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Challenges of the particle finite element method (PFEM) for modelling geotechnical problems

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ABSTRACT: Since its invention in 2004, the particle finite element method (PFEM) has being attracted increasing attention. So far, it has been demonstrated to be a robust and powerful numerical tool for handling various challenging engineering problems such as free-surface flow, solid-structure interaction, multiphase problems, melting problems with phase changes, etc. Nevertheless, several issues arise when adopting it for large deformation geotechnical problems. This is, to a large extent, due to the complex geomaterial behaviour. History dependency makes variable mapping between meshes inevitable if the classical PFEM is adopted. Linear elements used in the conventional PFEM do not work well for capturing soil behaviour. Although the smoothed particle finite element method, a variant version of the PFEM, allows the use of linear elements and alleviates the variable mapping requirement, stress oscillation occurs in its dynamic analysis. In this paper, challenges associated with the conventional PFEM for modelling geotechnical problems are explored followed by a new version of Nodal integration based PFEM (N-PFEM) proposed to overcome the issues. Numerical benchmarks demonstrate the correctness and robustness of the N-PFEM for dynamic analysis of geotechnical problems.

Keywords: PFEM; Nodal integration; Mixed variational principal; Granular flow

1 INTRODUCTION

There are many geotechnical problems where geomaterials undergo large deformation. Representative examples include cone penetration test, offshore foundation installation, embankment failure, among others. In the past decades, several numerical approaches have been developed and applied to large deformation geotechnical problems, such as the smoothed particle hydrodynamics (SPH) method (Bui and Nguyen 2021), the material point method (MPM) (Soga et al. 2016), the particle finite element method (PFEM) (Zhang et al. 2015), etc.

The PFEM is a hybrid method with the feature of the particle method for handling large deformation and the accuracy of the traditional Lagrangian finite element method. It was originally invented for modelling free-surface flow problems in the community of fluid dynamics in 2004 (Idelsohn et al. 2004). Soon after its invention, it was demonstrated to be a powerful and robust numerical tool for simulating many challenging problems such as fluid-solid interaction, multiphase flow, melting problems with phase change, etc.

To solve geotechnical problems, the PFEM has been adapted so that history-dependent behaviour, a typical characteristic of geomaterials, can be handled. So far, several challenging problems, such as cone penetration (Sabetamal et al. 2021), granular flow (Zhang et al. 2014, Zhang et al. 2016), landslides (Wang et al. 2021), etc., have been simulated successfully using the PFEM or its variant version. Despite that, issues still exist for the PFEM modelling of geotechnical problems which will be explored and discussed in this paper.

2 PARTICLE FINITE ELEMENT METHOD (PFEM)

2.1 PFEM steps for fluid dynamics

Treating mesh nodes as free particles at the end of Lagrangian finite element analyses is the fundamental of the PFEM. The free particles are used to re-construct a computational domain and meshes so that free-surface evolution such as water splashing can be captured. The simulation cycle of the PFEM modelling of fluid dynamic problems is as below (Figure 1):

i) Fill the domain with a set of particles (Figure 1(a)).
ii) Conduct Delaunay Triangulation using the particles (Figure 1(b)).
iii) Identify boundaries based on the triangles generated in ii) (Figure 1(c)).
iv) Solve the governing equations on the meshes.
v) Update node positions (Figure 1(d)).
vi) Erase mesh topology (Figure 1(e)) and go to ii) in which another Delaunay triangulation will be conducted leading to (Figure 1(f)).

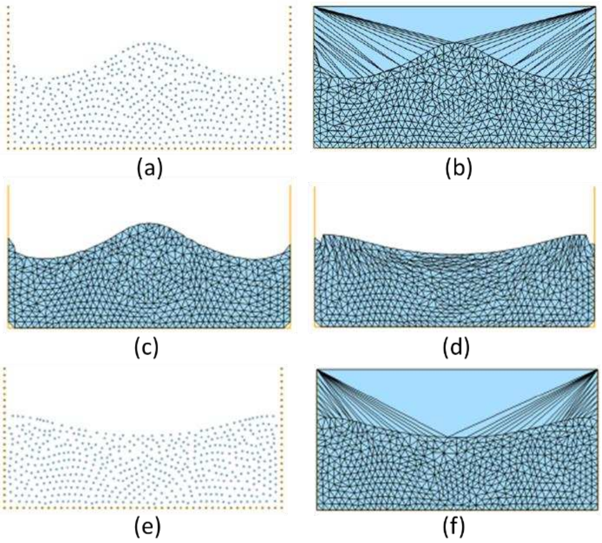

Figure 1. PFEM steps (after (Cremonesi et al. 2020))

The boundary identification in step iii) is achieved using the alpha-shape method whose implementation is forthright. The radius of the circumcircle of each triangle from ii) is first checked. The triangle is deleted if the radius is greater than αh, where α is an empirical factor and h is a characteristic length interpreted as the size of the used meshes; otherwise the triangle is retained. The retained triangles form the computational domain as seen in Figure 1(c) where boundaries of the domain are identified and also the mesh is ready for FE analyses to be carried out in step iv). By doing so, free surface evolution can be captured, even for problems with new free surface generation such as water splashing and wave breaking.

