Earthquake Geotechnical Engineering for Protection and Development of Environment and Constructions – Silvestri & Moraci (Eds)
 ISBN 978-0-367-14328-2

SSI system damping including radiation damping and plastic energy dissipation for shallow foundations

B.L. Kutter
University of California, Davis, USA

ABSTRACT: This paper revisits the method of determining the system period and damping by combining springs and dashpots associated with a single degree of freedom structure supported by a shallow foundation. Similar to previous approaches, the structure, foundation sliding, and foundation rocking flexibilities are modeled by a series system of spring and dashpot components. Different from previous approaches, the present study uses concepts from the theory of plasticity to further decompose the sliding and rocking components into plastic (hysteretic) and elastic (radiation) components. It is believed that the final form of the new equations takes a more intuitive form that might be used with more confidence in practice. For linear systems, the new equations are identical to the equations used in current design codes in the US. However, for nonlinear systems, the new equations produce significantly different system damping than those recommended by design codes such as ASCE (2017). For the new proposed equations, the system damping is sensitive to the plastic energy dissipation in the foundation; on the other hand, the soil hysteretic damping in ASCE (2017) depends only on the intensity of the design spectrum and site classification, irrespective of ductility demand on foundations.

1 INTRODUCTION

Several authors have developed expressions for stiffness and damping associated with single degree of freedom structures supported on shallow foundations (Givens et al. 2016). The objective is to combine a system of springs and dashpots from the foundation and structure into a single “equivalent” SDOF component with a known natural period and damping. Apparent differences arise in some published expressions depending on assumptions made regarding the assumed nature of the damping for the system (viscous or hysteretic) and the assumed nature of the damping for each component (viscous, hysteretic, or radiation). It is found that the formulation and resulting equations are simplified and intuitive if plastic mechanisms are separated from elastic mechanisms and if each component of damping is weighted by the deformation of that component.

This paper starts from first principles to rederive equations to combine viscous and hysteretic damping of a multi-component series system of springs and dashpots. Each term of the resulting equations has a similar form and hence, the new equations are believed to be easier to comprehend and to check. The last section of the paper shows that the new forms of the damping expressions are identical to those presented by NIST (2012) for linear systems, but there are significant differences when nonlinear hysteretic behavior is incorporated.

The lateral displacement of the mass of the SSI system shown in Figure 1 has contributions from three mechanisms: structure flexibility, foundation sliding and foundation rotation. These displacements are additive:

$$u_{sys} = u_{str} + u_x + h\theta \tag{1}$$

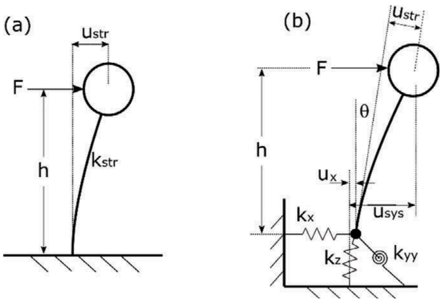

Figure 1. Schematic illustration of deflections caused by force applied to (a) fixed base structure; and (b) structure with vertical, horizontal and rotational flexibility at its base (after NIST 2012). Note that the mass of the footing is neglected in the present analysis.

Adopting the assumption of plasticity theory (Hill 1950), the deformation of each component of the footing deformation is decomposed into an elastic part (indicated by superscript *e*) and a plastic part (superscript *p*):

$$u_{sys} = u_{str} + u_x^e + u_x^p + h\theta^e + h\theta^p \tag{2}$$

On a related note, based on nonlinear dynamic analysis of soil-pile interaction, Wang et al. (1998) showed that linear viscous dashpots (representing radiation damping in the far field) should not be arranged in parallel with hysteretic dissipation elements.

The elastic components are presumed to dissipate energy by radiation damping only; the plastic components are presumed to dissipate energy by friction or hysteresis. Consistent with Equation (2) and the approach followed and tested by Deng et al (2014), each component of displacement is associated with a stiffness and a dissipation mechanism depicted in Figure 2a.

