Proceedings 10th NUMGE 2023
10th European Conference on Numerical Methods in Geotechnical Engineering
Zdravkovic L, Kontoe S, Taborda DMG, Tsiampousi A (eds)

https://doi.org/10.53243/NUMGE2023-162

A subloading surface clay and sand model

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ABSTRACT: The mechanical response of natural soils is characterized by a narrow elastic domain, a smooth transition between elastic to elasto-plastic regime, and hysteretic behaviour during cyclic loading. These realistic features can be efficiently introduced into an existing constitutive model by reformulating it in the framework of subloading surface plasticity. This unconventional elasto-plastic approach is advantageous over other techniques, since it allows to retain some stress history information only by adding an additional constitutive parameter and one hardening rule. In this paper, CASM, a state parameter dependent, critical state model, is reformulated in the framework of subloading plasticity. Several element tests are reported to illustrate the performance of the model.

Keywords: constitutive relations; elastoplasticity; CASM; subloading surface model

1 INTRODUCTION

The mechanical response of natural soils is affected by the recent stress history and is generally characterized by a progressive shift from elastic to elasto-plastic regime, a narrow elastic domain (Smith et al., 1992) and the accumulation of strains during cyclic loading (Wichtmann et al., 2005). Although these features characterize the mechanical response of natural soils, yet most of the constitutive models employed in current practice have a too large elastic domain, a drastic shift between elastic to elasto-plastic regimes and do not consider hysteretic behaviour during cyclic loading. These models, therefore, are thought to be suitable to model monotonic loading but might be imprecise for non-monotonic loading.

An existing constitutive model might be adapted to incorporate the effect of recent stress history by including elements of bounding surface plasticity (Dafalias, 1986; Rouainia and Muir Wood, 2000) or subloading surface plasticity (Hashiguchi, 1980; Hashiguchi, 1989; Hashiguchi et al., 2002). For example, the Modified Cam Clay model has been reformulated in both frameworks (Rouainia and Muir Wood, 2001; Hiley and Rouainia, 2008; Yamakawa et al., 2010). The adoption of one of these two approaches to enhance well-established constitutive models is advantageous as the modelling capabilities are much improved just by adding a small number of additional constitutive parameters and hardening rules and, at the same time, retaining all the knowledge on the reference constitutive model.

Subloading surface model (Hashiguchi, 1980; Hashiguchi et al., 2002; Pedroso, 2014) is an unconventional elastoplasticity framework specially designed to overcome the aforementioned shortcomings. This is accomplished by introducing an additional yield surface, the subloading surface, which has the same shape but different size than normal-yield surface, the relaxation of the Kuhn-Tucker conditions and the introduction of an additional hardening variable.

From a coding standpoint, extending a constitutive model in the framework of bounding surface plasticity is more laborious than doing the same employing a subloading approach. The former introduces a large number of internal variables and kinematic hardening relations to retain some of the stress history, whereas in the latter this is accomplished by just introducing an additional internal variable and one hardening rule. Moreover, explicit stress integration schemes of subloading plasticity are simpler than those of classical elastoplasticity, as they do not require a yield surface drift correction scheme, as the formulation is able self-correct the drift (Pedroso, 2014).

In this work, the Clay And Sand Model (CASM) (Yu 1998, 2006) is recast in the framework of subloading plasticity. CASM is a critical state-based, state parameter dependent elastoplastic model. CASM is able to represent a wide range of soil behaviours, ranging from strong undrained softening to ductile or dilatant behaviour. Moreover, by appropriately selecting some constitutive parameters, the original and modified Cam Clay models might be retrieved.

The paper is organized as follows: after a brief summary of conventional elastoplasticity, the main differences between elasto-plasticity and subloading plasticity are highlighted; then, the proposed constitutive model is fully described and, finally, the performance of the proposed model is examined in a set of element tests.

