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On the modelling of state and history dependency of small strain stiffness of soils with application to 1D settlement analysis

Sur la rigidité des sols en petite déformations et son application pour analyse de tassement unidimensionnelle

A.B. Tsegaye*, J.-S. L'Heureux
Norwegian Geotechnical Institute, Trondheim, Norway

**anteneh.biru.tsegaye@ngi.no*

ABSTRACT: Soil mechanics is increasingly focusing on understanding the deformation characteristics of soils under small strains. At small strains, soil stiffness is notably high, but its decay with increasing shear strains or stresses is rather fast and nonlinear. Accounting for small strain stiffness is crucial for accurately calculating deformations and minimizing the influence of distant fields in numerical simulations of boundary value problems. This paper offers a comprehensive overview of factors, including state and loading history dependencies, that impact the shear modulus of soils. It introduces an explicit 1D small-strain model, expanding upon Janbu's stress-dependent stiffness, to incorporate small strain considerations into 1D settlement analysis.

RÉSUMÉ: L'un des domaines qui attirent l'attention en mécanique des sols est le comportement des sols à petite déformations. À ce niveau, la rigidité des sols est relativement élevée et décroit non-linéairement avec l'augmentation des contraintes. Le module de cisaillement à petite déformation est un paramètre crucial pour calculer précisément les déformations et pour minimiser l'influence des conditions limite en analyse numérique. Cet article présente une synthèse des facteurs qui influencent le module de cisaillement des sols. Un modèle 1D à petites déformations est proposé en utilisant le modèle à contrainte dépendante de Janbu lors d'analyse de tassement.

Keywords: Small strain stiffness; stress dependency of stiffness; soil deformation.

1 INTRODUCTION

Researches on soil stiffness indicate that the tangent shear modulus is a function of several state variables and history of loading (Länsivaara, 1999; Benz, 2007; Tsegaye, 2014)

$$G = f(\sigma, e, \gamma, S_r, c_i, F, T, S_s, H_l), \tag{1}$$

where σ is effective stress, e is void ratio, γ is shear strain amplitude, S_r is saturation level, c_i are soil properties linked to grain shape, size and grading, F is loading frequency, T is temperature, S_s is soil structure, H_l is loading history. The relevant variables may be combined in several ways. Several authors in the literatures put the various factors in a multiplicative form. Accordingly, Tsegaye (Tsegaye, 2014) wrote the shear modulus, G, as

$$G = \alpha_G f_e f_\sigma f_H f_\varepsilon p_a, \tag{2}$$

where α_G is a dimensionless parameter, sometimes also called shear modulus number and p_a is atmospheric pressure. The soil constant α_G is may depend on grain characteristics such as grain shape, size and grading, (Wichtmann et al., 2011). The functions f_e, f_σ, f_H, and f_ε take into account the dependence of the shear modulus on void ratio, effective stress, loading history and strain amplitude respectively. The stress dependency and the void ratio dependency may be collectively called *state dependency* whereas the strain and loading history dependency may be called *memory dependency*.

This paper presents a synthesis of factors that influence the shear modulus of soils at small strain. Following the review, application of small strain stiffness is demonstrated by enhancing Janbu's stress dependent stiffness using novel small strain overlay formulations.

2 STATE DEPENDENCY

State dependency includes stress dependency and void ratio dependency of stiffness. Each is briefly discussed in the sub-sections below.

2.1 Stress dependency

According to (Tsegaye, 2014), the stress dependency factor may be multiplicatively split into effective confining pressure dependency factor, f_p, and stress-ratio dependency factor, f_η, say $f_\sigma = f_p f_\eta$.

The effective confining pressure dependency of stiffness of soils was proposed as early as the late 1930's by (Ohde, 1939). On the basis of numerous oedometer and triaxial compression tests, (Janbu, 1963) proposed a formulation of soil stiffness as power function of stress. The usual form of the stress-dependency function is of the form

$$f_\sigma = f_p = \left(\frac{p}{p_a}\right)^n, \tag{3}$$

where p is the effective confining stress, n is the Ohde-Janbu stress dependency parameter that ranges between 0.5 and 1 for several types of soils. A similar power law has been proposed to describe stress dependency of stiffness in the small strain regime as well, *e.g.*, (Hardin and Drnevich, 1969). Some researchers use the minor effective principal stress instead of the effective confining pressure, *e.g.*, (Duncan and Chang, 1970; Schanz et al., 1999), Janbu's formulation is in terms of the axial (vertical) stress. In the literature, the geometric mean stress $(\sigma_i \sigma_j)^{\frac{1}{2}}$ has also been used for sands instead of the arithmetic mean, p (Ni, 1987; Hardin and Blandford, 1989) where σ_i and σ_j are the principal stresses in the plane G is measured.

