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Simple probabilistic analysis of drained and undrained slope stability using Janbu‘s direct method

Analyse probabiliste simple de la stabilité en comportement drainé et non drainé d'une pente utilisant la méthode directe de Janbu

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ABSTRACT: It is well known that parameters characterizing the mechanical behavior of soils are highly uncertain. With the advancement of numerical methods and increasing computational capacity, probabilistic analysis is where geotechnics should be heading in the near future. In this short essay, focus will be on simple probabilistic analysis of slope stability problems using Janbu's stability charts in the spreadsheet platform. Both drained and undrained conditions will be considered. Furthermore, the results of the analysis are interpreted in terms of partial factors such that they can be used with code specifications either according to absolute partial factor requirement or the so-called «percentage improvement» requirement for natural slopes.

RÉSUMÉ: Il est bien connu que les paramètres caractérisant le comportement mécanique du sol sont très incertains. L'avancée des méthodes de calcul numérique et l'amélioration des capacités de traitement rendent possible, dans un futur proche, l'utilisation courante de l'analyse probabiliste en géotechnique. Dans cet article, l'attention est portée sur une analyse probabiliste de stabilité de pente basée sur des tableaux de stabilité implémentés dans des feuilles de calcul. Les comportements drainé et non drainé sont présentés. Les résultats sont présentés sous forme de coefficients partiels afin d'être utilisés en lien avec les exigences présentées dans les codes en terme de facteur de sécurité partiel ou dans l'approche nommée « percentage improvement » pour l'amélioration de la stabilité de pentes naturelles.

Keywords: Janbu‘s direct method; stability charts, probabilistic analysis; slope stability; partial factors; percentwise improvement

1 INTRODUCTION

Probabilistic methods give a scientific way of dealing with uncertainties. However, their use in the day today geotechnical engineering is limited. Part of the reason is the (real and assumed) longer calculation time required to perform probabilistic analyses and the limited experience to check the results. However, with the current development of computing machines and various easy to use tools this trend is likely to change.

Simple probabilistic models can be easily programmed in to a spreadsheet (Low 2003) and can be used for performing simple probabilistic analysis. This paper intends to present a probabilistic analysis of slope stability problems using the

Monte Carlo and the Hasofer-Lind probabilistic analysis methods. Janbus stability charts (Janbu 1954a) are used for defining the performance function related to the slope stability. For facilitating their use in a spreadsheet, some analytical forms which approximate the charts with reasonable accuracy were developed. Further, the linking of the results to partial factors is demonstrated.

2 JANBU'S STABILITY CHARTS

Janbu (1954a) established a number of charts, collectively known as stability charts, for a simplified calculation of factor of safety of slopes under drained and undrained conditions. Next, Janbu's stability charts are introduced and some approximating formulas are given.

2.1 Drained

For drained conditions, the factor of safety, F, considering a toe slip circle, is defined as:

$$F = \frac{N_{cf} c}{P_d} \quad (1)$$

where $N_{cf} = f(\lambda_{c\varphi} = \frac{P_e \tan(\phi)}{c}, cot\theta)$ is the stability number that is red from charts for a given slope angle θ and calculated $\lambda_{c\varphi}$. c and ϕ are the spatial average cohesion and the spatial average friction angle of the soil mass. $P_d = \frac{\gamma H + q - \gamma_w H_w}{\mu_q \mu_w \mu_t}$ is the driving force and $P_e = \frac{\gamma H + q - \gamma_w H'_w}{\mu_q \mu'_w}$ is the resisting force, γ is the unit weight of the soil mass, H is the slope height, q is the surcharge load, γ_w is unit weight of water and H_w is the tail water depth, μ_q, μ_w, μ'_w and μ_t are correction factors for surcharge, tail water, body water and tension cracks respectively. The stability charts include charts for these correction factors as well, See Figure 1 for illustration.

The following approximations were found to represent the various curves adequately.

$$\mu_w \approx a_w\,(H_w/H)^2 + b_w\,(H_w/H) + c_w \quad (2)$$

$$\mu_q \approx a_q\,(q/\gamma H)^2 + b_q\,(q/\gamma H) + c_q \quad (3)$$

$$\mu_t \approx b_t\,(H_t/H) + c_t \quad (4)$$

where all fitting parameters are functions of the slope angle, θ, Table 1. And the stability number, N_{cf}, is approximated according to

$$N_{cf} \approx a_N(\lambda_{c\varphi}) \cot\theta + b_N(\lambda_{c\varphi}) \quad (5)$$

N_{cf} is partitioned in $\cot\theta$ for producing simple equations that can adequately represent the curves, Table 2.