2 CONVENTIONAL ELASTOPLASTICITY FRAMEWORK

In the framework of small strain elastoplasticity (Simo, 1998; Simo and Hughes, 1998), the strain rate, $\dot{\boldsymbol{\epsilon}}$, can be additively decomposed into an elastic and a plastic part, $\dot{\boldsymbol{\epsilon}}^e$ and $\dot{\boldsymbol{\epsilon}}^p$:

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p \tag{1}$$

The relation between stresses and elastic strains can be expressed as:

$$\dot{\boldsymbol{\sigma}} = \mathbb{D}_e : \dot{\boldsymbol{\epsilon}}^e \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbb{D}_e$ is the fourth order elastic stiffness tensor.

The yield surface defines the elastic domain: inside the yield surface the material response is purely elastic and no irreversible deformations are produced:

$$f(\boldsymbol{\sigma}, \boldsymbol{z}) \leq 0 \tag{3}$$

where $f(\boldsymbol{\sigma}, \boldsymbol{z})$ is the yield surface and $\boldsymbol{z}$ are a set of internal plastic variables.

For a non-associative plastic model, the flow rule may be expressed as:

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\gamma}\boldsymbol{n} = \dot{\gamma}\frac{\partial g(\boldsymbol{\sigma}, \boldsymbol{z})}{\partial \boldsymbol{\sigma}} \tag{4}$$

where $\dot{\gamma}$ is the plastic multiplier, $\boldsymbol{n}$ is a direction of plastic strain rate and $g(\boldsymbol{\sigma}, \boldsymbol{z})$ is known as the plastic potential.

The evolution of plastic internal variables is governed by a set of hardening rules:

$$\dot{\boldsymbol{z}} = \dot{\gamma}\mathbf{h}(\boldsymbol{\sigma}, \boldsymbol{z}) \tag{5}$$

where $\mathbf{h}(\boldsymbol{\sigma}, \boldsymbol{z})$ are the set of hardening functions.

Finally, an elasto-plastic model must fulfil the Kuhn-Tucker conditions and the consistency condition, that read:

$$f(\boldsymbol{\sigma}, \boldsymbol{z}) \leq 0 \tag{6}$$

$$\dot{\gamma} \geq 0 \tag{7}$$

$$\dot{\gamma} f(\boldsymbol{\sigma}, \boldsymbol{z}) = 0 \tag{8}$$

$$\dot{\gamma} \dot{f}(\boldsymbol{\sigma}, \boldsymbol{z}) = 0 \tag{9}$$

3 SUBLOADING PLASTICITY

Subloading surface plasticity (Hashiguchi, 1980; Hashiguchi, 1989; Hashiguchi et al., 2002) is an unconventional plastic framework specially designed to introduce a smooth transition between elastic to elasto-plastic regime, plasticity inside of the yield surface and strain accumulation and hysteretic behaviour during cyclic loading. This is accomplished by using two yield surfaces -the normal yield surface and the sub-loading surface (see Figure 1)-, relaxing the Kuhn-Tucker conditions and introducing an additional hardening parameter to the formulation, the so-called similarity ratio, R_s.

Both the subloading and normal-yield surface have the same shape but different sizes; indeed, the ratio between the size of the two surfaces is described by the similarity ratio, R_s. The current stress state always lies on the subloading surface.

The original proposal of Hashighuchi (1980) assumes that both yield surfaces are described by homogeneous functions of first degree and expressed as:

$$F(\boldsymbol{\sigma}, \boldsymbol{z}) = \bar{f}(\boldsymbol{\sigma}) - f(\boldsymbol{z}) \tag{10}$$

$$f(\boldsymbol{\sigma}, \boldsymbol{z}) = \bar{f}(\boldsymbol{\sigma}) - R_s f(\boldsymbol{z}) \tag{11}$$

where it is clear that the similarity ratio, R_s, ranges between 0 and 1. $\bar{f}(\boldsymbol{\sigma})$ and $F(\boldsymbol{z})$ are homogeneous functions.