2.2 Challenges of PFEM for solid dynamics

The first attempt to make use of the PFEM for geotechnical problems was reported in (Carbonell et al. 2010) where ground excavation was concerned. The main motivation for using the PFEM to the excavation problems is its convenience in detecting the changing geometry of ground and the contact between ground and a road-header of complex geometry. In the simulation, the damage model was used, and when criteria were met geomaterials were removed to mimic excavation process. The deformation of geomaterial in (Carbonell et al. 2010) is relatedly small. To explore the capability of the PFEM for tackling mesh distortion issues in modelling large deformation geotechnical problems, a variant version of the PFEM was developed for in (Zhang et al. 2013). Unlike the cases considered in (Carbonell et al. 2010), geomaterials in (Zhang et al. 2013) underwent remarkably large deformation and were simulated using plastic constitutive models

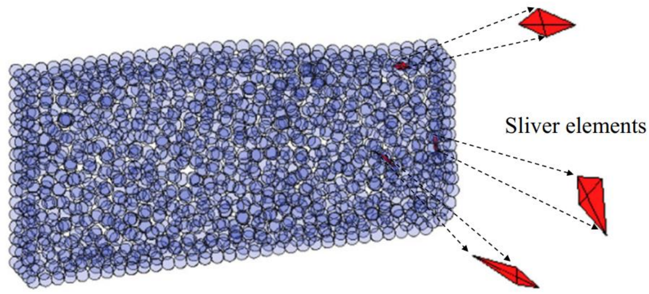

Figure 2. Sliver elements which are of low quality for FE analyses (after (Wang et al. 2022))

Challenges were encountered when extending the classical PFEM for geotechnical problems. This is, to a large extent, because of the complex soil behaviour. Three-node triangular elements in the conventional PFEM do not perform well for modelling elastoplastic materials like soils because of the associated volumetric locking issue. High-order elements were, thus, adopted in the version of the PFEM proposed in (Zhang et al. 2013). Given the low quality of the automatically generated meshes in the boundary identification process (particularly at shear bands as shown in Figure 2,), remeshing the complete identified domain using a new set of nodes has to be carried out to ensure simulation accuracy which, consequently, necessitates the mapping of variables at both quadrature points and mesh nodes. Alternatively, a mixed finite element formulation was adopted in the PFEM to overcome the volumetric locking issues (Monforte et al. 2017, Monforte et al. 2018). By doing so, the low-order element such as three-node triangular element can be adopted, but information should still be transferred from old to new meshes when handling history-dependent materials (Carbonell et al. 2022). A drawback of variable mapping is the accumulated error. Although mesh smoothing may increase the quality and somewhat alleviate the requirement of variable mapping, the quality of smoothed meshes is not guaranteed for three dimensional cases in which sliver elements exist (Wang et al. 2022).

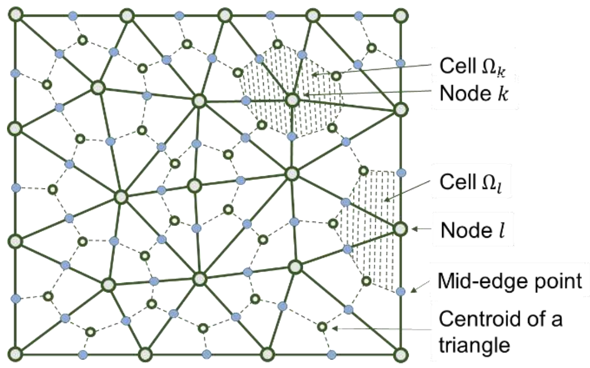

Figure 3. Cells constructed on triangles

A possible way to overcome this issue is the use of nodal integration leading to the so-called smoothed particle finite element method (SPFEM) (Zhang et al. 2018). In the SPFEM, nodal integration is carried out on cells (Figure 3) which are constructed based on triangles. It was shown in (Zhang et al. 2018) that good simulation results can be gained despite of low mesh quality. Moreover, linear triangles can be adopted without the volumetric locking issue. However, further investigation shows that the SPFEM developed on the conventional displacement-based finite element method suffers from the stress oscillation issue when modelling dynamic problems. *Ad hoc* regularisation techniques have to be employed to stabilise the stress field (Jin et al. 2021, Shafee and Khoshghalb 2022, Yuan et al. 2023).