Our goal is to represent the 5-component system by single component SDOF system (Figure 2(b)) that has equivalent system natural frequency and damping. For static loading the displacement of each component of the 5-component system may be represented by the force divided by stiffness:

$$F/k_{sys} = F/k_{str} + F/k_x^e + F/k_x^p + Fh^2/k_{yy}^e + Fh^2/k_{yy}^p \tag{3}$$

Cancelling F,

$$1/k_{sys} = 1/k_{str} + 1/k_x^e + 1/k_x^p + h^2/k_{yy}^e + h^2/k_{yy}^p \tag{4}$$

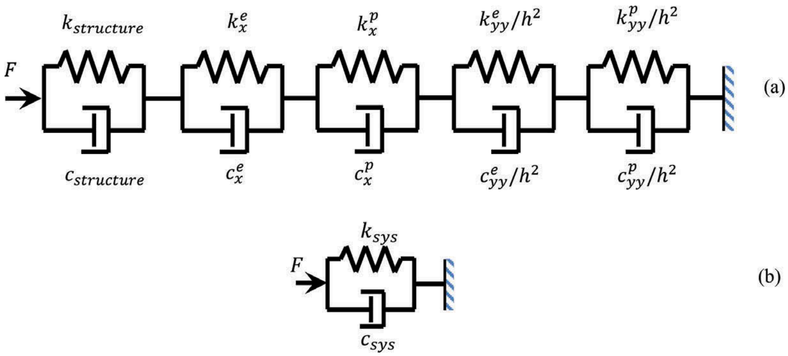

Figure 2. (a) Decomposition of a system into structural, sliding elastic, sliding plastic, rocking elastic, and rocking plastic components, and (b) an equivalent single component system.

Following and extending a convention adopted by NIST (2012), let us define

$T_{sys} = 2\pi\sqrt{\frac{m}{k_{sys}}}$ Natural period of the system with all of the components.

$T_{str} = 2\pi\sqrt{\frac{m}{k_{str}}}$ Natural period of the system if the structure was the only contributor to deformation (fixed base period).

$T_x^e = 2\pi\sqrt{\frac{m}{k_x^e}}$ Natural period of the system if elastic sliding was the only contributor to deformations. $k_x^e = F/u_x^e$.

$T_x^p = 2\pi\sqrt{\frac{m}{k_x^p}}$ Natural period of the system if plastic sliding was the only contributor to deformation. $k_x^p = F/u_x^p$ is the secant stiffness of the plastic sliding hysteresis loop.

$T_{yy}^e = 2\pi\sqrt{\frac{mh^2}{k_{yy}^e}}$ Natural period of the system if elastic rocking was the only contributor to deformations. $k_{yy}^e = Fh^2/u_{yy}^e$.

$T_{yy}^p = 2\pi\sqrt{\frac{mh^2}{k_{yy}^p}}$ Natural period of the system if plastic rocking was the only contributor to deformations. $k_{yy}^p = Fh^2/u_{yy}^p$ is the secant stiffness of the plastic rotational hysteresis loop.

With the above definitions, we see that the flexibility (*1/k*) of each component is proportional to the square of the period for that component, and (4) can be rearranged:

$$T_{sys}{}^2 = (T_{str})^2 + \left(T_x^e\right)^2 + \left(T_x^p\right)^2 + \left(T_{yy}^e\right)^2 + \left(T_{yy}^p\right)^2 \tag{5}$$

It is a little more complicated to derive the combination of springs with damping for dynamic loading. So before tackling the 5-component system, we consider 1- and 2-component systems in the next section.

2 ENERGY DISSIPATION IN 1- AND 2-COMPONENT SYSTEMS

To derive the damping of a series system of springs and dashpots, we ensure that the energy dissipated by the system is equal to the sum of the energy dissipated by each component. First let us review the computation of the energy dissipated by a single component consisting of a single linear spring and a linear viscous dashpot (Figure 3a).

The work dissipated in a cycle of harmonic motion of period $T = 2\pi/\omega$ is:

$$\Delta W_{vis} = \int_0^T F\, du = \int_0^{2\pi/\omega} F\, \frac{du}{dt} dt \tag{6}$$

$$\Delta W_{vis} = \int_0^{2\pi/\omega} \left(ku + c\frac{du}{dt}\right)\frac{du}{dt} dt \tag{7}$$

Let the displacement be of the form $u = A\ sin(\omega t)$, then, the velocity would be $du/dt = A\omega\ cos(\omega t)$. The energy dissipated becomes:

$$\Delta W_{vis} = \int_0^{2\pi/\omega} (kAsin(\omega t) + cA\omega\ cos(\omega t))A\omega\ cos(\omega t)dt \tag{8}$$

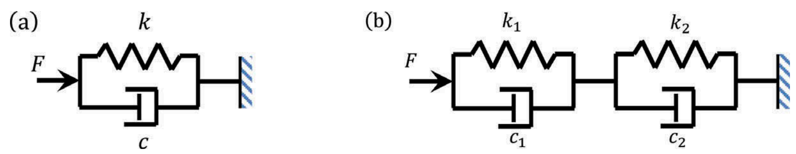

Figure 3. (a) General 1-component, and (b) 2-component dashpot systems.