In subloading plasticity, the Kuhn-Tucker conditions and consistency condition -Equation (6) to (9)- are replaced by a loading condition (Hashiguchi et al., 2002):

$$f(\boldsymbol{\sigma}, \boldsymbol{z}) = 0 \quad \dot{\gamma} \geq 0 \tag{12}$$

$$\dot{\gamma} = 0 \quad \text{if} \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \dot{\boldsymbol{\epsilon}} < 0 \tag{13}$$

This way, the current stress state always lays in the subloading surface, Equation (12), and plastic flow occurs when $\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \dot{\boldsymbol{\epsilon}} > 0$, thus, plastic straining occurs even if the current stress state is not in the normal-yield surface.

Finally, the formulation is closed by specifying a hardening law for the similarity ratio. This hardening law is better understood by studying the plastic modulus, (H), that reads:

$$H = -\frac{\partial f}{\partial \boldsymbol{z}} : \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{\epsilon}^p} : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial R_s}\frac{dR_s}{d\gamma} \tag{14}$$

Generally, $\frac{dR_s}{d\gamma}$ is described with a function that only depends on $\mathrm{R_s}$. This function has to be null when the normal-yield and subloading surface coincide ($R_s = 1$) so that the hardening of the reference constitutive model is retrieved, and should be infinity for the case of $R_s = 0$, so that the material behaves elastically.

The hardening law of the similarity ratio is generally described as:

$$\dot{R}_s = U(R_s)\dot{\gamma} \quad \text{if} \quad \dot{\gamma} \geq 0 \tag{15}$$

and a suitable expression is (Hashiguchi et al., 2002):

$$U(R_s) = \frac{u}{\tan\left(\frac{\pi}{2}R_s\right)} \tag{16}$$

where $U(R_s)$ is a monotonically decreasing function of R_s and u is a constitutive parameter controlling the magnitude of plastic straining; classical elastoplasticity is recovered as u tends to infinity.

4 FORMULATION OF THE CONSTITUTIVE MODEL

The reference constitutive model is the Clay And Sand Model (CASM) (Yu, 1998). The model is based on critical state theory and formulated in terms of the state parameter.

In the subloading plasticity framework, the normal yield surface corresponds to the yield surface of CASM:

$$F = \left(\frac{\sqrt{\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}}}{p' M_\theta}\right)^n + \frac{1}{\ln(r)}\ln\left(\frac{p'}{p'_o}\right) \tag{17}$$

where $\boldsymbol{s} = \boldsymbol{\sigma} - p'\mathbb{1}$ is the deviatoric stress tensor, $\mathbb{1}$ is the second order identity tensor, M_θ is the stress ratio at critical state, n is a parameter that regulates the shape of yield surface, r is a spacing ratio, that controls the location of the intersection of the CSL with the yield surface, p'_o stands for the preconsolidation pressure.

Differently from the original proposal of Hashighuchi (1980), here none of the yield surfaces are homogeneous functions, thus they are not expressed with the formal structure of Equation (10).

The subloading surface has the same shape as the normal yield surface, but different size (see Figure 1). This is accomplished by multiplying the preconsolidation pressure by the similarity ratio:

$$f = \left(\frac{\sqrt{\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}}}{p' M_\theta}\right)^n + \frac{1}{\ln(r)}\ln\left(\frac{p'}{R_s p'_o}\right) \tag{18}$$

A non-associative flow rule is assumed (Mánica et al., 2021):

$$g = \left(\frac{\sqrt{\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}}}{p' M_\theta}\right)^m + \mathrm{m} - \frac{y(m-1)}{p'} - 1 \tag{19}$$

where m is a parameter that controls the shape of the plastic potential function and y is a constant that must be solved for the current stress state.