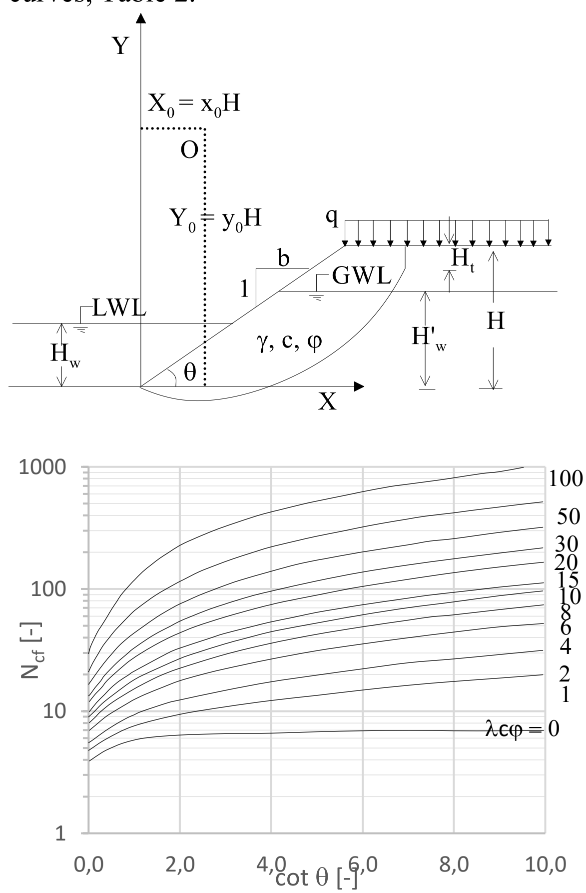

Figure 1. Sketch and plots for the drained stability number

Table 1. Values for fitting parameters

	θ^2	θ^1	1^0

a_w	-5.49×10^{-5}	1.1×10^{-2}	5.3×10^{-3}
b_w	5.07×10^{-5}	-1.09×10^{-2}	5.74×10^{-3}
c_w	0	0	1
a_q	3.564×10^{-5}	3.268×10^{-3}	3.3×10^{-3}
b_q	0	$-8{,}2\times10^{-3}$	-6.222×10^{-4}
c_q	0	0	1
b_t	-2.517×10^{-5}	6.671×10^{-3}	1.95×10^{-3}
c_t	0	0	1

Table 2. Values for fitting parameters a_N and b_N

$\cot\theta/\lambda_{c\varphi}$	a_N	b_N
<1	$0.92\lambda_{c\varphi}+1.712$	$0.5\pi(1-f_1)$
(1,2)/10	$1.585\lambda_{c\varphi}+5.395+0.5\pi(1-f_1)$	$-0.665\lambda_{c\varphi}-3.683-\pi(1-f_1)$
(1,2)/≥10	$0.93(\lambda_{c\varphi}+\frac{\sqrt{2}\pi}{3})$	$-0.5\pi(\sqrt{3}-f_2)$
≥2/<10	$1.023\lambda_{c\varphi}+0.271$	$-2a_N+2.505\lambda_{c\varphi}+7.107$
≥ 2/≥10	$1.023\lambda_{c\varphi}+0.271$	$-2a_N+2.073\lambda_{c\varphi}+11.95$

$f_1=\sqrt{4\lambda_{c\varphi}+13}$; $f_2=\sqrt{4\lambda_{c\varphi}+22}$

2.2 Undrained

For undrained conditions, the factor of safety, F, is defined as:

$$F=\frac{N_0 c_u}{P_d} \tag{6}$$

where the stability number N_0 is red from a chart, Figure 2, depending on the slope angle and the depth of hard bedding below the foot of the slope, c_u is the spatial average undrained shear strength of the soil and, P_d is the driving force calculated according to

$$P_d=\frac{\gamma H+q-\gamma_w H_w}{\mu_q\mu_w\mu_t} \tag{7}$$

The various variables and the correction factors are as defined before.