Evaluation of (8) over the limits of integration provides:

$$\Delta W_{vis} = \pi\omega c A^2 \tag{9}$$

From (9) we see that energy dissipation increases with damping coefficient, frequency, and the square of the amplitude for a visco-elastic component. Next consider the energy dissipation of a 2-component system depicted in Figure 3(b). The energy dissipated is the sum of the dissipation in the two components:

$$\Delta W_{sys} = \Delta W_1 + \Delta W_2 \tag{10}$$

If both components of a 2-component system are viscoelastic:

$$\pi\omega c_{sys} {A_{sys}}^2 = \pi\omega c_1 {A_1}^2 + \pi\omega c_2 {A_2}^2 \tag{11}$$

But, for a hysteretically damped system the energy dissipation should be frequency independent. We could approximate the hysteretic damping if we set the viscous damping coefficient to be an inverse function of frequency; i.e.: $c = 2k\beta/\omega$. Hence the energy dissipation in cycle for a hysteretically damped component would be:

$$\Delta W = 2\pi k \beta_{hys} A^2 \tag{12}$$

Where β_{hys} is the damping ratio of the hysteretically damped system. The amplitude of the displacement is related to the force F and the complex stiffness, $k^* = k + i\omega c = k(1 + 2i\beta_{hys})$ by

$$A = \frac{|F|}{|k^*|} = \frac{|F|/k}{\sqrt{\left(1 + \left(2\beta_{hys}\right)^2\right)}} \tag{13}$$

$$A^2 = \frac{F^2/k^2}{1 + 4{\beta_{hys}}^2} \tag{14}$$

The energy dissipation of this element becomes

$$\Delta W = 2\pi k \beta_{hys} \left(\frac{F^2/k^2}{1 + 4{\beta_{hys}}^2}\right) \tag{15}$$

Now, the energy dissipated by a 2-component system is the sum of the energy dissipated by the two components:

$$\Delta W_{sys} = \Delta W_1 + \Delta W_2 \tag{16}$$

or

$$2\pi k_{sys} \beta_{hys,sys} \left(\frac{F^2/{k_{sys}}^2}{1 + 4{\beta_{hys,sys}}^2}\right) = 2\pi k_1 \beta_{hys,1} \left(\frac{F^2/{k_1}^2}{1 + 4{\beta_{hys,1}}^2}\right) + 2\pi k_2 \beta_{hys,2} \left(\frac{F^2/{k_2}^2}{1 + 4{\beta_{hys,2}}^2}\right) \tag{17}$$

Which simplifies to:

$$\frac{\beta_{hys,sys}}{k_{sys}} \left(\frac{1}{1 + 4{\beta_{hys,sys}}^2}\right) = \frac{\beta_{hys,1}}{k_1} \left(\frac{1}{1 + 4{\beta_{hys,1}}^2}\right) + \frac{\beta_{hys,2}}{k_2} \left(\frac{1}{1 + 4{\beta_{hys,2}}^2}\right) \tag{18}$$

If the damping ratios, $\beta_{hys,sys}$,$\beta_{hys,1}$, and $\beta_{hys,2}$ are between 0 and 20%, the terms in parentheses take on values between 1.0 and 0.86. Hence the above may be approximated with less than 14% error by:

$$\frac{\beta_{hys,sys}}{k_{sys}} = \frac{\beta_{hys,1}}{k_1} + \frac{\beta_{hys,2}}{k_2} \tag{19}$$

And if $T_{sys} = 2\pi\sqrt{\frac{m}{k_{sys}}}$, $T_1 = 2\pi\sqrt{\frac{m}{k_1}}$, $T_2 = 2\pi\sqrt{\frac{m}{k_2}}$, and $\omega_2 = \sqrt{\frac{k_2}{m}}$, then:

$$\beta_{hys,sys} = \left(\frac{T_1}{T_{sys}}\right)^2 \beta_{hys,1} + \left(\frac{T_2}{T_{sys}}\right)^2 \beta_{hys,2} \tag{20}$$

