*The paper was published in the proceedings of the 7th International Conference on Earthquake Geotechnical Engineering and was edited by Francesco Silvestri, Nicola Moraci and Susanna Antonielli. The conference was held in Rome, Italy, 17 – 20 June 2019.*

*Earthquake Geotechnical Engineering for Protection and Development of Environment and Constructions – Silvestri & Moraci (Eds)*
 *ISBN 978-0-367-14328-2*

# Performance of velocity-based time-discontinuous Galerkin space-time finite element method in nonlinear elastodynamic analysis

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ABSTRACT: The safety of earthen structures, such as dams, tunnels and embankments, draws growing attention in Japan since tremendously big earthquakes have been occurring in the most earthquake-prone countries in the world. In order to assess the safety during earthquakes, the dynamic response analysis of the earthen structure is numerically analyzed commonly by the finite element method. In finite element framework, several methods have been proposed to achieve accurate results, and one of them is the velocity-based space-time finite element method, abbreviated by v-ST/FEM, which is based on time discontinuous Galerkin method. This method results in a high-order accurate and unconditionally stable algorithm. However, it has been applied only to linear problems. Hence, the objective of this article is to examine the performance of v-ST/FEM on nonlinear problems. In order to achieve this purpose, this paper presents the seismic response analysis of an earth dam considering its nonlinear stress-strain relationship. For simplicity, in the present analysis, it is assumed that (1) only shearing stress-strain relationship shows the nonlinear hysteretic behavior based on Hardin Drnevich model with modified Masing rule, which is widely used for practical analysis. (2) whole nonlinear analysis is approximated as a series of linear analysis and in one time interval $[t_n, t_{n+1}]$, constant stiffness matrix which corresponds to $t_n$ is used. From the results of analysis, it is found that shear stress-strain relationship successfully follows the HD model and it revealed that v-ST/FEM can be applied to such nonlinear problems. However, the algorithm used here leaves the plenty of room for improvement. First, due to the assumption (1), this method does not consider the nonlinear behavior due to an axial strain. Second, because of assumption (2), the results are dependent on the size of time steps.

## 1 INTRODUCTION

In order to compute the dynamic response of earthen structures, so-called semi-discretized FEM formulation is usually used. In this approach, the governing differential equations are first discretized in the space with finite element methods and then the resulting system of ODEs are integrated with respect to time. Several time integration schemes were proposed in the past, but to achieve high accuracy, conventional integration methods require small time steps, and this leads to expensive computation effort. Hence, to solve this problem, some alternative strategies have been proposed. One of them is v-ST/FEM (Sharma, 2018), which is based on time discontinuous Galerkin method. Although this method results in a high-order accurate and unconditionally stable algorithm, v-ST/FEM has been only applied to linear elastodynamic problems. The objective of this paper is to extend v-ST/FEM to nonlinear problems, and then to examine its performance for a nonlinear elastodynamic problems. Subsequently, seismic response analysis of an earth dam considering its nonlinear behavior is carried out. In order to describe the nonlinear stress-strain relationship of soil which constitutes the dam body, one-dimension skeleton curves are widely used. To use these skeleton curve models in multidimensional problems, constitutive laws should be extended multidimensionally. There are several methods which can be used to extend a constitutive law using a skeleton curve. In this study, the simplest way, which uses a skeleton curve only for shear stress and shear strain, is used. In this study, HD model is used as a skeleton

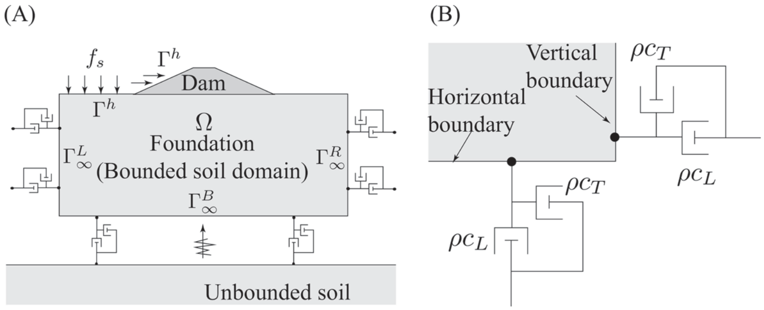


Figure 1. Schematic diagram of (A) finite element model for a dam-foundation interaction problem using Lysmer‘s viscous boundary condition, (B) dashpots and distribution of damping coefficients at Lysmer‘s standard viscous boundary.

curve model because it is the most used model for practical analysis. This paper presents the two-dimensional analysis of the seismic response of the dam with HD model applied only to the shearing stress-strain relationship.

