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Seismic response of n-layered soil: Exact analytical closed-form solution

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ABSTRACT: An original exact analytical closed-form solution for the seismic response of n-layers soil is exposed in this paper. Fundamental frequency of n-layers soil is obtained as a solution of a closed-form equation, having a condensed expression, explicit in terms of the soil properties, established for the multilayer soil. The displacement at any point of the soil is obtained by explicit formulas depending on the soil dynamic and geometrical properties, and fundamental mode frequency. Numerical applications for 2, 3, 4 and 26-layers soil profiles show comparative results with presently used approximate methods of Modules average, Dobry-Madera and Simplified Rayleigh Procedure. In fact, the presented method constitutes a general solution for n-layers soil seismic response and allows to compute frequency and modal shape for all eigenmodes. Results are shown for modes 1, 2 and 3 for the previously mentioned soil profiles.

1 BASIC EQUATIONS

The soil is considered as an elastic inhomogeneous medium of axes x, y, z (z vertical axis directed downwards). The problem is reduced to a problem of plane strain in the plane (x, z), and the displacement u is only horizontal. It is also considered that the displacement u_x does not depend on x. Hence, $u_x(x, y, z, t) = u_x(z, t)$.

We consider a soil formed of n deformable horizontal layers (n ≥ 2), resting on a rigid bedrock (layered soil). Each layer i is homogeneous and isotropic, with a constant density ρ_i. The thickness of the layer i is denoted H_i. The total thickness of layers 1 to n is denoted H.

Each soil layer being homogeneous and isotropic, one can synthetically write a linear elastic constitutive law for the layered medium, with piecewise constant elastic coefficients E(z) and υ (z). As a result, the velocity of elastic shear waves V is constant in each layer i.

Introducing the shear modulus: $G = \frac{E}{2(1+\upsilon)}$, the equation of motion is then written:

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial}{\partial z}\left(G \frac{\partial u_x}{\partial z}\right) \tag{1}$$

The discontinuity equations at each interface between two layers result in the following interface conditions (the symbols + and – refer to both sides of the interface):

– Continuity of the stress vector at the interface, which leads to:

$$G^{+}\left(\frac{\partial u_x}{\partial z}\right)^{+} = G^{-}\left(\frac{\partial u_x}{\partial z}\right)^{-} \tag{2}$$

Note that this equation implies the existence of a slope discontinuity of $u_x(z, t)$ at each interface between layers, in the ratio of the shear moduli G^{-}/G^{+}.
– continuity of the displacement at the interface:

$$u_x^{+}(z, t) = u_x^{-}(z, t) \tag{3}$$

The following boundary conditions result from the above interface conditions:

- Free surface condition at z = 0:

$$\frac{\partial u_x}{\partial z}\ (0,t)=0 \tag{4}$$

- Rigid bedrock condition at z = H:

$$u_x(H,t)=0. \tag{5}$$

2 EQUATIONS FOR EIGENMODES

Consider a function with separate variables, which satisfies the equation of motion. This function is of the form: $\Phi(z,t)=U_x(z).\ g(t)$. Mathematical developments, which we cannot detail here by lack of space, lead to the following results:

1. There is a number of discrete positive real values ω_j: $0<\omega_1\leq\omega_2\leq\omega_3\leq\ldots\leq\omega_j\leq\ldots$
   To each value ω_j correspond 2 functions $U_x^j(z)$ and $g_j(t)$, such as the function $u_x^j(z,t)=U_x^j(z).\ g_j(t)$ satisfies the equation of motion. The functions $U_x^j(z)$ are the eigenmodes and the values $\lambda_j=-\omega_j^2$ are the eigenvalues. The functions $g_j(t)$ are the periodical response functions applying to the eigenmodes. Their expression is: $g_j(t)=\ C_{1j}\sin(\omega_j t)+C_{2j}\cos(\omega_j t)$. C_{1j} and C_{2j} are constants and ω_j is the angular frequency of the eigenmode j.
2. For our problem, there is a countably infinite set of values ω_j; hence $j\in N^*$. This result will be established further in the text. The expression of the solution in displacement for our problem is:

$$u_x(z,t)=\ \sum_{j=1}^{\infty}U_x^j(z).g_j(t) \tag{6}$$

3. The eigenmodes satisfy the following differential equation, resulting from the PDE motion equation:

$$\frac{d^2U_x^j}{dz^2}+\frac{1}{G}\frac{dG}{dz}\frac{dU_x^j}{dz}+\ \frac{\rho\omega_j^2}{G}U_x^j=0 \tag{7}$$