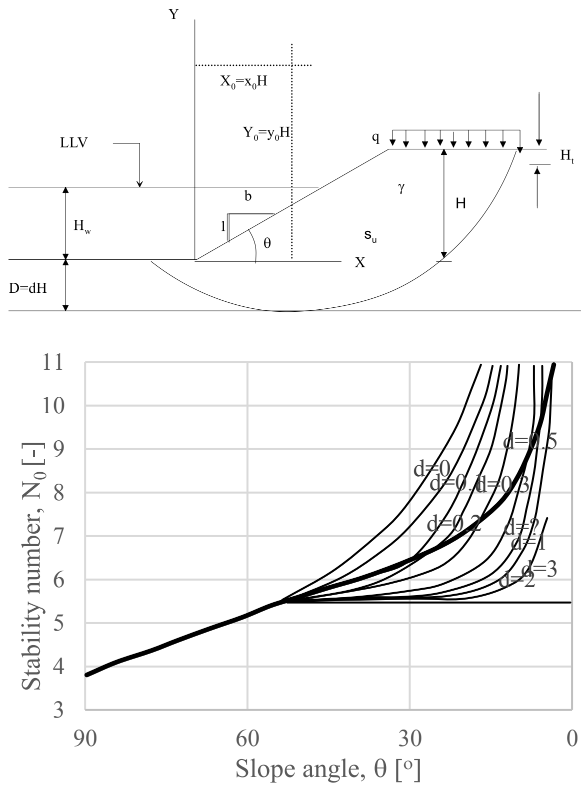

Figure 2. Sketch and plots of undrained stability number against slope angle.

3 USING THE MONTE CARLO METHOD

The Monte Carlo method is among the level III probabilistic methods and uses the possibility of drawing random numbers from a uniform probability density function between zero and one (CUR-publicatie 190. 1997).

The non-exceedance probability of an arbitrary random variable is uniformly distributed between zero and one, regardless of the distribution of the variable, i.e.

$$F_x(x)=x_u;\ x=F_x^{-1}(x_u), \tag{8}$$

where x_u is the uniformly distributed variable between zero and one; $F_x(x)$ is the non-exceedance probability $p(x<\text{x})$, F_x^{-1} is the inverse of the prob-

ability distribution of x. Using this formula a random number x can be generated from an arbitrary distribution $F_x(x)$ drawing a number x_u from the uniform distribution between zero and one.
By taking i number of realizations of the uniform probability distribution between zero and one, a value can be determined for every x_i,

$$x_i = F_{xi}^{-1}(x_{ui}/x_1, x_2, \dots, x_{i-1}) \quad (9)$$

For statistically independent base variables, this can be simplified to $x_i = F_{xi}^{-1}(x_{ui})$.
By repeating this procedure a large number of times and evaluating the performance function, the probability of failure can be estimated with

$$p_f \approx \frac{n_f}{n}, \quad (10)$$

where n is the total number of simulations, n_f is the number of simulations, for which the performance function Z<0. The reliability index is then calculated from the failure probability assuming normal distribution. The sensitivity of the reliability index to each partaking random variable can then be approximated by:

$$\alpha_i = -\frac{cov(x_i, Z)}{\sigma_{xi}\sigma_Z} \quad (11)$$

The required minimum number of simulations is related to the maximum relative error accepted, the reliability of the resulting failure probability and the failure probability itself.
The number of simulations can be reduced by applying importance sampling.

The following cookbook is followed for implementing Monte Carlo probabilistic assessment of slope stability in a spreadsheet using Janbu's direct method for defining the performance function.

- A uniformly distributed random number between 0 and 1 is generated for each random variable using the Excel function, Rand ().
- The normal inverse of the generated numbers is calculated and transformed into the specified distribution space according to the input mean and standard deviation of the variable in concern.
- A performance function, $Z = F - 1 = 0$ is defined according to Janbu's direct method; the performance function is then evaluated for the combination of the randomly generated input parameters.
- Out of a total of n realizations, the number of times Z < 0 is counted and the probability of failure is approximated according to Equation 10.
- The reliability index is calculated by taking the negative of *normsinv* of the failure probability. *Normsinv* is an inbuilt excel function for calculating the inverse of probability of an event for a normally distributed variable.
- The sensitivity coefficient of each partaking variable is calculated according to Equation 11.

Note that for the sake of simplicity the base variables are assumed to be statistically independent. Example calculations are presented next.