## 2 V-ST/FEM FOR NONLINEAR DAM-FOUNDATION INTERACTION PROBLEM

Figure 1 illustrates the finite element model for dynamic response analysis of dam-foundation system. Let $\Omega$ be the truncated computational domain that contains both dam and foundation domain (Fig. 1). $\Omega$ is described as the set of finite spacial elements $\Omega^{(e)}$, $e = 1, \cdots, n_{ne}$. Let $\Gamma_i^g$ and $\Gamma_i^h$ $(i = 1, 2)$ denote the boundary of $\Omega$ where spacial components of displacement and traction are prescribed, respectively. $\Gamma_\infty = \Gamma_\infty^L + \Gamma_\infty^R + \Gamma_\infty^B$ denotes the truncated boundary of $\Omega$ where the subscripts $\{L, R, B\}$ are used to denote the left, right and bottom viscous boundaries. In this study, viscous boundary conditions (Lysmer et al. 1969) are prescribed on $\Gamma_\infty$ to eliminate any spurious reflections at these boundaries. Furthermore it is assumed that the seismic input motions are represented by vertically propagating waves and entering at $\Gamma_\infty^B$.

Under these conditions, the strong form of initial-boundary value elastodynamic problem can be stated as

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho b_i = 0 \tag{1a}$$

$$u_i(\mathrm{x}, t) = g_i(\mathrm{x}, t), \quad \sigma_{ij} n_j = h_i(\mathrm{x}, t) \tag{1b}$$

$$u_i(\mathrm{x}, 0) = u_i^0(\mathrm{x}), \quad v_i(\mathrm{x}, 0) = v_i^0(\mathrm{x}) \tag{1c}$$

where, $b_i$ = body force density; $g_i$ = prescribed displacement; $h_i$ = prescribed boundary traction; $u_i^0$ = initial value of displacement; $v_i^0$ = initial value velocity; $\rho$ = mass density. For the present dynamic problem (Figure 1), traction boundary condition in Equation (1b) takes the following form:

$$\sigma_{ij} n_j = f_i^s \qquad on \quad \Gamma_i^h \tag{2a}$$

$$\sigma_{ij} n_j = -c_{ip}^v v_p \qquad on \quad \Gamma_\infty^L \cup \Gamma_\infty^R \tag{2b}$$

$$\sigma_{ij} n_j = -c_{ip}^h v_p + 2 c_{ip}^h v_p^{in} \quad on \quad \Gamma_\infty^B \tag{2c}$$

where, $v_i^{in}$ = incoming seismic wave velocity.

In these equations, $c_{ij}^v$ and $c_{ij}^h$ denotes the damping coefficients matrices for dashpots placed at vertical and horizontal viscous boundary, respectively:

$$\mathbf{c}^v = \begin{bmatrix} \rho c_L & 0 \\ 0 & \rho c_T \end{bmatrix} \quad \mathbf{c}^h = \begin{bmatrix} \rho c_T & 0 \\ 0 & \rho c_L \end{bmatrix} \tag{3}$$

where, $c_L$ and $c_T$ are the speed of P-wave and S-wave in the unbounded medium, respcetively. Following the v-ST/FEM formulation procedure (Sharma et al. 2018), Equation (1)-(3) results in system of algebraic equation as below

$$\begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathrm{V}_1 \\ \mathrm{V}_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{J}_1 \\ \mathrm{J}_2 \end{Bmatrix} \tag{4}$$

In this scheme, velocity is discontinuous at time $\{t_0, t_1, \cdots, t_N\}$, and $\{\mathrm{V}_1\}$ and $\{\mathrm{V}_2\}$ denote the space-nodal values of velocity at time $t_n^+$, $t_{n+1}^-$, respectively, where $t_n^+ = \lim_{\varepsilon\to 0}(t_n + \varepsilon)$, $t_{n+1}^- = \lim_{\varepsilon\to 0}(t_{n+1} - \varepsilon)$. Matrices $[K_{ij}]$ are given by following expressions.