4. The eigenmodes $U_x^j(z)$ satisfy boundary conditions and interface conditions resulting from the corresponding conditions satisfied by the displacement $u_x(z,t)$.
5. The eigenmodes $U_x^j(z)$ are defined only to a multiplicative constant. We decide to norm the eigenmodes with respect to their surface value: so, we put: $U_x^j(0)=1\forall j\in N^*$.
6. Soil deformation for eigenmode j: We suppose the ground vibrating in mode j alone. Let d_{max}^j the maximum displacement at the ground surface for this case. Let a_{max} the Peak Ground Acceleration (PGA) – the maximum acceleration at the ground surface. As a consequence of the motion equation, we have: $a_{max}=\ \omega_j^2\ d_{max}^j$. Then: $d_{max}^j=a_{max}/\omega_j^2=a_{max}.T_j^2/4\pi^2$, T_j being the period of eigenmode j. The soil deformation for mode j is by definition: $\overline{U_x^j(z)}=d_{max}^j.\ U_x^j(z)$. This definition considers that $U_x^j(z)$ is normed by putting $U_x^j(0)=1$.

3 SOLUTION FOR THE FUNDAMENTAL MODE

The approach is identical for all the eigenmodes, but in order to fix the ideas, we will make the calculation for the fundamental mode (mode 1). In the following paragraph, we will omit the index j of the eigenmode to facilitate the notations. The index i will refer to the soil layer.

$U_x(z)$ is the shape of the fundamental mode; ω is the angular frequency of the fundamental mode; the period of the fundamental mode is $T=\ 2\pi/\omega$.

3.1 *General expression of the solution in displacement for n-layers soil*

ρ, V and G are constant in each soil layer i, with their values ρ_i, V_i, G_i. Knowing that: $G_i = \rho_i V_i^2$ the equation (7) is reduced in each layer i, to:

$$\frac{d^2 u_{ix}}{dz^2} + \frac{\omega^2}{V_i^2} u_{ix} = 0 \tag{8}$$

where $u_{ix}(z)$ is the expression of displacement in the soil layer i.
The solution of this equation is of the form:

$$u_{ix}(z) = A_i \cos\left(\frac{\omega}{V_i} z + \varphi_i\right) \tag{9}$$

A_i, φ_i being constants depending on the soil layer.

3.2 *Resolution of equations for the n-layers soil*

1) The free surface condition is written for displacement in layer 1: $\frac{du_{1x}}{dz}(0) = 0$. This leads to $u_{1x}(0) = A_1$ and $u_{1x}(z) = A_1 \cos\left(\frac{\omega}{V_1} z\right)$ [The solution with $\varphi_1 = \pi \bmod 2\pi$ is identical to this one].

2) The conditions are written at the interface of layers 1 and 2, where z = H_1:

$$u_{1x}(H_1) = u_{2x}(H_1) \tag{10}$$

$$G_1 \frac{du_{1x}}{dz}(H_1) = G_2 \frac{du_{2x}}{dz}(H_1) \tag{11}$$

Deriving $u_{ix}(z)$ with respect to z, then making the ratio of (11) to (10), we obtain an equation at interface 1- 2, which leads to the expression of φ_2. Substituting this expression in (10), we can deduce the expression of A_2. We then deduce the expression of $u_{2x}(z)$:

$$u_{2x}(z) = A_2 \cos\left(\frac{\omega}{V_2}(z - H_1) + \arctan\left[\frac{\rho_1 V_1}{\rho_2 V_2} \tan\left(\frac{\omega}{V_1} H_1\right)\right]\right).$$

Note: As for φ_1, φ_2 is defined $\bmod \pi$ and hence A_2 can take 2 opposite values. But φ_2 is found in the expression of $u_{2x}(z)$ and the two cases lead to a unique expression of $u_{2x}(z)$, which is thus defined uniquely. This authorizes us for further calculations to retain a value of arctan[ ] between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ without affecting the generality of the results.