Recall that the above equation is determined assuming that component 1 and component 2 are both hysteretically damped. But, next suppose that component 1 is hysteretically damped while component 2 is viscously damped. If we desire to approximate the pair of components by an equivalent hysteretically damped system, the energy dissipation from Equation (11) becomes:

$$2\pi k_{sys}\beta_{sys}A_{sys}{}^2 = 2\pi k_1\beta_1 A_1{}^2 + \pi\omega c_2 A_2{}^2 \tag{21}$$

Inserting previously defined relations for the amplitudes, A, for each component.

$$2\pi k_{sys}\beta_{hys,sys}\left(\frac{(F/k_{sys})^2}{1+4\beta_{hys,sys}{}^2}\right) = 2\pi k_1\beta_{hys,1}\left(\frac{(F/k_1)^2}{1+4\beta_{hys,1}{}^2}\right) + \pi\omega c_2\left(\frac{(F/k_2)^2}{1+4(\beta_{vis,2}(\omega))^2}\right) \tag{22}$$

In (22), only the last term is frequency dependent. Simplifying by cancelling F and 2π from each term and again assuming that the denominator of each term in parentheses is close to one if the corresponding damping ratio less than 20%:

$$\frac{\beta_{hys,sys}}{k_{sys}} = \frac{\beta_{hys,1}}{k_1} + \frac{\omega c_2}{2k_2^2} \tag{23}$$

Using the relationship between damping ratio and damping coefficient $\left(\beta_{vis,2} = \frac{c_2}{2\sqrt{k_2 m}}\right)$ and the relation $m = k_2/\omega_2^2$ the damping coefficient may be expressed as

$$c_2 = \beta_{vis,2}\left(2\sqrt{k_2 m}\right) = \beta_{vis,2}\left(2\sqrt{k_2^2/\omega_2^2}\right) \tag{24}$$

Which, with (23) leads to the steps shown in Equations 25, 26, and 27.

$$\frac{\beta_{hys,sys}}{k_{sys}} = \frac{\beta_{hys,1}}{k_1} + \frac{\omega\beta_{vis,2}}{k_2\omega_2} \tag{25}$$

$$\beta_{hys,sys} = \frac{k_{sys}}{k_1}\beta_{hys,1} + \frac{k_{sys}}{k_2}\frac{\omega}{\omega_2}\beta_{vis,2} \tag{26}$$

$$\beta_{hys,sys} = \left(\frac{T_1}{T_{sys}}\right)^2 \beta_{hys,1} + \left(\frac{T_2}{T_{sys}}\right)^2 \frac{T_2}{T}\beta_{vis,2} \tag{27}$$

The last term in (27) shows that the equivalent damping ratio of the system is dependent on the period of shaking. If we are to find the system damping at the fundamental mode of the SSI system, we may set $T = T_{sys}$ into the above equation to obtain:

$$\beta_{hys,sys} = \left(\frac{T_1}{T_{sys}}\right)^2 \beta_{hys,1} + \left(\frac{T_2}{T_{sys}}\right)^3 \beta_{vis,2} \tag{28}$$

In effect, we sum the weighted damping coefficients of the components. A generalized equation, consistent with previous work (e.g., NIST 2012) for equivalent hysteretic damping of a multi-component series system would hence be:

$$\beta_{hys,sys} = \sum_{k=1}^{number\ of\ components} \left(\frac{T_k}{T_{sys}}\right)^{m_k} \beta_k \tag{29}$$

Where

$$\beta_k = equivalent\ hysteretic\ damping\ ratio\ of\ the\ k^{th}\ component$$

$$m_k = 2\ if\ component\ k\ is\ hysteretically\ damped$$

$$m_k = 3\ if\ component\ k\ is\ viscously\ damped$$

3 STRUCTURE ON NONLINEAR ROCKING AND SLIDING FOOTINGS

Finally, we return to the 5- component system sketched in Figure 2a. Using the result from Equation (29) above, and assuming that damping of each subsystem is hysteretic in nature ($m_k = 2$)

$$\beta_{sys} = \left(\frac{T_{str}}{T_{sys}}\right)^2 \beta_{str} + \left(\frac{T_x^p}{T_{sys}}\right)^2 \beta_x^p + \left(\frac{T_x^e}{T_{sys}}\right)^2 \beta_x^e + \left(\frac{T_{yy}^p}{T_{sys}}\right)^2 \beta_{yy}^p + \left(\frac{T_{yy}^e}{T_{sys}}\right)^2 \beta_{yy}^e \tag{30}$$

