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A semi-analytical model for axial soil-pile interaction in generalized inhomogeneous soils

Un modèle semi-analytique pour l'interaction axiale sol-pieu dans des sols inhomogènes généralisés

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ABSTRACT: The seminal work of Nogami & Novak in 1976 for the harmonic response of end-bearing piles in homogeneous soil strata, is extended in this study to generate a benchmark semi-analytical model for piles in vertically inhomogeneous soils. Following the classical dynamic model and previous work to treat soil inhomogeneity under static axial loads, the pile is modelled as a rod, and the soil as an approximate continuum of the Tajimi type. The approximation resides in reducing the number of the elastodynamic equations which govern the soil motion to one, which satisfies the equilibrium in the vertical direction, by eliminating certain stresses and displacements in the soil. This is based on the physically motivated assumption that the vertical normal and shear stresses in the soil are essentially controlled exclusively by the vertical component of the soil displacement, for the axisymmetric problem at hand. Soil inhomogeneity is introduced via a generalized power law variation of the soil shear modulus with depth. The proposed generalized formulation can treat any type of soil inhomogeneity by employing in the solution pertinent shape functions to capture the soil modes, which satisfy the boundary conditions of the pile-soil system. The solution is expressed in terms of series and includes: (i) displacements and stress fields in the soil and along the pile-soil interface; (ii) displacements and normal stresses along the pile length, and (iii) dynamic pile head stiffness and damping.

RÉSUMÉ: Le travail fondateur de Nogami et Novak en 1976 sur la réponse harmonique des pieux d'appui en sol homogène est étendu dans cette étude pour générer un modèle semi-analytique de référence pour les pieux dans les sols verticalement inhomogènes. En suivant le modèle dynamique classique et les travaux précédents sur le traitement de l'inhomogénéité du sol sous des charges axiales statiques, le pieu est modélisé comme une tige et le sol comme un continuum approximatif de type Tajimi. L'approximation réside dans la réduction du nombre d'équations élastodynamiques qui régissent le mouvement du sol à une seule, qui satisfait à l'équilibre dans la direction verticale, en éliminant certaines contraintes et déplacements dans le sol. Cela repose sur l'hypothèse physiquement motivée selon laquelle les contraintes normales et de cisaillement verticales dans le sol sont essentiellement contrôlées exclusivement par la composante verticale du déplacement du sol, pour le problème axisymétrique en question. L'inhomogénéité du sol est introduite par le biais d'une variation généralisée de la loi de puissance du module de cisaillement du sol avec la profondeur. La formulation généralisée proposée peut traiter tout type d'inhomogénéité du sol en utilisant, dans la solution, des fonctions de forme pertinentes pour capturer les modes du sol, qui satisfont aux conditions aux limites du système pieu-sol. La solution est exprimée en termes de séries et comprend: (i) les champs de déplacement et les champs de contrainte dans le sol et le long de l'interface pieu-sol; (ii) les déplacements et les contraintes normales le long de la longueur du pieu, et (iii) la rigidité et l'amortissement dynamiques de la tête de pieu.

Keywords: Complex dynamic pile stiffness; analytical model; generalized soil inhomogeneity; soil-pile interaction.

1 INTRODUCTION

The consideration of dynamic soil-structure interaction (SSI) is crucial to evaluate the seismic vulnerability of structures. It is well known that the assumption of a fixed-base condition at the foundation level causes an underestimation of the natural period of superstructures (Mylonakis and Gazetas, 2000). During the last decades, a variety of methods of various levels of accuracy and sophistication have been developed to take into account the effect of the foundation (e.g., Veletsos and Meek, 1974; Gazetas, 1984; Maravas et al., 2014; Anoyatis and Lemnitzer, 2017b; 2023; Cui et al., 2022). On one hand, powerful finite element commercial software can handle a variety of complex soil–pile configurations, such tools can be computationally expensive. On the other hand, analytical models provide a cost-effective alternative solution for soil–pile interaction analysis and allow engineers to test different design scenarios and optimize the pile and superstructure for given soil conditions. For dynamic analyses in particular, analytical models are further free of the selection of appropriate mesh size and absorbing boundaries (e.g. viscous boundaries, infinite elements, boundary elements) to control radiation conditions and avoid spurious reflections. The proposed solution models the pile as a rod, following the classical strength-of-materials solution, and assumes perfect bonding at the soil-pile interface. The soil is modelled as an approximate continuum of the 'Tajimi' type, applied by Novak and Nogami to piles in homogeneous soils under axial and lateral inertial loading (Nogami and Novak, 1976; Novak and Nogami, 1977). Such models reduce the number of dependent variables by eliminating certain stresses and displacements in governing elastodynamic equations.