3) Formulation of displacement for a layer i:

Let:

$$\alpha_i = \frac{\omega}{V_i} H_i + \beta_{i-1} \text{ for i } 1 \tag{12}$$

$$\beta_i = \arctan(r_i . \tan \alpha_i) \text{ for } 1 \le \text{i} \le \text{n}-1 \text{ and } \beta_0 = 0;\ \beta_i \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\tag{13}$$

With: $r_i = \frac{\rho_i V_i}{\rho_{i+1} V_{i+1}}$

α_i and β_i are angles and are expressed in radian. We have shown in § 3.2.2 that the hypothesis $\beta_i \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$ did not interfere with the generality of the solution.

Then the formulas for calculating the displacement in the soil layer i (with i ≥ 2) are:

$$A_i = A_{i-1} \frac{\cos \alpha_{i-1}}{\cos \beta_{i-1}} \tag{14}$$

$$\varphi_i = \beta_{i-1} - \frac{\omega}{V_i} z_i \tag{15}$$

With $z_i = \sum_{j=1}^{i-1} H_j$ (depth of the upper limit of layer i)

And:

$$u_{ix} = A_i \cos\left(\frac{\omega}{V_i}[z - z_i] + \beta_{i-1}\right) \tag{16}$$

We demonstrate these formulas by recurrence: verifying that they apply for i=2, then showing that if they are valid for (i), this implies that they are then valid for (i + 1), which would prove that they are valid ∀ i integer ∊ [2; n]. Demonstration: a) the verification for i=2 is immediate, knowing the expressions of φ_2 and A_2; b) in layer (i), we have Eq. (16) assumed to be valid. In layer (i + 1), we have the general expression of Eq. (9) with (i+1). We seek to determine A_{i+1} and φ_{i+1}. We write the conditions at the interface of layers (i) and (i+1), where $z = z_{i+1}$. Following a similar approach to the calculation in paragraph 3.2.2, we obtain the expressions of φ_{i+1} and A_{i+1}. Substituting by the expressions of α_i, r_i and β_i, these expressions become: $\varphi_{i+1} = \beta_i - \frac{\omega}{V_{i+1}} z_{i+1}$, and $A_{i+1} = A_i \frac{\cos\alpha_i}{\cos\beta_i}$ and consequently: $u_{(i+1)x} = A_{i+1}\cos\left(\frac{\omega}{V_{i+1}}[z - z_{i+1}] + \beta_i\right)$. QED. We note that the solution thus established, which gives the shape of the fundamental mode, is defined only to a multiplicative constant ($A_1 = u_{1x}(0)$).

4) Equation (E) for modal angular frequency ω: In layer n, by application of the preceding formulas, the displacement is written: $u_{nx}(z) = A_n \cos\left(\frac{\omega}{V_n}[z - z_n] + \beta_{n-1}\right)$. The condition at the interface with the rigid bedrock is written: $u_{nx}(H) = 0$. Hence, knowing that $H - z_n = H_n$: $\beta_{n-1} = \frac{\pi}{2} - \frac{\omega}{V_n} H_n + k\pi,\ k \in Z$. Since $\beta_{n-1} = \arctan(r_{n-1}.\tan\alpha_{n-1})$, we can then deduce the equation (E) for determining the modal angular frequency ω:

$$r_{n-1}.\tan\alpha_{n-1}(\omega) = \cot(t_n\omega) \tag{17}$$

Where $t_n = H_n/V_n$ and r_{n-1} are constants, and $\alpha_{n-1}(\omega)$ is the function of ω defined above in Eqs. (12) & (13).

Developed explicit formulation of Equation (E) for n = 2, 3, 4: Equation (E) is written:

- for n = 2: $\frac{\rho_1 V_1}{\rho_2 V_2}.\tan\left(\frac{H_1}{V_1}\omega\right) = \cot\left(\frac{H_2}{V_2}\omega\right) = 1/\tan\left(\frac{H_2}{V_2}\omega\right)$
  As can be seen from following §5, this formulation (providing that $\omega \neq k\pi V_2/H_2,\ k \in Z$) is equivalent to Madera's equation for a soil with 2 deformable layers above a rigid bedrock.
- for n = 3: $\frac{\rho_2 V_2}{\rho_3 V_3}.\tan\left\{\frac{H_2}{V_2}\omega + \arctan\left[\frac{\rho_1 V_1}{\rho_2 V_2}.\tan\left\{\frac{H_1}{V_1}\omega\right\}\right]\right\} = \cot\left(\frac{H_3}{V_3}\omega\right)$
- for n = 4:

$$\frac{\rho_3 V_3}{\rho_4 V_4}.\tan\left\{\frac{H_3}{V_3}\omega + \arctan\left[\frac{\rho_2 V_2}{\rho_3 V_3}.\tan\left\{\frac{H_2}{V_2}\omega + \arctan\left[\frac{\rho_1 V_1}{\rho_2 V_2}.\tan\left\{\frac{H_1}{V_1}\omega\right\}\right]\right\}\right]\right\} = \cot\left(\frac{H_4}{V_4}\omega\right)$$