It is argued that due to the repetitive structure of Equation (30), it is easier to understand than Equations (31) (equations 2-10 and 2-11a from NIST 2012):

$$\beta_{sys} = \beta_f + \frac{1}{\left(T_{sys}/T_{str}\right)^2}\beta_{str} \tag{31a}$$

$$\beta_f = \frac{\left(T_{sys}/T_{str}\right)^2 - 1}{\left(T_{sys}/T_{str}\right)^2}\beta_s + \frac{1}{\left(T_{sys}/T_x\right)^2}\beta_x + \frac{1}{\left(T_{sys}/T_{yy}\right)^2}\beta_{yy} \tag{31b}$$

In Equation (31b), NIST (2012) defines β_s as the "soil hysteretic material damping". Equations (31a) and (31b) may be rearranged to obtain (32) that is a more transparent but equivalent to (31a) and (31b):

$$\beta_{sys} = \left(\frac{T_{str}}{T_{sys}}\right)^2 \beta_{str} + \left(\frac{T_x}{T_{sys}}\right)^2 (\beta_x + \beta_s) + \left(\frac{T_{yy}}{T_{sys}}\right)^2 (\beta_{yy} + \beta_s) \tag{32}$$

The proposed equation (30) is identical to both forms of the NIST (2012) equations (either Equation (31a) and (31b) or Equation (32)) if hysteretic damping of the soil $\beta_s = 0$ and the plastic energy dissipation terms (terms with p superscript) are neglected.

4 COMBINATION OF LINEAR AND NONLINEAR DAMPING

Although Equations (30) and (32) are in complete agreement for linear behavior, they are not equivalent for nonlinear behavior. Equation 30 was derived assuming that deformation is the sum of the elastic and plastic components of deformation, consistent with thermodynamic requirements of rate-independent plasticity theory (e.g., Hill 1950). On the other hand, Givens et al (2016) state without explanation that "soil damping (βs) can be included in the system by simply adding it to the translational and rotational radiation damping terms"; however, they do reference (Wolf 1985) to support this claim. Similar to NIST (2012) and Givens et al. (2016), Wolf (1985, p.41) suggests (without explanation) that material damping may be introduced "in and approximate manner" by adding the hysteretic (material) damping and elastic (radiation) damping.

NIST (2012) assumes without explanation that the same soil material damping factor, β_s, applies to both the rocking and sliding mechanisms of deformation; this assumption contradicts the observations from experiments by Gajan et al. (2008) that hysteretic damping for rocking is different than that for sliding. Furthermore, NIST (2012) assumes that the damping ratios are correlated to the site class and S_{DS} (the short period spectral response acceleration for a design spectrum). On the other hand, it is easy to imagine that heavily loaded foundations subject to small accelerations but long period displacements could produce significant plastic energy dissipation.

5 QUANTITATIVE COMPARISONS FOR SPECIAL CASE

Some simplifying assumptions are made to enable quantitative comparison of the new equations and the NIST (2012) equations. Consider a foundation that has a large sliding stiffness such that $T_x \ll T_{sys}$ so all the T_x terms in Eqs. (30) and (32) are negligible. For this special case, the proposed equation (30) becomes:

$$\beta_{sys} = \left(\frac{T_{str}}{T_{sys}}\right)^2 \beta_{str} + \left(\frac{T_{yy}^p}{T_{sys}}\right)^2 \beta_{yy}^p + \left(\frac{T_{yy}^e}{T_{sys}}\right)^2 \beta_{yy}^e \tag{33}$$

T_{yy} in (32) represents the period due to both elastic and plastic flexibilities for associated with rotational deformations, hence we set $\left(T_{yy}\right)^2 = \left(T_{yy}^p\right)^2 + \left(T_{yy}^e\right)^2$ in (32) to facilitate comparison with (33). T_{yy} may be thought of as a "rigid structure period", that includes the elongated period associated with modulus degradation. Hence the NIST (2012) equation (32) becomes:

$$\beta_{sys} = \left(\frac{T_{str}}{T_{sys}}\right)^2 \beta_{str} + \frac{\left(T_{yy}^p\right)^2 + \left(T_{yy}^e\right)^2}{\left(T_{sys}\right)^2}\left(\beta_{yy} + \beta_s\right) \tag{34}$$