3.1 Example probabilistic analysis of drained stability

An example slope of height 10.3 m and inclination 26 degrees with soil parameters and their distribution in Table 3 is considered. Monte Carlo simulation was carried out in a spreadsheet by generating a total of 150,000 uniformly distributed random numbers between 0 and 1 for each parameter. The generated random numbers show some weak correlation but not that significant. The variation of the various quantities are presented in Figures 3, 4, 5 and 6. The probability of failure is then calculated to be about 0.034 which corresponds approximately to a reliability index of 1.82. The calculated sensitivity coefficients and the design values of each partaking variable is presented in Table 4. For the calculated example, the dominant strength variable is the friction angle and the dominant load variable is the body water height, H'_w.

Table 3. Input parameters for the example slope

Variable	Dist.	mean	stdv.
Unit weight of soil, γ_s [kN/m^3]	LN	20	1
Cohesion [kPa]	LN	2.9	0.6
Friction angle, φ [º]	LN	30	1.7
Surcharge load, q [kPa]	LN	10	4
Tension cracks, H_t [m]	LN	0.5	0.1
Tail water depth, H_w [m]	LN	2	0.2
Body water height, H_w' [m]	LN	6	0.6

LN =Log Normal

Table 4. Sensitivity and value of each partaking random variable at the design point

Variable	α	v*
Unit weight of soil, γ_s [kN/m^3]	-0.1	20.1
Friction angle, φ [º]	0.76	27.7
Cohesion [kPa]	0.40	2.5
Surcharge load, q [kPa]	-0.02	10.2
Tension cracks, H_t [m]	-0.02	0.5
Tail water depth, H_w [m]	0.1	2
Body water height, H_w' [m]	-0.5	6.5

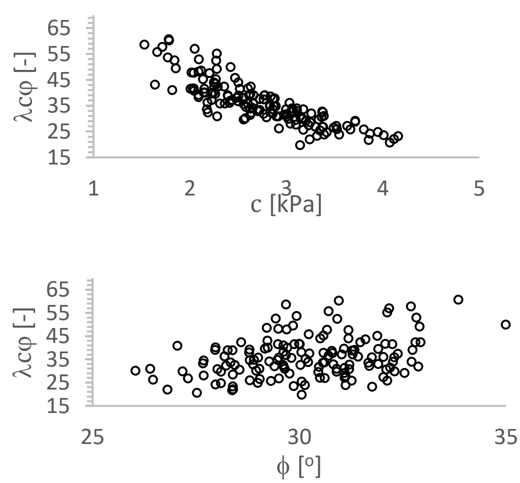

Figure 3. The variation of the variable $\lambda_{c\varphi}$ with cohesion and friction during the Monte Carlo simulations

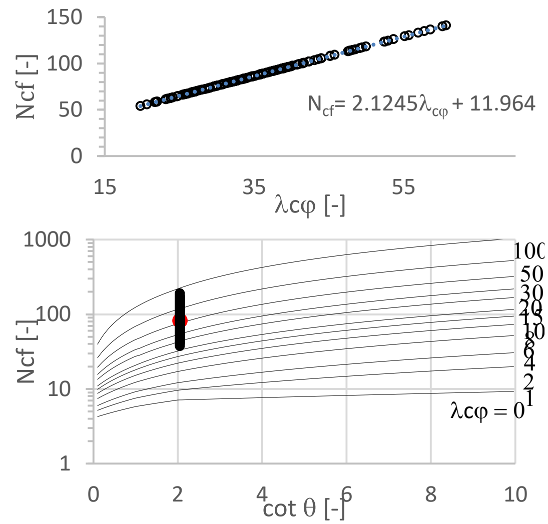

Figure 4. Variation of the stability number with $\lambda_{c\varphi}$ during the Monte Carlo simulations.