6) Number of roots of Equation (E): Putting Eq. (E) in the form: $\varphi(\omega) = 0$, we observe that $\varphi(\omega)$ consists of the sum of 2 functions. The function $-\cot(t_n\omega)$ is periodic, of period $\mathcal{T}_n = \pi/t_n$, admitting a countably infinite number of branches separated by singular values, and varying monotonously in each branch from $-\infty$ to $+\infty$. The 2nd function $r_{n-1}\tan\alpha_{n-1}(\omega)$ is pseudo-periodic, with a similar behavior, although it is more complex to determine the distribution of its singular values or its pseudo-periodicity. Consequently, for each branch of each of these two functions, there is at least one root of Eq. (E). Hence, Eq. (E) has a countably infinite number of discrete roots. As ω intervenes only by its square ω^2 in the equation of motion, we consider only the positive roots of Eq. (E), ordered in ascending order: $0 < \omega_1 \leq \ldots \omega_j \leq \ldots$

7) Solving Equation (E): Equation (E) is an implicit equation for ω, which can be solved by a numerical method with any precision defined in advance. By definition, ω is the smallest positive root of Equation (E). We note that 0 is not a solution of Eq. (E).

8) Condensed explicit formulation of Equation (E): considering the expressions of α_i and β_i, we can write a condensed explicit expression of Eq. (E) for modal angular frequency, as follows.

Let the sequence of functions $\left(\xi_{\omega,i}\right)_{i\in[1;n-1]}$:

$\xi_{\omega,i} : x \mapsto \xi_{\omega,i}(x) = r_i \tan\{t_i\omega + \arctan(x)\}$ for $1 \leq i \leq n-1$, with $t_i = H_i/V_i$; $r_i = (\rho_i V_i)/(\rho_{i+1}V_{i+1})$; $\arctan(x) \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$. Then the function $r_j \tan\alpha_j(\omega)$ can be expressed as:

$r_j \tan \alpha_j(\omega) = (\xi_{\omega,j} \circ \ \xi_{\omega,j-1} \circ \ldots \circ \xi_{\omega,1})(0) = \left(\overset{j}{\underset{k=1}{\mathcal{C}}} \ \xi_{\omega,k}\right)(0)$ with $\circ$ being the operator of composition of functions and $\overset{j}{\underset{k=1}{\mathcal{C}}} \ \xi_{\omega,k}$ a condensed notation for multiple composition of functions. Eq. (E) for modal angular frequency can then be written in a condensed form:

$$\left(\overset{n-1}{\underset{k=1}{\mathcal{C}}} \ \xi_{\omega,k}\right)(0) = \cot(t_n\omega) \tag{18}$$

3.3 *Fundamental mode shape. Soil deformation*

Value of A_1: The solution established in §3.2 is defined only to a multiplicative constant (A_1). Following §2, we put: $U_x(0) = 1$. This defines the constant A_1: $A_1 = 1$.

Fundamental mode shape: To calculate the displacement $U_x(z)$, which gives the fundamental mode shape, one proceeds as follows: the determination of the modal angular frequency ω by the resolution of Eq. (E) and the hypothesis $A_1 = 1$ make it possible to calculate explicitly the parameters A_i and φ_i defined in Eqs. (14) & (15) for each layer i (i ≥ 2), starting from the surface. For layer 1, the formula giving the displacement, established in §3.2.1, becomes with $A_1 = 1$: $u_{1x}(z) = \ \cos\left(\frac{\omega}{V_1} z\right)$. Hence it is possible to calculate the displacement $u_{ix}(z)$ in each soil layer, and at each layer interface. We can check that: $u_{nx}(H) = \ 0$. As appears from the expressions of A_i and φ_i, the displacement in a layer i depends only on the mechanical properties of layers 1 to i, and the thicknesses of layers 1 to (i - 1), as well as the angular frequency ω.