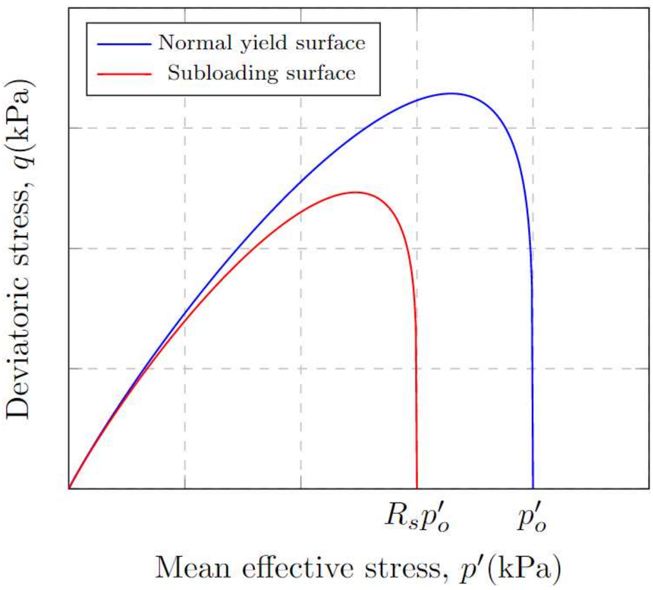

Figure 1. Normal yield surface and subloading surface of subloading CASM

CASM adopts the same hardening rule as the modified Cam-Clay model:

$$\dot{p}'_o = \frac{p'_o}{\lambda^* - \kappa^*}\dot{\epsilon}_v^p \tag{20}$$

where ϵ_v^p volumetric plastic strain, $\lambda^* = \frac{\lambda}{1+e_o}$ stands for the slope of normally-consolidated in a volumetric strain-logarithmic mean stress compression plane, e_o is an initial void ratio, $\kappa^* = \frac{\kappa}{1+e_o}$ is the slope of the swelling line in a volumetric strain-logarithmic mean stress compression plane.

The elastic regime is characterised by a pressure-dependent hypoelastic model, which is described by:

$$K = \frac{p'}{\kappa^*} \qquad G = \frac{3(1-2\mu)}{2(1+\mu)}K \tag{21}$$

where K is the bulk moduli, G is the shear moduli and μ stands for the Poisson's ratio.

As a consequence of the previous equations, the critical state line (CSL) in the compression plane, $\upsilon - \ln(p')$, is a straight line:

$$\upsilon_{csl} = \Gamma - \lambda\, ln(p') \tag{22}$$

where υ_{csl} is the specific volume at the CSL, p' is the mean effective stress, Γ is the specific volume on the CSL for $p' = 1$ kPa and λ is the slope of the CSL.

The plastic modulus of the model can be obtained as:

$$H = -\frac{\partial f}{\partial p_c}\frac{\partial p_c}{\partial \epsilon_v^p}\frac{\partial g}{\partial p'} - \frac{\partial f}{\partial R_s}\frac{dR_s}{d\gamma} \tag{23}$$

Noting the similitudes and differences between the first and second term of the plastic modulus, the term $\frac{d\mathrm{R_s}}{d\gamma}$ is written as:

$$\dot{R}_s = U(R_s)\frac{\partial g}{\partial p\prime}\dot{\gamma} \quad \text{if} \quad \dot{\gamma} \geq 0 \qquad (24)$$

This way the plastic modulus of the model reads:

$$H = -\frac{\partial f}{\partial p_c}\frac{\partial p_c}{\partial \epsilon_v^p}\frac{\partial g}{\partial p'} - \frac{\partial f}{\partial R_s}U(R_s)\frac{\partial g}{\partial p\prime} \qquad (25)$$

Therefore, both terms, that arising from the reference constitutive model and that originating from subloading, plasticity have a similar formal structure and, more importantly, the same order of magnitude.

5 REPRESENTATIVE NUMERICAL SIMULATIONS

To showcase the performance of the model, a set of element tests are presented, employing the constitutive parameters reported at Table 1.

5.1 Isotropic compression loading (case I)

The first numerical analysis of this work corresponds to the isotropic loading. The initial stress state of the soil is characterized by $p' = 20$ kPa and an over-consolidation ratio equal to 10.