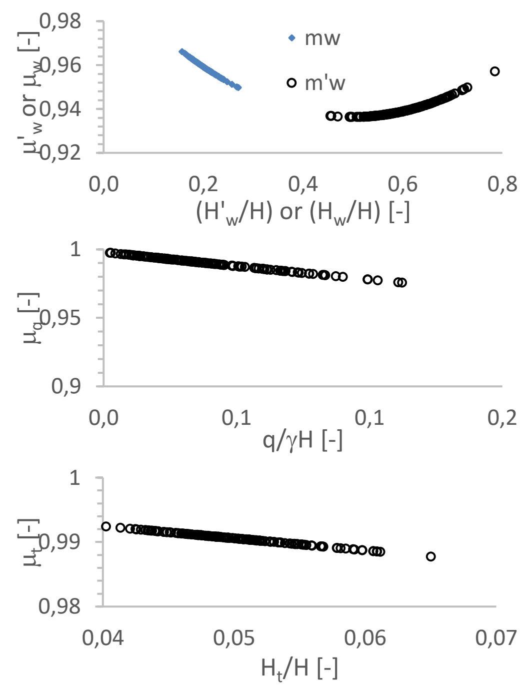

Figure 5. Variation of the correction factors during the Monte Carlo simulations

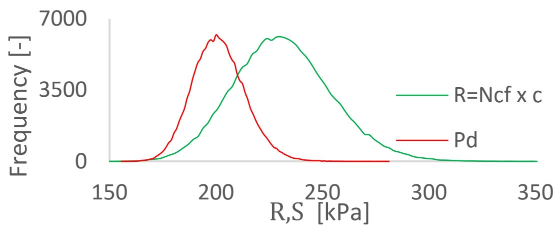

Figure 6. Frequency analysis results of the resistance and the load variables.

3.2 *Example probabilistic analysis of undrained stability*

Here, we consider the slope geometry in the previous example. Input parameters are presented in Table 5. Furthermore, assuming toe circle, the stability number, $N_0 = 6.6$ is red from the corresponding diagram, Figure 2. The depth parameter can also be considered as a stochastic variable.

Monte Carlo simulation was carried out in a spreadsheet by generating a total of 107278 uniformly distributed random numbers between 0 and 1 for each parameter. The generated random numbers show some weak correlation but not that significant. The probability of failure is then calculated to be about 6.8 x 10^{-2} which corresponds approximately to a reliability index of 1.49. The frequency distribution of the various parameters, the resistance and the load and the performance function are presented in Figure 7. Note that the region $S \geq R$ $(F - 1 \leq 0)$ is where failure is expected. Table 6 contains the influence coefficient of each partaking variable and their value at the design point.

Table 5. Input parameters for the example slope

Variable	Dist.	mean	stdv
Unit weight of soil, γ_s[kN/m^3]	LN	20	1
Undrained shear strength, c_u [kPa]	LN	44	8.8
Surcharge load, q [kPa]	LN	10	5
Tension cracks, H_t [m]	-	-	-
Tail water depth, H_w [m]	-	-	-

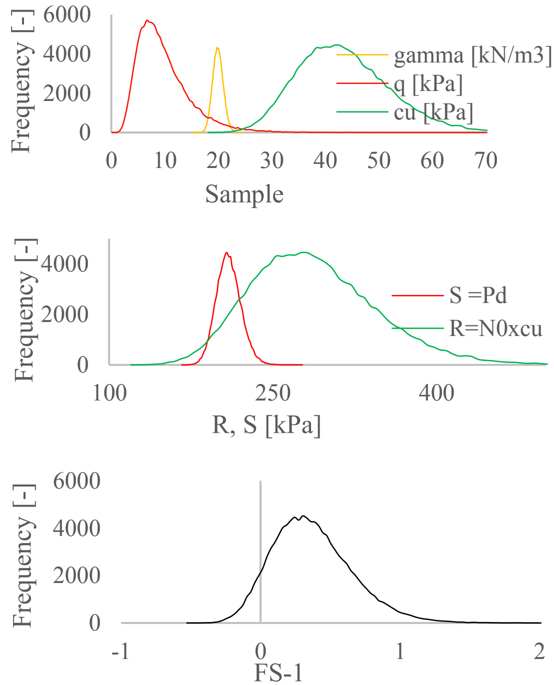

Figure 7. The frequency distribution of the input variables (γ, q and c_u), resistance ($R=N_0c_u$) and loads ($s=P_d$) and the performance function (Z=F-1).

Table 6. Sensitivity and value of each part

Variable	α	v*
Unit weight of soil, γ_s[kN/m^3]	0.23	20.3
Undrained shear strength, c_u	-	31.3
[kPa]	0.97	
Surcharge load, q [kPa]	0.11	10.9
Tension cracks, H_t [m]	-	-
Tail water depth, H_w [m]	-	-

α =sensitivity, v*=values of the variables at the design point.