3 NODAL INTEGRATION BASED PFEM (N-PFEM)

In this section, a variant version of nodal integration based PFEM (N-PFEM) is introduced. This version inherits the nodal integration feature as the SPFEM but solves the governing equations in mathematical programming based on a generalised Hellinger-Reissner variational principle.

3.1 Min-max problem

According to (Zhang et al. 2019), the time discretised governing equations for dynamic analysis of elastoplastic models with volume Ω and boundary Γ are equivalent to the following min-max problem with the use of generalised Hellinger-Reissner variational principle

$$\begin{aligned} &\underbrace{min}_{\Delta\boldsymbol{u}}\ \underbrace{max}_{\boldsymbol{\sigma}_{n+1},\boldsymbol{r}_{n+1}}\ -\frac{1}{2}\int_\Omega \boldsymbol{\Delta\sigma}^T\,\mathbb{C}\boldsymbol{\Delta\sigma}d\Omega \\ &+\int_\Omega \boldsymbol{\sigma}_{n+1}^T\nabla\Delta\boldsymbol{u}d\Omega+\frac{1-\theta_1}{\theta_1}\int_\Omega \boldsymbol{\sigma}_n^T\nabla\Delta\boldsymbol{u}d\Omega \\ &-\int_\Omega \tilde{\boldsymbol{t}}_n^T\Delta\boldsymbol{u}d\Gamma-\int_\Omega \tilde{\boldsymbol{b}}_n^T\Delta\boldsymbol{u}d\Omega \\ &-\frac{1}{2}\int_\Omega \boldsymbol{r}_{n+1}^T\frac{\Delta t^2}{\tilde{\rho}}\boldsymbol{r}_{n+1}d\Omega+\int_\Omega \boldsymbol{r}_{n+1}^T\Delta\boldsymbol{u}d\Omega \\ &subject\ to{:}\quad F(\boldsymbol{\sigma}_{n+1})\le 0 \end{aligned} \tag{1}$$

in which

$$\tilde{\boldsymbol{t}}_n=\frac{1}{\theta_1}\bar{\boldsymbol{t}} \tag{2}$$

$$\tilde{\boldsymbol{b}}=\frac{1}{\theta_1}\boldsymbol{b}+\tilde{\rho}\frac{v_n}{\Delta t}\ \text{with}\ \tilde{\rho}=\frac{\rho}{\theta_1\theta_2} \tag{3}$$

In min-max problem (1), the independent master fields consist of the displacement increment, $\Delta\boldsymbol{u}$, the stress, $\boldsymbol{\sigma}_{\mathrm{n+1}}$, and the inertial force, $\boldsymbol{r}_{\mathrm{n+1}}$. Subscripts n and n+1 refer to the values at known and unknown steps, respectively. The material density is ρ; the body force is $\boldsymbol{b}$; the velocities are $\boldsymbol{v}$; ∇ is the gradient operator; $\bar{\boldsymbol{t}}$ is the traction force imposed on the boundary Γ_{t}; $\mathbb{C}$ is the elastic compliance matrix, and F is the yield function. The time increment is Δt. The parameters θ_1 and θ_2 are in between 0 and 1.

3.2 Nodal integration

On each cell, both stress and strain distribution are uniform since the three-node triangular element is adopted (Figure 3). Nodal integration is then conducted over cells for min-max problem (1) leading to

$$\begin{aligned} &\underbrace{\max}_{\boldsymbol{\sigma}_{\mathrm{n+1}},\boldsymbol{r}_{\mathrm{n+1}}}\quad -\frac{1}{2}\boldsymbol{\Delta\hat{\sigma}}^{\mathrm{T}}\boldsymbol{C\Delta\hat{\sigma}}-\frac{1}{2}\hat{\boldsymbol{r}}_{\mathrm{n+1}}^{\mathrm{T}}\boldsymbol{D}_r\hat{\boldsymbol{r}}_{\mathrm{n+1}} \\ &\text{subject to}\ \begin{cases}\overline{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}_{\mathrm{n+1}}+\boldsymbol{A}^{\mathrm{T}}\hat{\boldsymbol{r}}_{\mathrm{n+1}}=\tilde{\boldsymbol{f}} \\ F^i(\hat{\boldsymbol{\sigma}}_{\mathrm{n+1}})\le 0 \qquad i=1,2,\cdots,NN\end{cases} \end{aligned} \tag{4}$$

in which $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{r}}$ are the vectors consisting of stress components and inertial force components at nodes; $(\cdot)^i$ represents the value of $(\cdot)$ at ith node if not otherwise specified; and NN is the total number of nodes, which is also equal to the total number of cells implying imposition of the yield criterion on all nodes. Readers are referred to (Meng et al. 2021, Zhang et al. 2022) for other symbol definitions.