Table 1. CASM parameters, material constant and initial values

	Case I	Case II	Characteristic
M	-	1.56	CASM parameter
r	2	7	CASM parameter
m	1.745	1.745	CASM parameter
n	-	5.16	CASM parameter
κ	0.001	0.008	CASM parameter
μ	-	0.4	CASM parameter
e_o	1	1	CASM parameter
λ	0.02	0.04	CASM parameter
u	60-1000	300-140000	Material constant
p'	20	20-200	Initial value (kPa)
p'_o	200	200	Initial value (kPa)

Figure 2 depicts the evolution of the void ratio, e, in terms of the mean effective stress, p', for several values of the constitutive parameter u. At low mean effective stresses, the stiffness is similar to the elastic one. As the soil element reaches mean effective stresses in the vicinity of the preconsolidation stress, stiffness tends to smoothly reduce, reaching the elasto-plastic stiffness at mean effective stresses much higher than the initial consolidation stress. Therefore, all simulations show a smooth transition from elastic to elasto-plastic state, and the rate of degradation of stiffness is rapid as larger values of u are considered. Moreover, classical elasto-plasticity is recovered as u tends to infinity (Hashigushi, 2009).

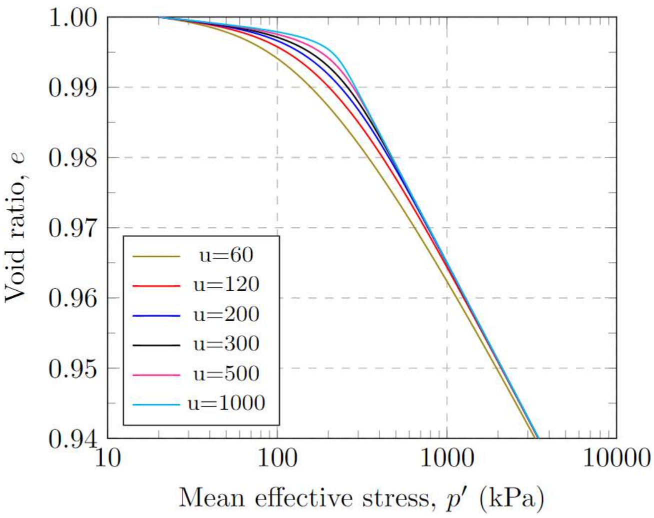

Figure 2. Influence of constant material, u, on curvature: void ratio, e, vs. mean effective stress, p'

In order to study the hysteretic behaviour during cyclic loading, an additional simulation is reported considering cycles of loading, unloading and reloading (Figure 3). During unloading, purely elastic behaviour is recovered, governed by κ. Remarkably, during the subsequent reloading the material has a lower stiffness than during unloading and stiffness continuously reduces until reaching the plastic stiffness.

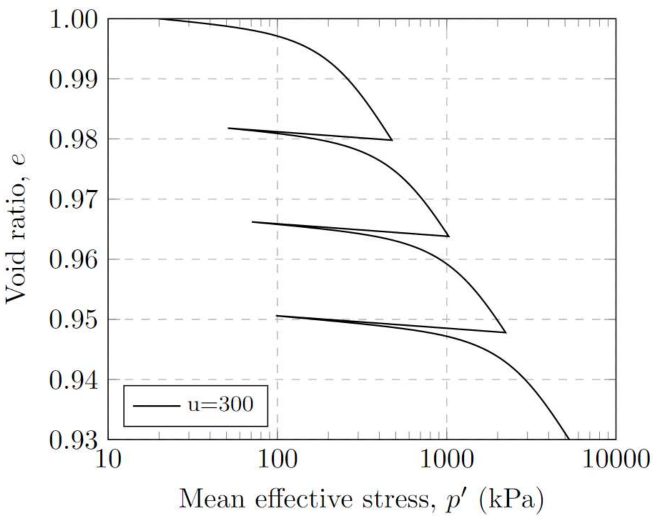

Figure 3. Isotropic loading. Void ratio, e, vs mean effective stress, p', during cyclic loading

5.2 Undrained triaxial compression test (case II)

The second numerical analysis corresponds to undrained triaxial loading, employing the constitutive parameters reported in Table 1.