4 USING THE HASOFER-LIND METHOD

The Hasofer-Lind Method (Hasofer and Lind, 1974), falls in the category of Level II reliability methods. It entails linearizing the performance function at a carefully selected point and approximating the probability of distribution of each variable by a standard normal distribution function. The methodology is summarized in the following three steps:

- At the design point, the distribution of each variable is approximated by a normal distribution function - according to the so-called normal tail approximation (Rackwitz and Fiessler, 1978).
- Then, the value of each variable at the design point is transformed to a standard normal space. The transformation rule is according to

$$u_i = \frac{v_i - \mu_i}{\sigma_i} \quad (12)$$

  where v_i is a vector of random variables, and μ_i and σ_i are respectively their mean and their standard deviation.
- The minimum distance from the edge of the limit state surface (Z=0) to the origin in the transformed co-ordinate system is then the Hasofer-Lind reliability index and is obtained by solving the optimization problem:

$$\beta = \min_{z=0} \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{u}} \quad (13)$$

where, **u** (Equation 12) is the vector of the transformed random variables and **R** is the correlation matrix. In so doing, we find the most probable failure point.

The sensitivity coefficients are calculated according to

$$\alpha_i = -\frac{v_i^* - \mu_i}{\beta\sigma_i} \quad (14)$$

where v_i^* are values of the partaking variable at the design point. In which we adopt the sign convention that α_i is positive for strength like variables and negative for load like variables.

The Hasofer-Lind method is implemented in excel spreadsheet together with Janbu's direct method for both short-term (undrained) and long-term (drained) reliability analysis of stability of slopes. The inbuilt Excel solver is used for performing the optimization. Next, the previous examples are recalculated using the Hasofer-Lind method.

4.1 Probabilistic analysis of drained stability

Input parameters are given in Table 3. The calculated reliability index is 1.89. The probability of failure is then approximated as $p(F \leq 0) \approx \Phi(-\beta) = 0.03$. The resulting sensitivity coefficients and values at the design point are presented in Table 7. The results compare well with those obtained using the Monte Carlo method.

The sensitivity coefficients imply that the dominant strength variable is the friction angle and the dominant load variable is the body water height.

Table 7. Sensitivity coefficients and value of each partaking random variable at the design point

variable	α	v*
Unit weight of soil, γ_s [kN/m^3]	-0.01	20
Friction angle, φ [°]	0.74	27.7
Cohesion [kPa]	0.38	2.5
Surcharge load, *q* [kPa]	-0.13	10
Tension cracks, H_t [m]	-0.03	0.5
Tail water depth, H_w [m]	0.14	1.94
Body water height, H_w' [m]	-0.52	6.6

4.2 Probabilistic analysis of undrained stability

The example slope used in the Monte Carlo simulation with the input parameters in Table 5 is considered. The calculated reliability index is 1.33. The probability of failure is then approximated as $p(F \leq 0) \approx \Phi(-\beta) = 0.091$. The resulting sensitivity and values at the design point are presented in Table 8. These results compare well with those obtained from the Monte Carlo simulation.

Some analytical solutions which can be used for a quick check are presented in Appendix.

Table 8. Sensitivity and value of each part

Variable	α	V*
Unit weight of soil, γ_s[kN/m^3]	-0.23	20
Undrained shear strength, c_u [kPa]	0.97	33
Terrain load, q [kPa]	-0.10	10
Tension cracks, H_t [m]	0	-
Tail water depth, H_w [m]	0	-

5 RELATING TO THE PARTIAL FACTOR APPROACH

5.1 *Partial factors*

In the limit state approach, the representative values of the variables associated with the resistance are divided by a factor greater than one, say γ_R, while representative values of the variables associated with loads are multiplied by a factor less than one, say γ_S. The multiplying factors, γ_R and γ_S are called partial factors, Figure 8.

The partial factors associated with each variable can be calculated according to

$$\gamma_i = \frac{1-k_iV_i}{1-\alpha_i\beta V_i} \tag{15}$$

where V_i is the coefficient of variation and defined as the quotient of standard deviation to mean, k_i depends on the definition of the characteristic value of each parameter. The definition of the characteristic strength is somewhat arbitrary in the geotechnical engineering practice. For instance mean value, a low quantile in the probability distribution (5% quantile, 2.3% quantile) and the most probable value have been recommended (DNVGL-RP-C207, 2017).