When the stress dependency function is a function of the effective confining pressure alone, the resulting stiffness is none-conservative and may result in physically incorrect solution during repeated actions of loading-unloading (Zytynski et al., 1978; Lade and Nelson, 1987; Houlsby et al., 2005). To this end, several conservative stress-dependent elastic stiffness formulations have been proposed, *e.g.*, (Vermeer, 1980; Lade and Nelson, 1987; Molenkamp, 1988; Einav and Puzrin, 2004; Houlsby et al., 2005). For example, (Lade and Nelson, 1987) derived a conservative stress-dependent elastic shear modulus with the stress-dependency function specified as

$$f_\sigma = \left(\frac{p}{p_a}\right)^{2\lambda_1} f_\eta^{L-N}, \tag{4}$$

wherein f_η^{L-N} is the Lade-Nelson stress ratio dependency function and is given by

$$f_\eta^{L-N} = (9 + R\eta^2)^{\lambda_1}, R = 2\frac{1+\upsilon}{1-2\upsilon}, \tag{5}$$

R and λ_1 are material parameters. R is given in terms of Poisson's ratio, υ, (which is assumed constant.) According to (Lade and Nelson, 1987), all these parameters can be determined from conventional triaxial compression tests with unloading-reloading cycles. The Lade-Nelson exponent may be estimated with $\lambda_1 = \frac{n}{2}$, leading to a range of λ_1 roughly between 0.25 and 0.5. (Lade and Nelson, 1987) reported values of λ_1 between 0.2 and 0.3 for different types of sands.

2.2 Void ratio dependency

Interlocking and intergranular contacts decrease with increasing void ratio, e. It is generally known that the denser the soil, the higher is its shear stiffness. (Hardin and Richart, 1963) proposed $f_e = \frac{(a_e - e)^2}{1+e}$, where a_e depends on the nature and grading of soils (Hicher, 1996). In general, a_e in the range 1.4 - 7.3 has been reported, *e.g.*, (Hardin and Richart, 1963; Hardin and Black, 1968; Kokusho et al., 1982; Wichtmann and Triantafyllidis, 2004). The higher values are reported for clayey soils. The other redundantly appearing equation in the literature is an exponential function of the form, $f_e = e^{-x_e}$. Values of x_e between 1 and 2 have been reported, *e.g.*, (Lo Presti et al., 1993; Lo Presti and Jamiolkowski, 1998; Fioravante, 2000).

3 MEMORY DEPENDENCY

Soil stiffness is known to depend on both strain magnitude, stress and loading direction. This dependency may in general be referred to as memory dependency. This is briefly discussed in this section.

3.1 Strain dependency

The stiffness of soils is strain dependent in general. But the decay is highly nonlinear and faster in the small strain region. It is often postulated that the small strain region can be defined with a certain limit, say $f_\varepsilon \geq f_{\varepsilon-lim}$ although there is no agreement in the literature on the question "how small is small strain?" For example, (Atkinson, 2000) states that the strain range can be sub-divided into very small ($\gamma \leq 10^{-6}$), small ($10^{-6} \leq \gamma \leq 10^{-3}$) and large strain ($\gamma \geq 10^{-3}$). Moreover, (Bui, 2009) proposed sub-divisions and different ranges, namely: small strain ($\gamma \leq 10^{-5}$), medium strain ($10^{-5} \leq \gamma \leq 10^{-3}$), large strain

($10^{-3} \le \gamma \le 10^{-2}$) and failure strain ($\gamma > 10^{-2}$). The complexity of the stiffness-strain relationship even in this range is difficult to account for using proper elastic formulations.

There are various proposed formulations that aim at capturing the nonlinear decay of secant shear modulus with shear strain in the small strain regime. Several of those are in the form

$$f_\varepsilon = \frac{1}{1+a_\gamma\left(\frac{\gamma}{\gamma_{ref}}\right)^{\alpha_\gamma}}, \qquad (6)$$