Key to the Tajimi approach in axial mode is the physically motivated assumption that the vertical normal, σ_z, and vertical shear stresses, τ_{rz}, in the soil, are controlled solely by the vertical displacement u_z (Mylonakis, 2001; Anoyatis and Mylonakis, 2012; Anoyatis et al., 2013, 2019, 2023). The influence of radial displacement is neglected by Nogami and Novak, 1976 and is considered to be vanishingly small by Mylonakis, 2001. This reduces the two equations of classical axisymmetric elasticity to one. The solution derived in this work extends the study of Anoyatis et al., 2023 to treat any type of soil inhomogeneity, applying pertinent mathematical techniques.

The soil-pile system response is expressed in terms of a generalized Fourier series of eigenfunctions (soil modes) along the vertical coordinate. The soil modes are obtained by employing appropriate shape functions to tackle soil inhomogeneity. Apart from its intrinsic theoretical interest, the proposed model has distinct advantages over other simple analytical models, e.g., Winkler-type solutions, (Novak, 1974; Scott, 1981; Anoyatis and Lemnitzer, 2017a) and more advanced models (e.g., Anoyatis et al., 2019, 2023) as it tackles any type of soil inhomogeneity in the vertical direction, consider the continuity of the medium in the vertical direction, and is free of empirical constants. The predictive power of the model is verified against the recent work of Anoyatis et al., 2023 for given types of inhomogeneity.

2 SOIL-PILE SYSTEM

The problem investigated in this study is depicted in Figure 1: A single, solid vertical pile of length L and circular cross section of diameter d is embedded in an inhomogeneous soil layer that rests on a rigid rock. The pile is subjected to an axial harmonic force of amplitude P and cyclic frequency ω, that is $P\,e^{i\omega t}$.

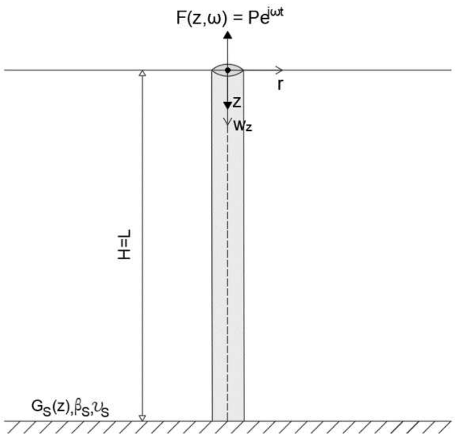

Figure 1. Problem under investigation.

The pile is modelled as an elastic rod of Young's Modulus E_p and mass density ρ_p. The soil layer is characterized by its soil mass density ρ_s, Poisson's ratio ν_s, and hysteretic material damping β_s. Soil inhomogeneity is considered through a depth-dependent complex-valued shear modulus $G_s^* = G_s(z)(1 + 2i\beta_s)$. The variation of soil shear modulus with depth is expressed by a power law function:

$$G_s^* = G_{sH}^* \left[b + (1-b)\left(\frac{z}{H}\right)\right]^n \tag{1}$$

In Equation (1), z is the vertical coordinate, n and b are dimensionless inhomogeneity parameters. b links the stiffness at the surface of the layer G_{s0} to the stiffness at its base G_{sH}, that is, $b = (G_{s0}/G_{sH})^{1/n}$.

3 SOIL ANALYSIS

The equilibrium of vertical forces acting on a soil element in axisymmetric conditions in cylindrical coordinates yields the governing differential equation:

$$\frac{\partial(\tau\, r)}{\partial r} + r\,\frac{\partial \sigma}{\partial z} + r\,\rho_s\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{2}$$

where $u = u_z(r,z,\omega)$ is the vertical displacement, $\tau = \tau_{rz}(r,z,t)$ and $\sigma = \sigma_z(r,z,t)$ are the vertical soil shear and normal stresses. The first and the second terms represent axisymmetric shearing and compression stresses, respectively, while the third term accounts for the development of inertial force due to vertical oscillations. The stress-displacement relations for axisymmetric deformations in a Tajimi - type continuum are written as in Nogami and Novak, 1976; Anoyatis et al., 2023:

$$\sigma \approx -\eta_s^2\, G_s^*\,\frac{\partial u}{\partial z} \tag{3a}$$

and

$$\tau \approx -G_s^*\,\frac{\partial u}{\partial r} \tag{3b}$$

where $\eta_s^2 = 2(1-\nu_s)/(1-2\nu_s)$ is a compressibility parameter related to the Tajimi approximation, as discussed in the earlier works of Mylonakis, 2001 and Anoyatis et al., 2019. Substituting Equations (3) into Equation (2), and considering harmonic displacement $u = u(r,z,\omega)e^{i\omega t}$ the equilibrium equation is rewritten in terms of soil displacements as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \eta_s^2\,\frac{\partial}{\partial z}\left[\frac{G_s^*(z)}{G_{sH}^*}\frac{\partial u}{\partial z}\right] + \left(\frac{\omega}{V_{sH}^*}\right)^2 u = 0 \tag{4}$$

where $V_{sH}^* = V_{sH}\sqrt{1+2i\beta_s}$ is the complex shear wave velocity in the soil at depth $z = H$.

The solution to Equation (4) can be expressed as the product of the solutions of the two ordinary differential equations that arise from the application of the method of separation of variables:

$$R'' + \frac{1}{r}R' - q^2 R = 0 \tag{5a}$$

and

$$(\bar{G}_s \Phi)' + \lambda \bar{G}_s \Phi + \left(\frac{\omega}{V_{pH}^*}\right)^2 \Phi = 0 \tag{5b}$$

where $\bar{G}_s = G_s^*(z)/G_{sH}^*$ expresses in a dimensionless form the profile of soil stiffness and V_{pH}^* is the complex propagation velocity of P-waves in the soil medium at depth $z = H$. q and λ are variables that emerge from the application of the method of separation of variables, and they are linked through the compressibility coefficient as $q = \eta_s\sqrt{\lambda}$. The general solution to Equation (5a) has been obtained in the past work of (Mylonakis, 2001). Equation (5b) is a general Sturm-Liouville equation and $\Phi = \Phi(z,\omega)$ are the associated soil modes, the solution of which has been found in the recent work of (Anoyatis et al., 2023) for two patterns of soil inhomogeneity $n = 1$ (linear) and $n = 2$ (parabolic). The solution derived in this paper can treat any type of inhomogeneity. The technique used to find the soil modes will be shown in detail in Section 3.1.

Displacements and shear stresses in the soil are expressed in terms of generalized Fourier series as:

$$u = \sum_{m=1}^{\infty} B_m\, K_0(q_m r)\,\Phi_m \tag{6}$$

and

$$\tau = \sum_{m=1}^{\infty} G_s^*\, B_m\, q_m\, K_1(q_m r)\,\Phi_m \tag{7}$$

where B_m are Fourier coefficients, and K_0 and K_1 are the modified Bessel functions of the second kind, zero and first order, respectively.

3.1 Soil modes

Equation (5b) can be solved by expressing the soil modes $\Phi = \Phi(z,\omega)$ in terms of a Fourier series as:

$$\Phi_m(z,\omega) = \sum_{j=1}^{\infty} S_j(\omega)\,\Psi_j(z) \tag{8}$$

in which S_j are the Fourier coefficients to be determined, and $\Psi_j(z)$ is a shape function of the form $cos(\alpha_j z)$ in which $\alpha_j = (\pi/2H)(2j-1)$.

Fundamental to this analysis, is the orthogonality identity of the shape functions:

$$\int_0^H \Psi_j\,\Psi_k\,dz = 0,\ \ j \neq k \tag{9}$$

Substituting Equation (8) into Equation (5b), making use of Equation (9) and, using the concept of the weak formulation, yields the following equation:

$$\sum_{j=1}^{\infty} S_j^m \int_0^H \bar{G}_s\,\Psi_j'\,\Psi_k'\,dz - \sum_{j=1}^{\infty} S_j^m\,\lambda_m \int_0^H \bar{G}_s\,\Psi_j\,\Psi_k\,dz - \left(\frac{\omega}{V_{pH}^*}\right)^2 S_k^m \int_0^H \Psi_k^2\,dz = 0 \tag{10}$$

The eigenvalues, λ_m, and the corresponding eigenvectors, S_j^m, depend on the frequency and can be obtained by solving the system of equations which emerges from Equation (10), ω time. The real and the imaginary part of the first soil mode at $\omega = 0$, $\omega = 0.5\,\omega_1$, $\omega = \omega_1$ for a damped soil medium ($\beta_s = 0.05$), $b = 0.25$ and $n = 1$ are illustrated in Figure 2, using $N = 50$ modes. Each mode is normalised by its value at $z = 0$.