Soil deformation for the fundamental mode: We suppose the soil vibrating in mode 1 alone. Following §2, the maximum surface displacement for mode 1 is: $\underline{d_{max}} = a_{max}.T^2/4\pi^2$. Soil deformation for mode 1 is by definition: $\overline{U_x(z)} = d_{max}.U_x(z)$. Thus, $\overline{U_x(z)}$ can be calculated explicitly at each point in the n-layers soil, starting from the surface.

4 SOLUTION FOR HIGHER MODES

As a result of §2, and as indicated in §3, the approach for the solution is identical for all eigenmodes. Hence, the expression of the displacement for any mode j is similar to that for the fundamental mode, in Eq. (9). Thus, in each layer of soil i: $u^j_{ix}(z) = \ A^j_i \cos\left(\frac{\omega_j}{V_i} z + \varphi^j_i\right)$ A^j_i, φ^j_i being constants depending on soil layer, and calculated for mode j, and ω_j being the angular frequency for mode j. We can then conclude that Equation (E) is valid for any mode j. The successive positive roots of Equation (E) define the values of the angular frequencies ω_j, which can be ordered by increasing values starting from ω, the angular frequency of mode 1.

5 APPROXIMATE METHODS FOR N-LAYERED SOIL

We mention the following methods:

- Weighted average of modules and densities: the fundamental period of layered soil is given by: $T = \ 4H/\sqrt{\sum G_i H_i / \sum \rho_i H_i}$; this method cited in Dobry et al. (1976) uses a thickness-weighted average of soil layers properties to relate to the homogeneous soil case.
- Dobry-Madera Method (1970/1976): An exact analytical closed-form solution for the fundamental period of two-layers soil was exposed by Madera (1970). This period is given by an implicit equation: $\tan\left(\frac{\pi}{2}\frac{T_A}{T}\right).\tan\left(\frac{\pi}{2}\frac{T_B}{T}\right) = \frac{\rho_B H_B}{\rho_A H_A}\frac{T_A}{T_B}$. $T = 2\pi/\omega$ is the fundamental period of 2-layers soil; with A upper layer, B lower layer laying on a rigid bedrock; $T_A = \ 4H_A/V_A$; $T_B = \ 4H_B/V_B$. A graphical solution chart was given in Dobry et al. (1976). For n-layers soil, an iterative method, proposed by Dobry and Madera, consists in the successive application of this solution to soil layers. The successive modification of the boundary conditions (position of the rigid bedrock) makes it possible to obtain only an approximate value of the soil fundamental period. This method was validated by comparative studies in Madera (1970) and in Dobry et al. (1976).

- Simplified Rayleigh Procedure (1976): This procedure was proposed by Dobry et al. (1976). It allows, in addition to the fundamental period, to calculate approximate values of soil displacement at each interface between two soil layers. It is assumed that soil layers have a constant density ρ, and that the first mode can be expressed as: $X_{i+1} = X_i + \frac{H-z_{mi}}{V_i^2} H_i$; with $X_i,\ X_{i+1}$ values of first mode shape at lower and upper limits of layer i, z_{mi} midpoint of layer i. X(z) can be estimated at layers interfaces by applying this formula starting from bedrock (X = 0). Other approximations allow to calculate the angular frequency of the fundamental mode for the multilayered soil using the formula: $\omega^2 = 4\sum_1^n \left(\frac{H-z_{mi}}{V_i}\right)^2 H_i / \sum_1^n (X_i + X_{i+1})^2 H_i$.

  This method was validated by a comparative study in Dobry et al. (1976).

6 NUMERICAL APPLICATIONS

6.1 *Comparative application of the methods for computing period and soil deformation for the fundamental mode*

In the following, we present the application of the previous approximate methods and the exact analytical solution developed above, to 4 cases of soil profiles, with 2, 3, 4 and 26 layers of deformable soils. The applied methods are: Modules average Method; Simplified Rayleigh Procedure; Dobry-Madera Method; Exact Analytical Method. The studied soil profiles are: Profile 1: 2 layers (Clay 1: 24 m - Sand 1: 8 m); Profile 2: 3 layers (Clay 1: 12 m - Clay 2: 12 m – Sand 1: 8 m); Profile 3: 4 layers (Clay 1: 12 m - Sand 1: 4 m - Clay 2: 12 m - Sand 2: 4 m); Profile 4: 26 layers (alternation of sands and clays on a layer of marl). For profiles 1, 2, 3, H = 32 m. For profile 4: H = 68 m and the marl layer is 22 m thick. The G_{max} shear modules are (in MPa): Clay 1: 40; Clay 2: 80; Sand 1 = Sand 2: 180; Marl: 190 to 670. The G_i modules were taken as 0.5 G_{max}. The density values ρ_i are (in tons/m^3): Clay 1 = Clay 2: 1.8; Sand 1 = Sand 2: 2; Marl: 2.15. Maximum acceleration at ground surface (PGA): a_{max} = 2.1 m/s^2.