where γ is the shear strain, γ_{ref} is a reference shear strain, and a_γ, α_γ and γ_{ref} are model parameters (Tsegaye, 2014). Early proposition by (Hardin and Drnevich, 1969), for example, was $a = 1$, $\alpha_\gamma = 1$ and $\gamma_{ref} = \tau_{max}/G_{max}$, where τ_{max} is the maximum shear stress. (Stokoe et al., 2004) considered $a = 1$, $\alpha_\gamma = 1$ and $\gamma_{ref} = \gamma_{0.5}$. (Santos and Correia, 2001) modified the same equation considering $a_\gamma = 0.385$, $\alpha_\gamma = 1$ and $\gamma_{ref} = \gamma_{0.7}$. The reference shear strains $\gamma_{0.5}$ and $\gamma_{0.7}$ are shear strains at which the stiffness has decayed by 50% and 30% respectively. In (Benz, 2007), $\gamma_{0.7}$ is determined at a slightly different point, where the stiffness has degraded 27.8%. When a_γ and α_γ are fixed, only one parameter, namely γ_{ref} is required to define the strain dependency curve. A cut-off is required for using Eq. (6) as an overlay for enhancement of elastoplastic models. The cut-off criterion can be either in terms of stiffness or in terms of strain. Where there is lack of data, defining $\frac{\gamma_{cut-off}}{\gamma_{ref}} = 10$, the cut-off stiffness for the small strain overlay may be set $f_\varepsilon \ge (f_{\varepsilon-cut-off} \approx 0.2)$.

3.2 Loading history dependency

According to (Atkinson et al., 1990), the stiffness of overconsolidated soils depends not only on the current state but also on the stress history. They distinguish the stress history into the overall history "which may be best described by overconsolidation ratio" and the recent history "which may be related to a sudden change of direction or a period of time at constant stress." (Tsegaye, 2014) formulated this as

$$f_H = f_{OCR} f_\theta, \qquad (7)$$

where f_{OCR} accounts for overconsolidation history dependency and f_θ considers recent loading history (in terms of magnitude and direction). (Hardin and Black, 1968) proposed $f_{OCR} = OCR^{n_o}$, where n_o is a material parameter. For granular materials such as sands, (Tsegaye, 2014) suggested that the ratio $\frac{e_c}{(e_c - \langle\Psi\rangle)}$ may be considered instead, where e_c is the critical state void ratio and Ψ is the state parameter according to (Been and Jefferies, 1985).

(Atkinson et al., 1990) performed laboratory tests on reconstituted overconsolidated samples of London Clay aiming at influence of rotation of stress on subsequent stiffness. Their test results demonstrated clearly that change of loading direction has a significant effect on stiffness of soils, *Figure 1b*.

At small stress ratio, it is seen that the stiffness increases linearly with the magnitude of the rotation angle of the stress path, the highest being for the 180 degrees, i.e., load reversal. However, this difference vanishes at higher mobilizations.

On the theoretical end, (Simpson, 1992) proposed a physical analogue to describe the effect of arbitrary changes of loading direction on stiffness of soils. He considered a man dragging a series of bricks that are tied on separate strings, *Figure 1a*. According to Simpson, if the man "walks continuously in one direction the brick will line up behind him and follow…if he turns back the bricks initially do not move then the ones on a shorter string start to move, gradually followed by the longer strings…if he turns 90^0, the bricks initially keep moving in their previous direction but gradually swing round behind him". Simpson formulated his ideas in strain space such that the man is taken a point in strain space, each brick represents the proportion of a soil element while the movement of a brick represents plastic strains and the difference between the movement of the man and the sum of the movements of the bricks represents elastic strains. (Länsivaara, 1999) proposed two hypotheses which are in line with Simpson's brick analogue. Namely, soil remembers stress history and that its stiffness depends thus on how the direction of the stress increment relates to the previous states; and that the recent stress history has the greatest influence on stiffness, while the effect of older stress states slowly diminishes to zero, *i.e.*, more weight should be placed on recent history. Few models have been developed based on the very recent stress (strain) history, *e.g.*, (Niemunis and Herle, 1997; Länsivaara and Nordal, 2001; Benz, 2007).

The higher stiffness observed at the beginning of loading and during the change of loading direction may also be alternatively reasoned as follows (Tsegaye, 2014). At the rest state of the soil mass that is in equilibrium, the soil grains are arranged in a stable configuration state. The degree of stability generally depends on the arrangement, the time it stayed in the given configuration state and the loading history it has been subjected to. Initialization of deformation requires destabilizing this stable configuration and the initial resistance can be significantly high. Upon continuous loading, the order and interaction of particles changes (Klausner, 1991), the established resistance gives way to a new order in an attempt to form a new stable state that accommodates the external action if the action is sustained. Changing the loading direction would be once again met with a higher resistance until such a state change is accommodated. This concept is called *resistance of an established state to change* in (Tsegaye, 2014).