Equation 15 implies that γ_i is greater as:

- the absolute value of the influence coefficient α_i is greater
- the desired reliability β is higher.
- the variance coefficient v_i is greater.

Partial factors are calculated for the undrained Monte-Carlo simulation is presented in Table 9, for instance.

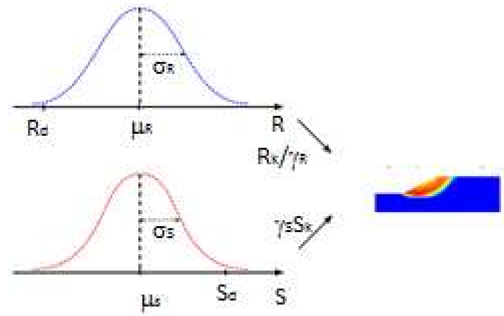

Figure 8. Illustration of the partial factor principle (R-resistance, S-load)

Table 9. Input parameters for the example slope

Variable	γ_i
Unit weight of soil, γ_s[kN/m^3]	0.98
Undrained shear strength, c_u [kPa]	1.40
Surcharge load, q [kPa]	0.92

5.2 *Required absolute partial factor*

Once the results are interpreted in terms of partial factors, they can be compared to code specified values. Design codes provide partial factors for resistances and loads. The partial factors derived from the probabilistic analysis calculated according to Equation 15 can be compared to those provided in the codes. The slope can be said to be safe enough if the calculated partial factors are higher than those required in the codes. Furthermore, the required absolute partial factor can be calculated based on target reliability index according to the reliability class of the project (Länsivaara and Poutanen 2013) or depending on acceptable consequences.

5.3 *Percentage improvement for undrained slopes*

For natural slopes, the philosophy of percentage improvement has been proposed (NVEs 7/2014). The required percentage improvement is defined as

$$\% - imp. = \frac{\gamma_{imp}-\gamma_{initial}}{\gamma_{initial}} \tag{14}$$

where γ_{imp} is the partial factor for the resistance after the improvement. Note that – partial factors can be improved by improving one or more of the following derived quantities - reliability index, sensitivity, and coefficient of variation of the given parameter. These quantities can further be related to the geometry of the slope (height and slope angle), the mean value of the shear strength and its standard deviation, the unit weight of the soil, other external loads and restraints. Improving one or more of these inputs should improve the reliability of the slope.

The procedures for linking results from probabilistic analysis with the percentage improvement concept through the method partial factors is as follows.

- Calculate the initial partial factor of the desired parameter, say the undrained shear strength.
- Obtain (from chart, eg. Figure 10) the percent improvement required and thus the desired partial factor which shall be obtained after improvement.
- Perform reliability analyses that includes improvement and calculate the new partial factor depending on the newly obtained reliability index and compare it to the desired partial factor.
- If the stabilizing measure is satisfactory, then propose the measure, if not propose an additional measure, or a new measure until required the partial safety factor is achieved.

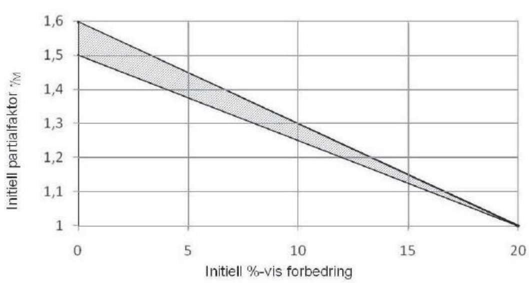

Figure 10: Required % improvement of stability conditions for total stress analyses. NVE 7/2014.

6 SUMMARY AND CONCLUSIONS

Simple spreadsheet based probabilistic analyses of slope stability problems using the Monte Carlo method and the Hasofer-Lind method were presented.

The performance function is defined according to Janbu's direct method which entails the use of a number of charts. For making the spreadsheet implementation possible, analytical approximations of some of the curves of Janbu's stability charts were developed.

From the probabilistic slope stability analysis which is done in a matter of seconds, one obtains the reliability index, the probability of failure, the sensitivity coefficients of each partaking variable, and various important plots as demonstrated using examples.

From the calculated sensitivity coefficients of the partaking variables one can easily notice variables that dominate the slope stability problem more objectively.