Numerical applications show that the Exact Analytical Method gives results for period and displacement in the range of the above-cited 3 methods, however with more accuracy.

Table 1 shows the results for the period T and maximum surface displacement d_{max}. Figures 1 and 2 show computed soil deformations for the fundamental mode, for 3 of the studied profiles.

6.2 *Application of the exact analytical method for computing period and soil deformation for higher modes*

For the same soil profiles studied in the previous paragraph, and the same value of PGA, we present below the results of the computation by the exact analytical method of the periods and shapes of modes 2 and 3, corresponding to the following roots ω_2 and ω_3 of Equation (E).

Table 2 shows the results for the period T and maximum surface displacement d_{max} for modes 1, 2 and 3. In Figures 3 and 4, soil deformations for modes 1, 2 and 3 are shown on the same graph, for 3 of the studied soil profiles.

Table 1. Comparative results for fundamental mode computations.

		Modules Average	Simplified Rayleigh Procedure	Dobry-Madera	Exact Analytical Method
Profile 1:	T (s)	0.90	0.83	(*)	0.98
2 layers	dmax (cm)	4.30	3.62	(*)	5.10
Profile 2:	T (s)	0.82	0.75	0.79	0.79
3 layers	dmax (cm)	3.58	3.02	3.34	3.30
Profile 3:	T (s)	0.82	0.78	0.83	0.82
4 layers	dmax (cm)	3.58	3.25	3.70	3.57
Profile 4:	T (s)	1.22	1.34	1.36	1.31
26 layers	dmax (cm)	7.86	9.49	9.80	9.18

* For 2-layers soil, the Madera Method coincides with the exact analytical solution.

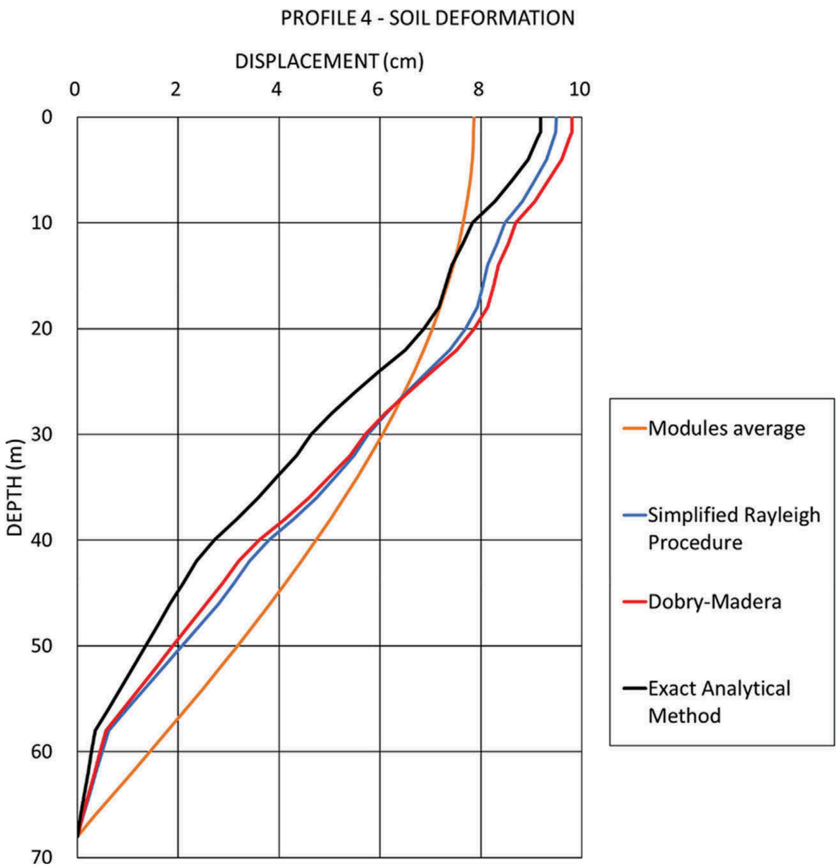

Figure 1. Comparative graphs for soil deformation - profile 4 (26-layers soil).