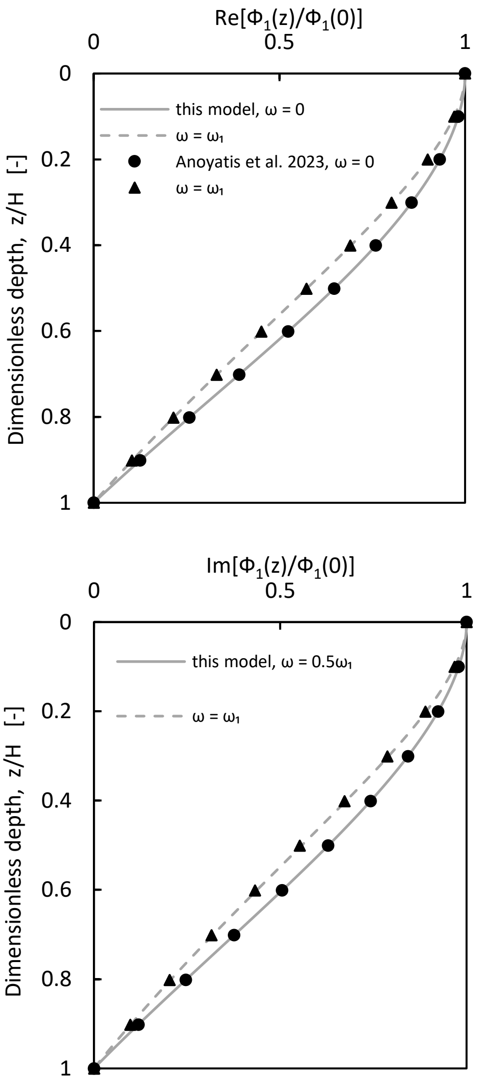

Figure 2. Real and imaginary parts of the first soil mode. Comparison with the closed form solution of Anoyatis et al., 2023.

Figure 3 shows the derivative of the first soil mode, $\Phi_1'(z,\omega)$, which has been normalised respect to its value at $z = H$. The dynamic effect on the soil mode can be clearly seen at the resonance frequency ($\omega = \omega_1$). Evidently, the results obtained by using the approximate proposed model with the semi closed-form solution developed by Anoyatis et al., (2023) are in excellent agreement.

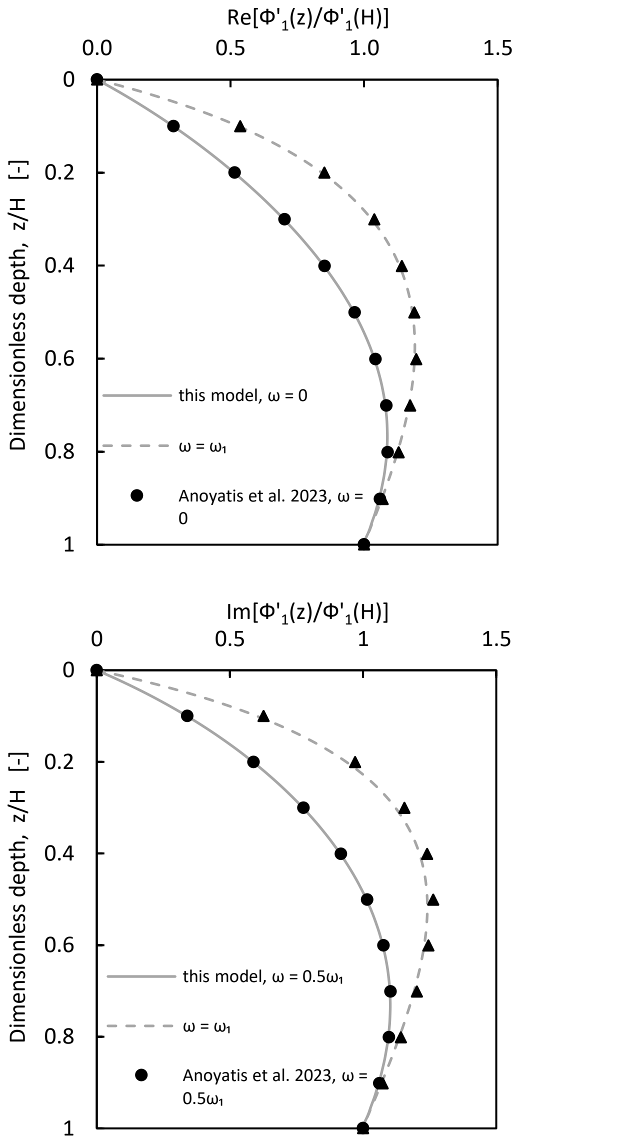

Figure 3. Real and imaginary parts of the derivative of the first soil mode. Comparison with the closed form solution of Anoyatis et al., 2023.