$$[\mathrm{K}_{11}] = \frac{1}{2}[\mathrm{M}] + \frac{3\Delta t_n^2}{24}[\mathrm{K}] + \frac{\Delta t_n}{3}[\mathrm{C}_\infty] \tag{5a}$$

$$[\mathrm{K}_{12}] = \frac{1}{2}[\mathrm{M}] + \frac{\Delta t_n^2}{24}[\mathrm{K}] + \frac{\Delta t_n}{6}[\mathrm{C}_\infty] \tag{5b}$$

$$[\mathrm{K}_{21}] = -\frac{1}{2}[\mathrm{M}] + \frac{5\Delta t_n^2}{24}[\mathrm{K}] + \frac{\Delta t_n}{6}[\mathrm{C}_\infty] \tag{5c}$$

$$[\mathrm{K}_{22}] = \frac{1}{2}[\mathrm{M}] + \frac{3\Delta t_n^2}{24}[\mathrm{K}] + \frac{\Delta t_n}{3}[\mathrm{C}_\infty] \tag{5d}$$

where, [M] = constant mass matrix; [K] = constant stiffness matrix.
$[C_\infty]$ is the matrix due to the dashpots along viscous boundary, which is expressed as

$$[\mathrm{C}_\infty] = \begin{bmatrix} \mathrm{C}_\infty^{11} & 0 \\ 0 & \mathrm{C}_\infty^{22} \end{bmatrix} \tag{6}$$

in which,

$$[\mathrm{C}_\infty^{ij}] = \int_{\Gamma_\infty^L \cup \Gamma_\infty^R} c_{ij}^v N^I N^J ds + \int_{\Gamma_\infty^B} c_{ij}^h N^I N^J ds \tag{7}$$

where, $N^I$ denotes the shape functions at spacial node $I\ (= 1, \cdots, n_{ne})$. $N^J$ also denotes the shape functions at spacial node $J\ (= 1, \cdots, n_{ne})$. The right hand side of Equation (8) are given by

$$\{\mathrm{J}_1\} = [\mathrm{M}]\{\mathrm{V}_0\} + \frac{\Delta t_n}{2}\{\mathrm{F}_1^{ext}\} + \frac{\Delta t_n}{2}\{\mathrm{F}_1^{in}\} - \frac{\Delta t_n}{2}[\mathrm{K}_n]\{\mathrm{U}_0\} \tag{8}$$

$$\{\mathrm{J}_2\} = \frac{\Delta t_n}{2}\{\mathrm{F}_2^{ext}\} + \frac{\Delta t_n}{2}\{\mathrm{F}_2^{in}\} - \frac{\Delta t_n}{2}[\mathrm{K}_n]\{\mathrm{U}_0\} \tag{9}$$

where, $\{\mathrm{V}_0\}$ and $\{\mathrm{U}_0\}$ denotes the space-nodal value of velocity and displacement at time $t_n^-$. $\{\mathrm{F}_a^{ext}\}$ is the force vector due to the external body and surface forces:

$$F_a^{ext}(i, I) = \int_{-1}^{+1}\int_{\Gamma_i^h} T_a N^I h_i ds d\theta + \int_{-1}^{+1}\int_{\Omega_h} T_a N^I \rho b_i d\Omega d\theta \tag{10}$$

and $\{F_a^{in}\}$ is the effective generalized earthquake force vector due to the incoming seismic excitations:

$$F_a^{in}(i,I) = \int_{-1}^{+1} \int_{\Gamma_\infty^B} N^I T_a 2c_{ij}^h v_j^{in} ds d\theta \tag{11}$$

$T_a$ denotes the shape functions for time. $T_a$ is given as

$$T_1(\theta) = \frac{1-\theta}{2} \quad T_2(\theta) = \frac{1+\theta}{2} \quad \theta \in [-1, 1] \tag{12}$$

## 3 DYNAMIC ANALYSIS OF AN EARTH DAM

In this section, seismic response analysis of earth dam is conducted considering its nonlinearity, whereby the performance of v-ST/FEM in nonlinear elastodynamic problem is investigated.