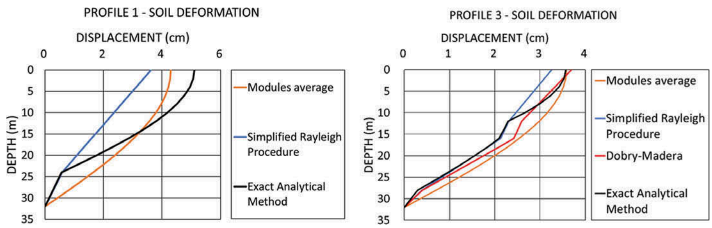

Figure 2. Comparative graphs for soil deformation - profiles 1 (2-layers soil) and 3 (4-layers soil).

Table 2. Computation of modes 1, 2 and 3.

		Mode 1	Mode 2	Mode 3
Profile 1:	T (s)	0.98	0.33	0.20
2 layers	dmax (cm)	5.10	0.58	0.22
Profile 2:	T (s)	0.79	0.31	0.18
3 layers	dmax (cm)	3.30	0.50	0.17
Profile 3:	T (s)	0.82	0.32	0.17
4 layers	dmax (cm)	3.57	0.55	0.16
Profile 4:	T (s)	1.31	0.51	0.35
26 layers	dmax (cm)	9.18	1.37	0.64

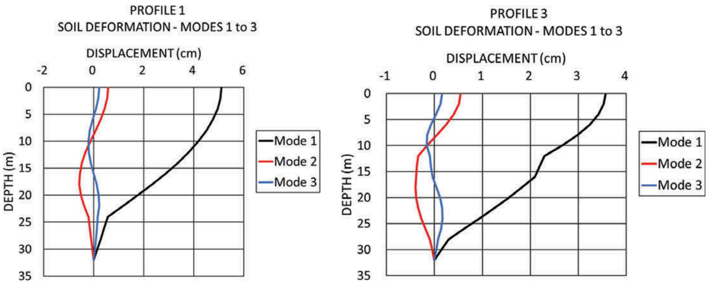

Figure 3. Modes 1, 2 and 3 for profiles 1 (2-layers soil) and 3 (4-layers soil).

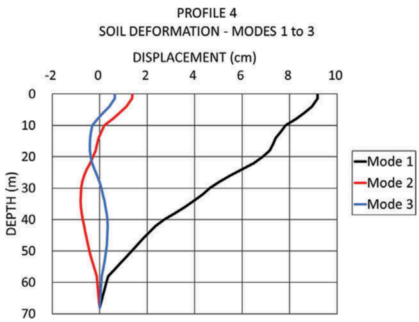

Figure 4. Modes 1, 2 and 3 for profile 4 (26-layers soil).

7 CONCLUSION

We have thus established an exact analytical closed-form solution allowing to calculate angular frequencies and mode shapes for all eigenmodes for a multilayer soil, in particular for the fundamental mode.

The established equation (E) for modal angular frequencies of the n-layers soil has a condensed expression, explicit in terms of soil properties.

The displacement is given by explicit formulas for the multilayer soil profile.

As an exact analytical closed-form solution obtained without any simplifying assumption, this solution does not need to be validated by comparative studies, except for the adequacy of the basic model of n-layered soil with respect to reality.

The approximate methods used so far to compute the fundamental period for a multilayer soil integrate simplifying assumptions, in addition to the basic model of the problem, allowing then to calculate an approximate value of the fundamental mode period, with a priori unknown precision. For this reason, it was necessary to validate these methods by comparative studies (such as the study of Madera 1970, and that of Dobry et al. 1976). Numerical applications show that the new method gives results for fundamental period and displacement in the range of the cited approximate methods, however with more accuracy.

We also note that the established solution allows to calculate the displacement at any point of the soil, including when the soil layers have a large thickness, without the need for cutting these layers into thin sub-layers, as for the approximate methods.

The calculation of soil deformation is directly applicable for computing the effects of kinematic interaction in the justification of pile foundations in a seismic context.

Being an analytical solution, the new method can be developed in addition to introduce viscous damping. Constituting a general solution for the layered soil seismic response, this method can furthermore naturally be used to calculate the response of the multilayer soil (including displacement, velocity and acceleration) in complex analyses including time-history analyses.

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