4 SOIL-PILE INTERACTION ANALYSIS

The equilibrium of vertical forces acting on a pile segment yields the equation of the pile motion.

$$-E_p A_p \frac{\partial^2 w}{\partial z^2} + \pi d \tau_0 - \omega^2 \widetilde{m}_p w = F \qquad (11)$$

where A_p is the pile cross sectional area, $\widetilde{m}_p = \rho_p\, A_p$ is the pile material density, and $\tau_0 = \tau(r = d/2, z, \omega)$ is the vertical shear stress at the soil-pile interface. $F = F(z,\omega)$ can be expanded in Fourier series as:

$$F(z,\omega) = \sum_{m=1}^{\infty} f_m\, G_s^*\, \Phi_m(z,\omega) \qquad (12)$$

where the coefficients f_m are obtained by applying the orthogonality of the soil modes:

$$f_m = \int_0^H F(z,\omega)\,\Phi_m\,dz \,/ \int_0^H G_s^* \Phi_m^2\,dz \quad (13)$$

Substituting Equations (6), (7) and (12) into Equation (11), considering perfect bonding at the soil-pile interface and utilizing the weak form, yields the following equation which allows the evaluations of the coupled B_m Fourier coefficients:

$$E_p A_p \sum_{m=1}^{\infty} B_m\,K_0(s_m) \int_0^H \Phi_m' \Phi_k'\,dz \; + \; 2\pi\, s_k\, B_k\, K_1(s_k) \int_0^H G_s^* \Phi_k^2\,dz \; - \; \omega^2\,\widetilde{m}_p \sum_{m=1}^{\infty} B_m\,K_0(s_m) \int_0^H \Phi_m \Phi_k\,dz = P\,\Phi_k(0) \quad (14)$$

where $s_m = q_m d/2$ is a dimensionless parameter.

The pile response w is given by the following expression:

$$w(z,\omega) = \sum_{m=1}^{\infty} B_m\,K_0(s_m)\,\Phi_m(z,\omega) \quad (15)$$

5 PILE STIFFNESS AND DAMPING

The pile head stiffness is defined as the ratio of the force P acting on the pile head over the corresponding displacement:

$$K^* = P \Big/ \sum_{m=1}^{N} B_m\,K_o(s_m)\,\Phi_m(z,\omega) \quad (16)$$

where $K^* = K^*(\omega)$ is a complex-valued stiffness. The real part represents the dynamic stiffness, and the ratio of the imaginary part over twice its real part, the damping ratio $\zeta(\omega) = Im[K^*(\omega)]/2Re[K^*(\omega)]$.

Results of the normalised dynamic pile head stiffness with normalised excitation frequency, for two values of E_p/E_{sH}, for a pile slenderness ratio $L/d = 25$, and for three types of inhomogeneity i.e., $n = 0.5$; $n = 1$, and $n = 2$ are shown in Figure 4. The soil has a stiffness ratio between the surface and the bottom $b = 0.25$ (Eq. (1)), damping ratio $\beta_s = 0.05$ and Poisson's ratio $\nu_s = 0.4$. The values for the pile-soil material density ratios are $\rho_p/\rho_s = 1.25$ for $E_p/E_{sH} = 100$ and $\rho_p/\rho_s = 1.5$ for $E_p/E_{sH} = 1000$. The number of modes used is 50.

The proposed predictions are compared against results from the work of Anoyatis et al., 2023. It is shown that the models are in perfect agreement. Some general trends are observed: for frequencies below the resonance, the dynamic stiffness decreases monotonically with frequency. Such attenuation is stronger for stiffer soils, i.e., $E_p/E_{sH} = 100$. At resonance, a drop in stiffness is observed. Beyond the resonance, the stiffness increases. For $n = 0.5$ the values of the dynamic pile stiffness are higher than the case of $n = 1$ and 2 since the corresponding soils are stiffer. As a consequence, the resonance is reached at higher values of excitation frequency.