### 3.1 *Stress-strain relationship*

In Equation (5a)-(5d), mass matrix and viscous boundary matrix are constant throughout the analysis; however stiffness matrix is dependent on the current state due to the nonlinear behavior of soil. In order to consider such nonlinear problem, whole analysis is approximated as a series of linear systems. Hardin Drnevich (HD) model with modified Masing criterion (Kramer, 1969) is adopted to describe the nonlinear behavior of soil, which makes it possible to describe shear modulus $G$ as a function of current shear strain. For simplicity, in time interval $[t_n, t_{n+1}]$, $G$ which corresponds to shear strain at time $t_n$ is used and it is assumed that shear modulus is kept constant within this time interval. From Equation (13) and (14), stiffness matrix for this time interval can be obtained.

$$D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right) \tag{13}$$

in which $\delta_{ij}$ = Kronecker-delta function, $D_{ijkl}$ = elastic modulus, $\lambda$, $\mu$ = Lame parameters.

$$\mu = G, \quad \lambda = \frac{\nu E}{(1-2\nu)(1+\nu)} \tag{14}$$

In the above equation, $G$ = shear modulus, $\nu$ = Poisson's ratio. In HD model, $G$ can be calculated by using Equation (15).

$$G = \frac{G_{max}}{\left(1 + \frac{|\gamma|}{\gamma_r}\right)^2} \tag{15}$$

where, $\gamma$ and $\gamma_r$ denotes shear strain and the reference strain respectively. In this study, $\lambda$ is computed from Equation (14), and kept constant in whole analysis, while $\mu$ is computed from Equation (14) and Equation (15) in every time step.

### 3.2 *Settings for simulation*

In the present simulation, only dam domain is assumed to have nonlinear elastic stress-strain relationship of HD model, and the foundation domain is assumed to be linear elastic material.

Table 1. Material parameters of the dam and foundation domain.

| | Unit weight $\gamma$ (kN=m$^3$) | Maximum shear modulus $G_{max}$ (Mpa) | Poisson's ratio $\nu$ | Reference strain $\gamma_r$ | Elastic modulus $E$ (Mpa) |
|---|---|---|---|---|---|
| Dam | 21.5 | * | 0.4 | 0.00033 | - |
| Foundation | 19 | 217 | 0.49 | - | 646.7 |

* $G_{max}$ for dam is calculated for each layers in Figure 2 following Equation (16)

| | Dam5 (Mpa) | Dam4 (Mpa) | Dam3 (Mpa) | Dam2 (Mpa) | Dam1 (Mpa) |
|---|---|---|---|---|---|
| $G_{max}$ | | 173.0 | 221.0 | 270.0 | 318.0 |

Finite element mesh for the present problem is described in Figure 2(A) and the computation includes not only the dam but also a foundation domain which is truncated by viscous boundaries. Reservoir is considered as empty and both stress and strain starts from the origin. For the foundation domain, the length in horizontal direction and in vertical direction are 540 (m) and 60 (m), respectively.

The parameters used in the analysis, mass density, $\rho$, maximum shear modulus, $G_{max}$, Poisson'ratio, $\nu$ and reference strain, $\gamma_r$ are given in Table 1. For HD model, $G_{max}$ and $\gamma_r$ are needed.

For simplicity, the present dam is divided into 5 layers (Figure 2(B)) and it is assumed that $G_{max}$ is a function of z described by Equation (16) (Okamoto, ) and this time the middle depth of each layers z are used, which is 14, 10, 6, 2 (m) from the bottom side layer.

$$V_s = 218 + \frac{822z}{z + 72.8} \quad G_{max} = \rho V_s^2 \tag{16}$$

where, $z$ = depth from the surface (m).
$G_{max}$ value for each element are given in Table 2.
$G_{max}$ value for each 5 layers

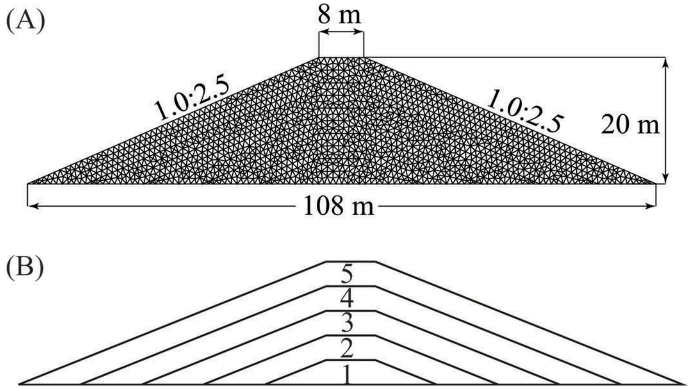


Figure 2. Schematic diagram of (A) Finite element mesh for earth dam, (B) 5 sections with different maximum shear modulus and shear wave velocity of Dam body

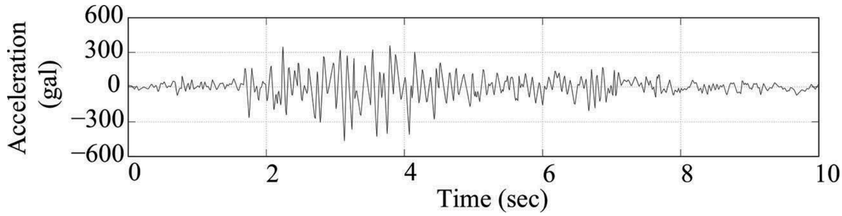


Figure 3. Input wave acceleration data in horizontal direction.