Maximsation problem (4) can be resolved in mathematical programming using the advanced primal-dual interior point method after being reformulated as a standard second-order cone programming problem which is

$$\begin{aligned} &\min \qquad c^T\boldsymbol{x} \\ &\text{subject to}\ \begin{cases}\boldsymbol{Ax}=\boldsymbol{b} \\ \boldsymbol{x}_i\in\mathcal{K}_q, with\ i=1,\dots,n \\ \boldsymbol{x}_j\in\mathcal{K}_r, with\ j=1,\dots,m\end{cases} \end{aligned} \tag{5}$$

where the cones are

$$\mathcal{K}_q=\left\{x_1\ge\sqrt{x_2^2+x_3^2+\cdots+x_n^2}\right\} \tag{6}$$

and

$$\mathcal{K}_r=\left\{2x_1x_2\ge\sqrt{x_3^2+x_4^2+\cdots+x_m^2}\right\} \tag{7}$$

Both x_1 and x_2 should be non-negtive in (7).

In the N-PFEM modelling, maximsation problem (4) is resolved to gain the variable states, such as displacements, stresses, strains, inertial forces, etc., at the next time step. Since the state of all variables are calculated and stored at nodes, variable mapping is not required in the N-PFEM modelling. Furthermore, the generalised Hellinger-Reissner variational principal underpins the finite element algorithm in mathematical programming meaning no stress regularisation technique is required in dynamic analysis.

4 NUMERICAL EXAMPLE

The classical granular column collapse problem is studied using the proposed N-PFEM. A cylindrical column of granular materials is released. The radius of the bottom of the column is $r_0 = 3.9$ cm and the height of the column is $h_0 = 7.8$ cm. The granular materials are modelled as purely frictional with following material parameters: density $\rho = 1.8\ \mathrm{g/cm^3}$, Young's modulus $E = 10$ MPa, Poisson ratio $\upsilon = 0.3$, friction angle $\phi = 30°$, and dilation angle $\psi = 0°$. Due to the symmetry, only a quarter of the geometry is simulated which is discretised using a total of 10,759 nodes and 54,350 elements. The time step used in the simulation is 0.002 s with the time integration parameters $\theta_1 = \theta_2 = 1.0$. Figure 4 shows the collapse process from the N-PFEM modelling where the vertical stress distribution is plotted. The normalised simulated time $\bar{t} = t/\sqrt{h_0/g}$. It can be seen the complete collapse process is captured successfully without stress oscillation even though no regularisation technique is adopted.

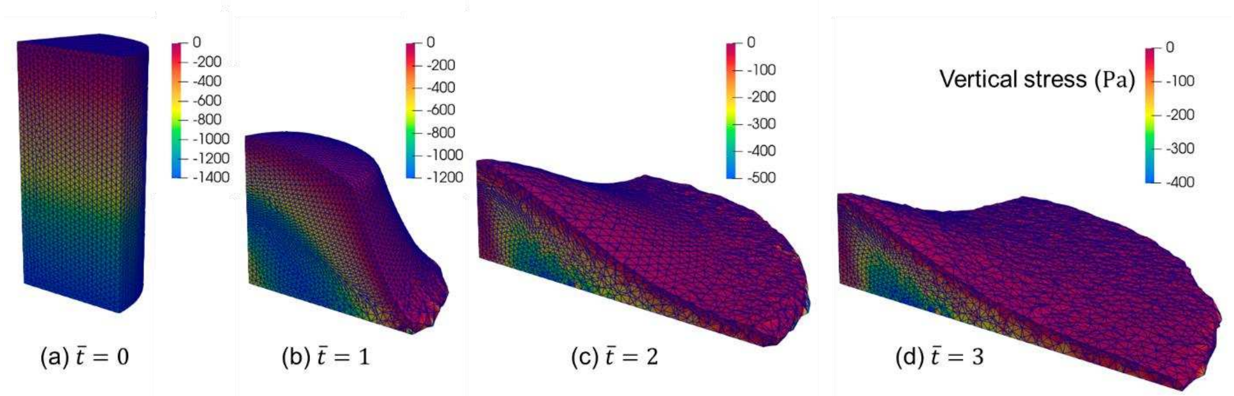

Figure 4. Collapse of a granular column at normalised time (a) $\bar{t} = 0$; (b) $\bar{t} = 1$; (c) $\bar{t} = 2$; (b) $\bar{t} = 3$

5 CONCLUSIONS

The PFEM is a novel numerical approach for simulating engineering problems with large material deformation. This paper discusses the challenges of adopting the PFEM for modelling history-dependent materials in geotechnical engineering such as requirement of variable mapping, volumetric locking, and stress oscillation. An improved nodal integration based PFEM overcoming these issues is then proposed with an example showing its robustness. It is shown that the developed N-PFEM is particularly suitable for modelling soil flow problems which are commonly encountered in geotechnical and geological engineering.

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