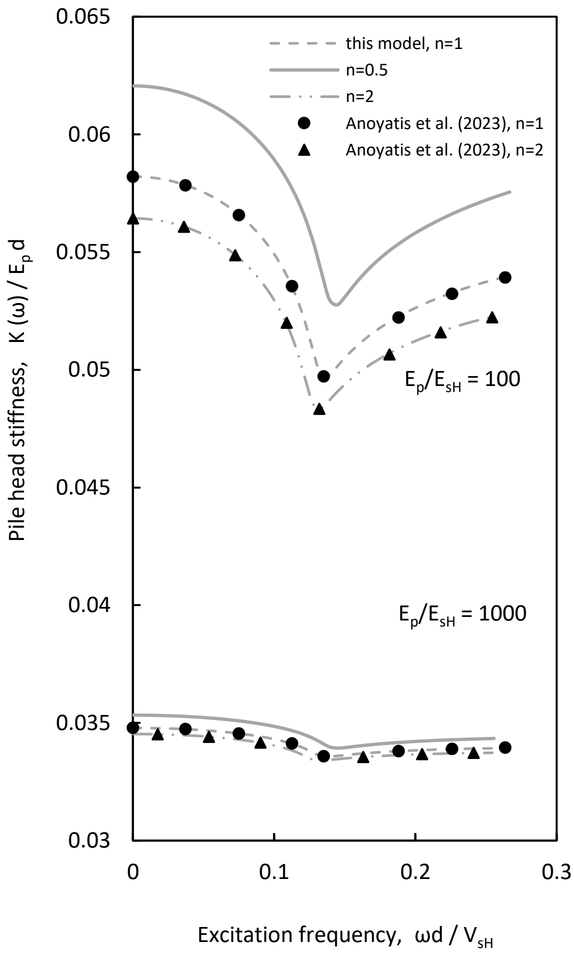

Figure 4. Variation of normalized dynamic pile head stiffness with normalized excitation frequency, for different types of soil inhomogeneity. Comparison with results from Anoyatis et al., 2023.

Figure 5 plots the variation of the pile head damping $\zeta(\omega)$ with the dimensionless frequency ω/ω_1 for $n = 0.5$ and for selected values of pile slenderness ratios. For all cases examined, the damping is practically unaffected by frequency in the low-frequency range. In the vicinity of the first resonance, it exhibits a jump, marking the emergence of travelling waves in the medium, and thus radiation damping.

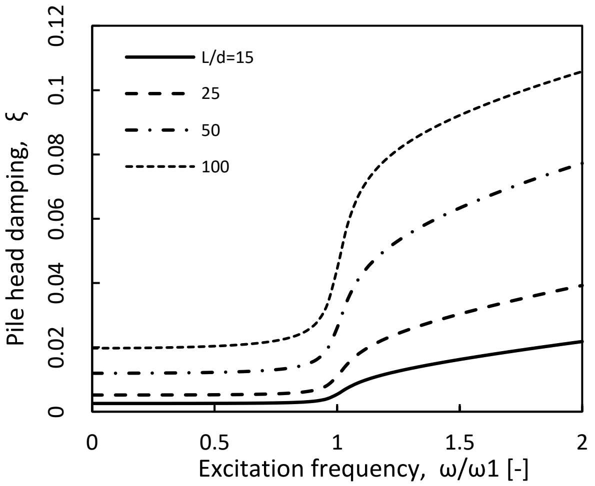

Figure 5. Variation of damping ξ with excitation frequency ω/ω_1. $n = 0.5, b = 0.25,\ \beta_s = 0.05,\ \nu_s = 0.4, E_p/E_{sH} = 1000,\ \rho_p/\rho_s = 1.5$.

6 CONCLUSIONS

A novel semi-analytical solution is derived for the analysis of end-bearing piles that are embedded in a generalized vertically inhomogeneous soil stratum over a rigid base that is subjected to an axial harmonic force. The proposed approximate-continuum solution uses pertinent trigonometric functions to approximate the actual shape functions. This approach is shown to generate results that are in perfect agreement with results obtained from the more rigorous model developed by Anoyatis et al, 2023, limited to specific types of soil inhomogeneity, linear and parabolic variation of the soil stiffness with the depth. The proposed model, with minor modifications, could further take into account transient loads and generate time histories of the pile response, displacement and stress fields.

The Python scripts and data of this study will be made openly available on the KU Leuven webpage titled 'Soil–Structure Interaction Toolkit', at the URL: https://bwk.kuleuven.be/projects/SSItk/, which aims to host scripts and data for a wide range of commonly met soil-pile interaction problems.

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