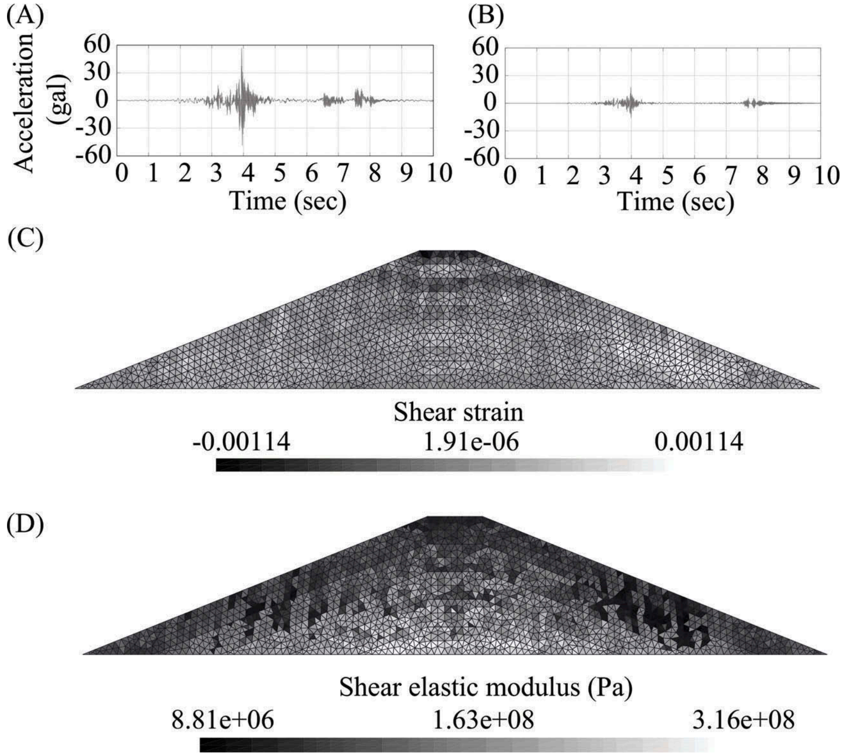


Figure 4. the result acceleration on the left top of the dam (A) in horizontal direction, (B) in vertical direction and schematic diagram of distribution of (C) the shear strain, (D) shear elastic modulus at 3.96 seconds, respectively.

In this study, $\lambda$ is calculated only for first time interval using Equation (14) and kept constant through the analysis, while $\mu$ will change depend on current state. For foundation domain, $c_L$ and $c_T$ in Equation (3) can be written as below

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad c_T = \sqrt{\frac{\mu}{\rho}} \tag{17}$$

Numerical simulation is carried out for the horizontal seismic motion recorded at some control point on the free surface. Total time duration of ground motion is 10 seconds in which the seismic acceleration comes to an end after 10 seconds. Figure 3 depicts the time history of the recorded acceleration; the maximum and minimum values are 357.47 (cm/s$^2$) and -464.78 (cm/s$^2$) respectively.