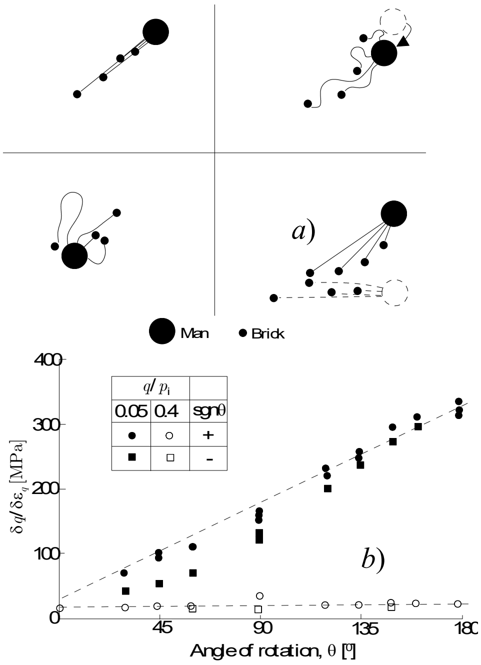

Figure 1. a) Simpson's brick illustration (Simpson, 1992)), b) Variation of stress path shear stiffness with change of direction in constant effective confining pressure laboratory tests on reconstituted London Clay after (Atkinson et al., 1990).

Let $G^{\theta} = m_{\theta} G_r$, where G^{θ} is the shear stiffness immediately after a change in the loading direction and G_r be a certain reference shear modulus. The reference shear modulus may be selected at the strain level where the influence of stress rotation has vanished by more than 90%, *i.e.*, swept out of memory, as (Gudehus, 1977) calls it. Let the factor m_{θ} at $\theta = 0^0$, 90^0 and 180^0 be given respectively by 1, m_{90} and m_{180} with $1 \geq m_{90} \geq m_{180}$. Assuming linear interpolation between these boundary conditions (Tsegaye, 2014) found:

$$m_{\theta} = m_{90} + (1 - 2m_{90} + m_{180})\theta^{+} \cos\theta + (m_{90} - m_{180}) \cos\theta, \quad \theta^{+} = \langle \operatorname{sgn}(\cos\theta) \rangle \tag{8}$$

where θ is the angle between the memorized previous loading direction and the current loading direction. Considering a linear increase of stiffness with $\cos\theta$ one may set $m_{90} = 1 + 0.5(m_{180} - 1)$.

The stiffness interpolation function can then be completely specified by specifying m_{180}. Where data is lacking $m_{180} = 5$ may be set. In addition, for virgin loading, $m_i G_r$, where $m_i = m_{180}$ may be considered.

We may further consider (Niemunis and Herle, 1997) intergranular strain formulation for the modelling the stiffness decay with strain. See (Tsegaye, 2014) for application of the intergranular strain concept to elastoplastic models. For the same, Eq. (**5**) may be extended as

$$m_{\theta} = m_{90\rho} + (m_{i\rho} - 2m_{90\rho} + m_{180})\theta^{+} \cos\theta + (m_{90\rho} - m_{180}) \cos\theta, \quad \theta^{+} = \langle \operatorname{sgn}(\cos\theta) \rangle \tag{9}$$

where $m_{90\rho} = m_{180} - \rho_I^{\chi}(m_{180} - m_{90})$, $m_{i\rho} = m_{180} + \rho_I^{\chi}(1 - m_{180})$, ρ_I is the normalized magnitude of the intergranular strain accroding to (Niemunis and Herle, 1997).

If the most recent memorized direction is say $\hat{\mathbf{h}}$ and the current loading direction is, say $\mathbf{d}$, the $\cos\theta$ in Eqs. (8) and (9) can be replaced by the scalar product $\hat{\mathbf{h}}:\mathbf{d}$, Figure 2.

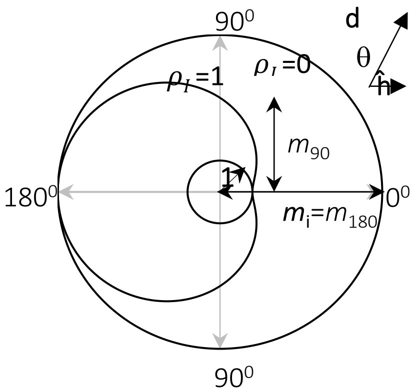

Figure 2. m_{θ} according to Eq. (9), $m_{90} = 1 + 0.5(m_{180} - 1)$, $m_i = m_{180} = 5$ considered.

