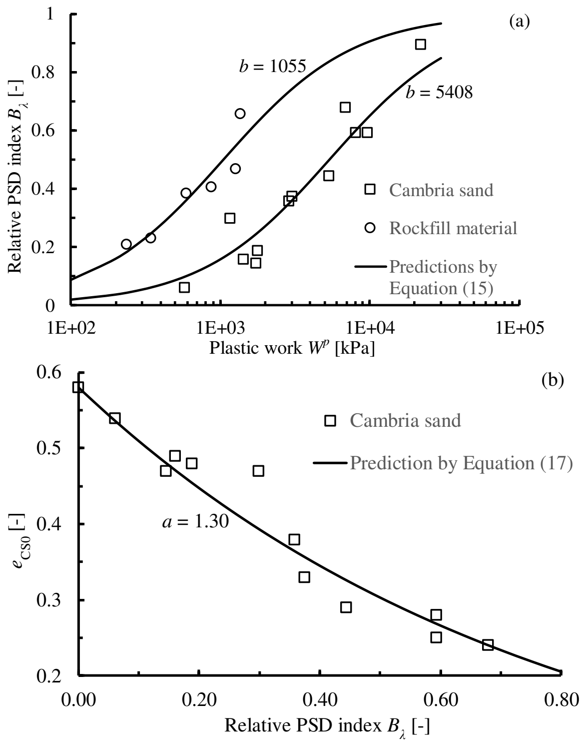

Figure 3. Calibration of breakage parameters: (a) relative PSD index B_λ versus plastic work, (b) e_{cs0} versus relative PSD index B_λ.

7 CONCLUSIONS

In this study, a simple constitutive model with consideration of the main properties of granular soils is present within the framework of Li & Dafalias (2000). These main properties include the nonlinear CSL and ICLs in the e - ln(p) space, the state-dependent behaviour, and the particle breakage and its influence on the stress-strain behaviour. A double logarithmic approach for modelling the nonlinearity of the CSL, ICLs in the e - ln(p) space is adopted, based on which, a new RCL intersects with the CSL at p = 0 has been developed. The modified state parameter is defined as the difference between the current void ratio and void ratio on the RCL at the same mean effective stress.

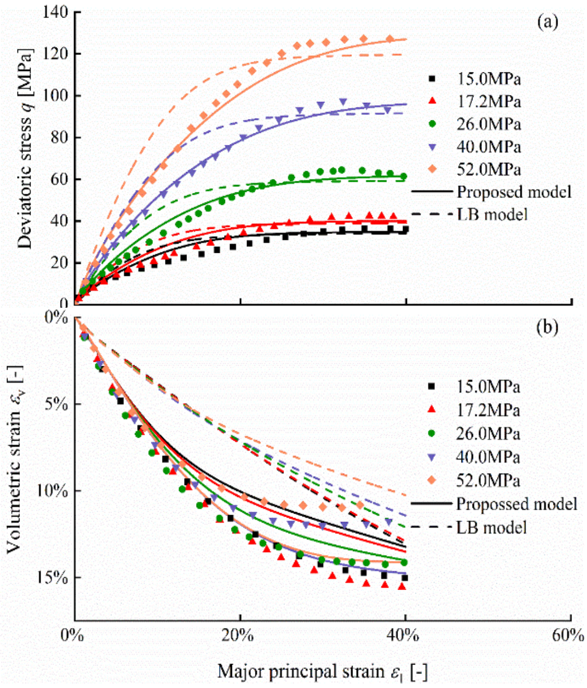

Figure 4. Measured and predicted drained shearing results of the Cambria sand with confining pressure varying between 15.0 MPa and 52.0 MPa: (a) deviatoric stress; and (b) volumetric strain relations

A simple dynamic evolution law of the CSL, LCL, and RCL with increasing particle breakage is proposed. The relative PSD index B_λ is employed as a measurement of particle breakage, which can be calculated from the input plastic work. The initial position of CSL in the e - ln(p) space moves downwards with increasing B_λ. However, it cannot be a parallel shift, and all the CSLs with various B_λ will eventually converge at high stresses, because particle breakage will complete and is no longer the main deformation mechanism of granular soils. The RCL evolves similarly with that of the CSL, i.e., shifts downwards from the initial position and converges eventually as particle breakage processes, while the LCL and the critical state stress ratio are independent of particle breakage. Such an evolution in the RCL has been incorporated into the proposed constitutive model by the concept of the proposed modified state parameter.

The proposed model was compared with the LB model and validated against experimental results of drained triaxial tests on the Cambria sand. It has been shown that the proposed model is superior to the LB model in capturing the nonlinearity of CSL, and state-dependent behaviour of granular soils.

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