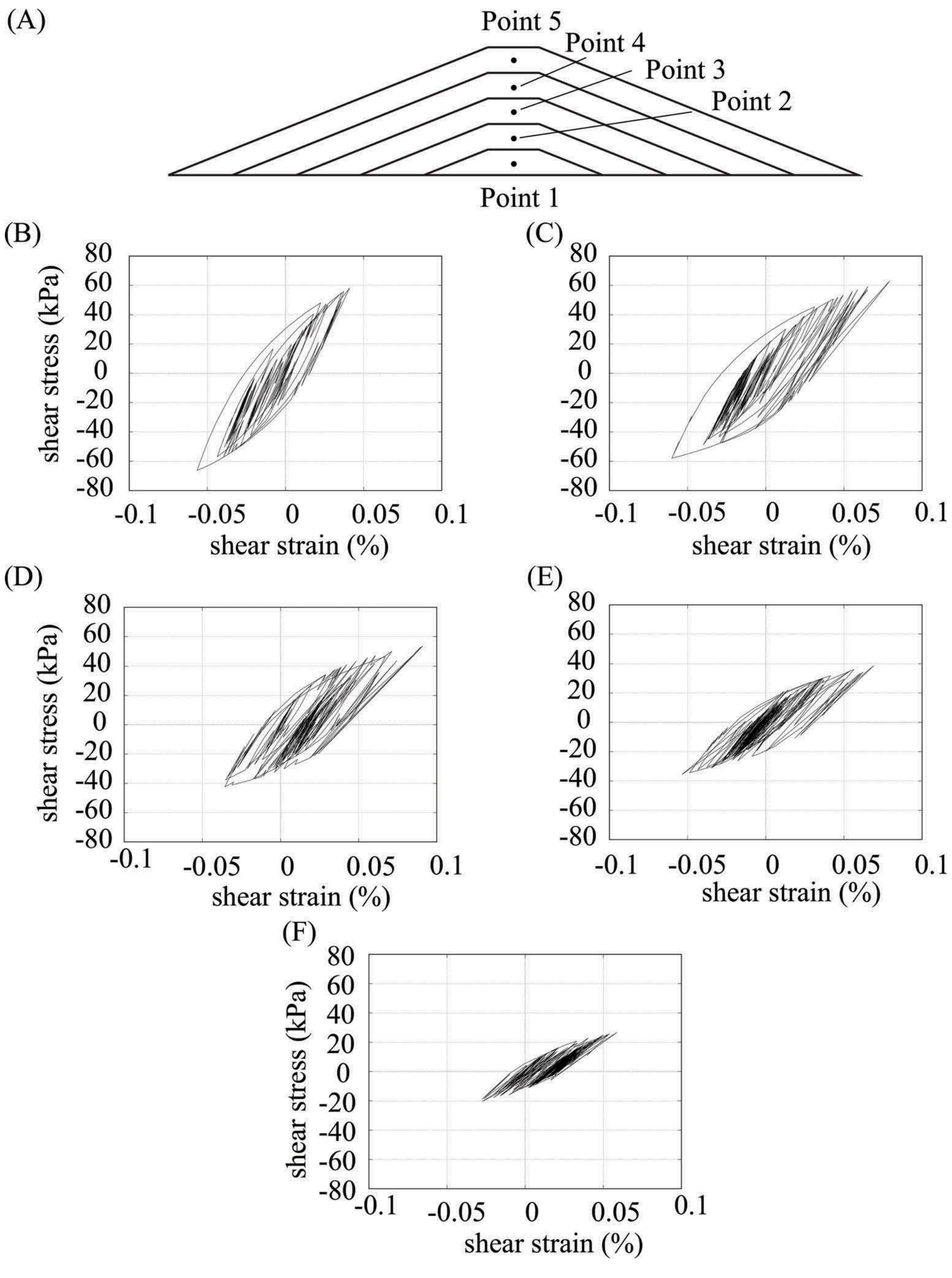


Figure 5. (A) 5 points which were focused. Computational results of relationship between shear stress and strain (B) at point 1 (C) at point 2 (D) at point 3 (E) at point 4 (F) at point 5

## 4 RESULTS AND CONCLUSION

Figure 4(A), (B) shows the response acceleration in horizontal and vertical direction, respectively. In these figures, it can be seen that response acceleration in both directions take maximum values around at 4 seconds, and Figure 4(C), (D) shows the distribution of shear stress and shear elastic modulus, respectively at 3.96 seconds when the acceleration response of dam top in horizontal direction became the biggest. Figure 5 shows the computational results of the relationship between shear stress and shear strain at Point 1-5 in Figure 5 respectively. From these results, it can be seen the dam‘s shear stress-strain relationship follow the HD model based on the modified Masing rule.

This reveals that v-ST/FEM can be introduced to this type of nonlinear elastodynamic problems. However, in the present analysis, hysteretic nonlinear constitutive model is only used for shear stress and shear strain relationship, and in this way hysteretic behavior due to axial strain is not considered. In order to obtain more accurate results, axial strain also should be taken into account. Furthermore, shear modulus at time  is used, so the stress value is dependent on the size of time intervals. These things should be taken into account in next step.

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