4 APPLICATION OF SMALL STRAIN OVERLAY TO JANBU'S 1D SOIL STIFFNESS FORMULATION

In this section, we would like to demonstrate the small strain overlay approach in a 1D. For simplifying the treatment, let us set f_e and f_H to unity. An enhanced form of Janbu's stress dependent stiffness is considered such that the vertical stress increment, $\mathrm{d}\sigma_z$, is given by:

$$\mathrm{d}\sigma_z = \alpha_G r_v f_{\varepsilon} f_{\sigma} p_a \mathrm{d}\varepsilon_z, r_v = \frac{1-v}{1-2v}. \tag{10}$$

where $d\varepsilon_z$ is the vertical strain increment, v is Poisson's ratio, α_G is considered a soil constant, and f_ε and f_σ are defined respectively by

$$f_\varepsilon = \begin{cases} \dfrac{1}{\left(1 + a_\gamma a\left(\gamma/\gamma_{0.7}^{ref}\right)\right)^2}, G \geq G_r \\ 1, G < G_r \end{cases} \quad (11)$$

$$f_\sigma = (\sigma_z/p_a)^n, 0 \leq n \leq 1 \quad (12)$$

where G is the shear modulus at a given shear strain within the small strain limit. From experience, n may be set 0.5 for sandy and coarser materials and 1 for clay soils. The stress dependency equation can lead to a numerical issue when $\sigma_z = 0$. This may be resolved by adding attraction to both the numerator and the denominator in Eq (12).

Upon integration, for $0 \leq n < 1$, we have:

$$\left(\frac{\sigma_z}{p_a}\right)^{1-n} - \left(\frac{\sigma_{z,0}}{p_a}\right)^{1-n} = \frac{\alpha_G r_v (1-n) {\gamma_{0.7}^{ref}}^2 (\varepsilon_z - \varepsilon_{z,0})}{\left(a_\gamma \varepsilon_{z,0} + \gamma_{0.7}^{ref}\right)\left(a_\gamma \varepsilon_z + \gamma_{0.7}^{ref}\right)} \quad (13)$$

and for $n = 1$,

$$\frac{\sigma_z}{\sigma_{z,0}} = \exp\left\{\frac{\alpha_G r_v \gamma_{0.7}^{ref} (\varepsilon_z - \varepsilon_{z,0})}{\left(a_\gamma \varepsilon_{z,0} + \gamma_{0.7}^{ref}\right)\left(a_\gamma \varepsilon_z + \gamma_{0.7}^{ref}\right)}\right\} \quad (14)$$

It is natural to assume that the initial state is unstrained and hence $\varepsilon_{z,0} = 0$ which simplifies the equations further. Note that Eqs. (13) and (14) can be easily inverted.

During 1D settlement calculation, it is customary to limit the stress increment that causes settlement in an arbitrary depth where $\Delta\sigma_z \geq 0.2\sigma_{z0}$. This can now be more objectively set using a strain limit.

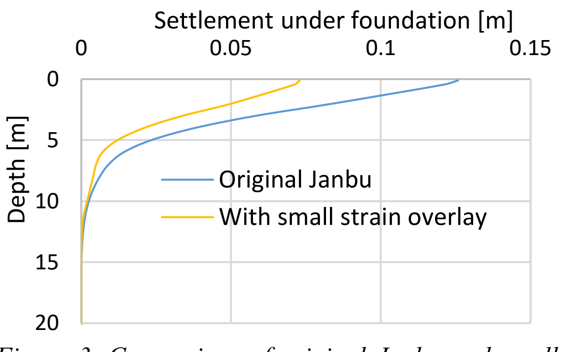

Figure 3. Comparison of original Janbu and small strain enhanced Janbu formulations in a 1D settlement analysis ($\alpha_G = 286$, $n = 1$, $\gamma_{0.7}^{ref} = 10^{-4}$, $a_\gamma = 0.385$. Janbu's original formulation is used beyond the small strain limit).

5 SUMMARY

This paper undertakes a comprehensive examination of soil stiffness at small strains, categorizing the factors influencing it into state and memory dependency. State dependency pertains to variables independent of a reference system, such as stress and void ratio, while memory dependency does not generally align with this criterion. Various formulations are explored to encompass the dependency of soil stiffness on both state and memory dependency, considering monotonic loading as well as changes in loading directions. Additionally, a straightforward 1D small strain overlay model is introduced to augment Janbu's stiffness formulation, illustrating the integration of small strain models with those designed for analysis of soil deformation at larger strains.

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