Figure 4 reports several simulations for a broad range of initial over-consolidation ratios, using to distinct values of the parameter u: 300 and 140000, which is considered the reference elasto-plastic solution. Of course, the obtained solution for a normally-consolidated soil is independent of the parameter *u*, as the similarity ratio is always equal to unity. For over-consolidated soils, subloading plasticity can properly represent the smooth transition between elastic to elasto-plastic regimes generally attributed to natural soils.

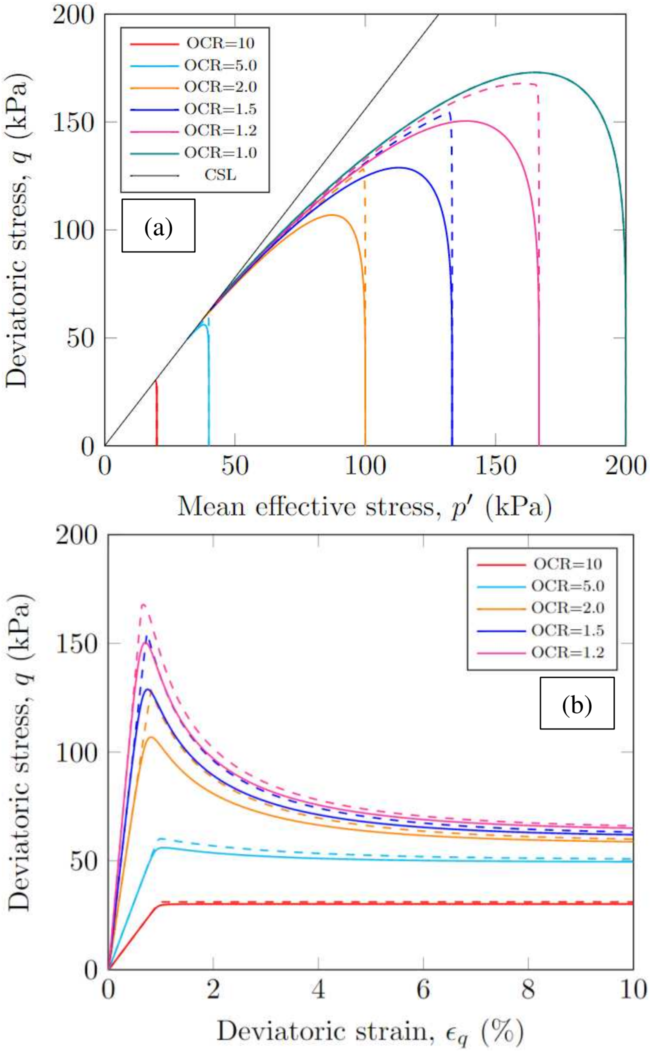

Figure 4. Undrained triaxial loading. Deviatoric stress vs mean effective stress, (a), and deviatoric stress vs deviatoric strain (b). Dashed lines (conventional elastoplasticity) and solid lines (subloading surface model).

A low value of the constitutive variable *u* seems to slightly reduce the residual undrained shear strength; however, at 10% of deviatoric deformation, none of the simulations has reached critical state. Moreover, at the end of the simulation, the similarity ratio has not yet reached a value of 1 (see Figure 5).

Finally, the performance of the model during cyclic undrained triaxial loading is showcased in Figure 6. Once recasted in the framework of subloading plasticity, the constitutive response depends on the recent stress history: compared to classical elasto-plasticity, during subsequent reloading, the soil has a much reduced stiffness and yields at lower deviatoric stress.