Probabilistic approaches offer us a powerful framework for looking at and taking into consideration of uncertainties in various parameters systematically. The existing experience and recommendations in design codes can be maintained by evaluating the results in terms of partial factors. Interpretation of the results of probabilistic analyses in terms of partial factors allows the practicing engineer to evaluate them according to either absolute partial factor requirement or the %-improvement requirement. However, for using the partial factor concept, the characteristic value must be clearly and precisely defined.

The approach being accessible to the Excel platform can be a handy tool for the practicing engineer. The results would fall short in areas where Janbu's stability charts are not adequate enough. Janbu's stability charts were derived assuming circular slip surfaces and can therefore be more applicable to slope stability problems which can be adequately solved with circular slip surfaces. One way to enhance the model without significantly changing the approach is considering a variable for model uncertainty into the performance function.

In this paper, lognormally distributed random variables were considered. However, other distributions can be easily implemented.

7 REFERENCES

CUR-publicatie 190. 1997. Probabilities in civil engineering, Part 1: Probabilistic design in theory. Stiching CUR, Gouda.

DNVGL-RP-C207. 2017. Statistical representation of soil data. Recommended practice.

Hasofer, A.M, Lind, N.C. 1974. Exact and invariant second-moment ode format. *J. Engrg Mechanics* **100** 111-121.

Janbu, N. 1954a. Stability analysis of slopes with dimensionless parameters. Thesis for the Doctor of Science in the Field of Civil Engineering, *Harvard University Soil Mechanics Series* **46**.

Low, B.K. 2003. Practical probabilistic slope stability analysis. *Proceedings, Soil and Rock America 2003, 12th Pan-American Conference on Soil Mechanics and Geotechnical Engineering and 39th U.S. Rock Mechanics Symposium*, 2777-2784. Verlag Glükauf GmbH Essen, M.I.T., Cambridge, Massachusetts.

NVE Veileder 7/2014. 2014. Sikkerhet mot kvikkleireskred, Vurdering av områdestabilitet ved arealplanlegging og utbygging i områder med kvikkleire og andre jordarter med sprøbruddegenskaper, ISSN 1501-0678

Rackwitz, R., and Fiessler, B. 1978. Structural reliability under combined random load sequences. *Computers & Structures* **9**, 484-494.

Länsivaara, T., Poutanen T. 2013. Slope stability with partial factor method. *Proceedings of the 18th International Conference on Soil Mechanics and Geotechnical Engineering*, 1823-1826. Paris.

APPENDIX

For undrained slope stability problems, the performance function is linear. If the parameters are assumed to be normally distributed, the reliability index can be solved analytically and is given as:

$$\beta = \frac{N_0\mu_{cu} - H\mu_\gamma - \mu_q}{\sqrt{\boldsymbol{\sigma}^T \mathbf{R} \boldsymbol{\sigma}}} \tag{1}$$

where **R** is the correlation matrix, **σ** is the vector of standard deviation of each random variable multiplied by constants that appear before each variable in the performance function. A reliability index of 1.258 is obtained for the example in section 4.2. It is known that considering normal distribution gives a conservative estimate of the reliability index. Furthermore, if the variables were uncorrelated, i.e., $\rho = 0$, the influence factors, α for each variable can then be shown to be:

$$\alpha_{cu} = \frac{N_0\sigma_{cu}}{\sqrt{N_0^2\sigma_{cu}^2 + H^2\sigma_\gamma^2 + \sigma_q^2}} \tag{2}$$

$$\alpha_\gamma = -\frac{H\sigma_\gamma}{\sqrt{N_0^2\sigma_{cu}^2 + H^2\sigma_\gamma^2 + \sigma_q^2}} \tag{3}$$

$$\alpha_q = -\frac{\sigma_q}{\sqrt{N_0^2\sigma_{cu}^2 + H^2\sigma_\gamma^2 + \sigma_q^2}} \tag{4}$$

It can be easily shown that $\alpha_{cu}^2 + \alpha_\gamma^2 + \alpha_q^2 = 1$. For the input parameters in Table 5, the influence factors are calculated to be $\alpha_{cu} = 0.98$, $\alpha_\gamma =$ -0.169 and $\alpha_q =$ -0.084. Analytical solution can be formulated if the variables were lognormally distributed by considering the performance function

$$Z = lnN_0 + lnc_u - lnP_d. \tag{5}$$

A reliability index of 1.31 was found for the example in section 4.2, assuming load and strength variables are lognormally distributed.