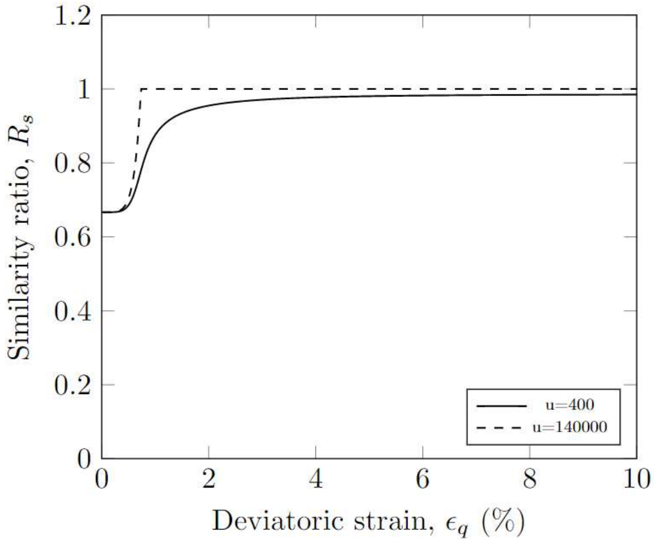

Figure 5. Undrained triaxial loading. Evolution of the similarity ratio. OCR = 1.5.

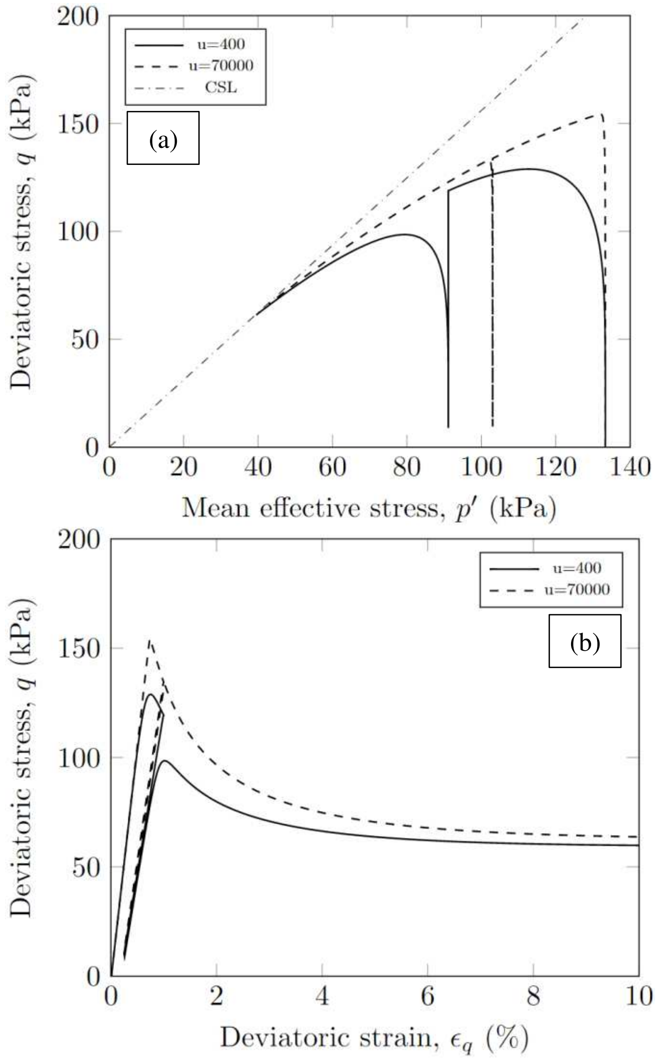

Figure 6. One cycle during undrained triaxial loading. Deviatoric stress vs mean effective stress, (a), and deviatoric stress vs deviatoric strain (b). Dashed lines (conventional elastoplasticity) and solid lines (subloading surface model).

6 CONCLUSIONS

The mechanical response of soils is characterized by a small elastic domain, a smooth transition between elastic and elasto-plastic regime and the accumulation of strains during cyclic loading. These features can be introduced to a constitutive model by reformulating it in the framework of unconventional elastoplasticity.

In this work, a critical state, state parameter dependent constitutive model has been recasted in the framework of subloading surface plasticity. The performance of the model has been showcased in a set of numerical simulations.

7 ACKNOWLEDGEMENTS

The first author acknowledges the support of the 'Beca Generación del Bicentenario' Scholarship Program of Ministry of Education, Peru.

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