Eqs. (33) and (34) will clearly produce different results. To quantitatively compare Equations (33) and (34) in Figure 4, we consider a special case with these typical values: $\beta_{yy} = 0.02$, $\beta_{yy}^e = 0.02$, $\beta_{str} = 0.05$ and $T_{yy}^e = 0.2$ *s*. $\beta_{yy}^p = 0.15$ is adopted as a reasonable value for non-linear rocking foundations based upon recommendations of Deng et al. (2014) or Gajan and Kutter (2008). And finally, we use ASCE (2017) recommendations to determine the value of β_s in (34); based upon ASCE (2017), the soil hysteretic damping value depends only upon the peak acceleration of the design spectrum, S_{DS}, and on the site classification. For Site Classes C and D, and $S_{DS}/2.5$ greater than 0.1, β_s varies between 0.01 and 0.15. $\beta_s = 0.05$ is assumed for generation of Figure 4. The horizontal axis of Figure 4 is the footing rotational ductility demand (rotation of the footing)/(rotation required to mobilize the capacity of the footing) which is directly computed from:

$$\textit{Footing rotational ductility demand} = \frac{\left(T_{yy}^e\right)^2 + \left(T_{yy}^p\right)^2}{\left(T_{yy}^e\right)^2} = \frac{u_{yy}^e + u_{yy}^p}{u_{yy}^e} \tag{35}$$

The fixed base period of the structure was toggled from 0.1 to 0.4 s as indicated in the legend of Figure 4. For this special case, that the proposed equations produce significantly less damping for low footing rotation ductility demand and greater damping for high ductility demand when compared to ASCE (2017) equations.

It is interesting that the ASCE (2017) equations are very insensitive to ductility demand on the rocking footing. This is because the soil hysteretic damping in ASCE (2017) depends only upon the peak acceleration of the design spectrum, S_{DS}, and on the site classification; it is independent of ductility demand on the footing. On the other hand, the foundation hysteretic damping is reasonably sensitive to the footing ductility demand for the proposed relationships.

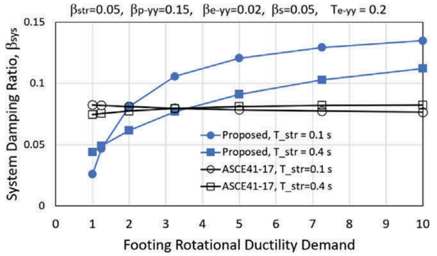

Figure 4. (a) Comparison of proposed damping equations as a function of the ductility demand on the footing for an example special case.

It should be noted that determination of the value of k^p_{yy} (the secant stiffness of the plastic rotation element) requires knowledge of the moment capacity of the footing and the shape of the hysteresis curve. For the above example a linear elastic-perfectly plastic (bilinear) moment rotation relationship was assumed and hence $k^p_{yy} = M_c/u^p_{yy}$ *and* $k^e_{yy} = M/u^e_{yy}$, where M is the moment on the footing and M_c is the moment capacity of the footing. Deng et al. (2014) and ASCE (2017) recommend a trilinear model instead of a bilinear model. Because the elasto-plastic response is "nonlinear", the solution of the problem would theoretically require an iterative solution (e.g., guess the system stiffness and damping, use the design spectrum to determine the total demand, distribute the demand to each component and then correct the stiffness and damping of each component, compute system stiffness and damping, and repeat until convergence is obtained).

6 CONCLUSIONS

The equations for damping for an SDOF structure supported on a rocking foundation were rederived using the concept from the theory of plasticity (Hill 1950) which indicates that the total deformation can be composed into elastic and plastic parts. The damping for each part is weighted by the flexibility of that part and summed to determine the damping of the system. The resulting equations (30) and (32) are believed to be of a more transparent form compared to those presented in NIST (2012) or ASCE (2017) (Equations 31(a) and 31(b)). Hopefully a more intuitive form will lead to more confidence for practicing engineers to use the equations and to check the designs. For elastic problems, with elastic radiation damping only and no hysteretic damping, the new equations are mathematically identical to those of NIST (2012). For nonlinear problems, however, NIST (2012) assumes that the hysteretic damping is identical for rocking and sliding modes and that soil hysteretic damping depends only on the amplitude of the design spectrum and on the site classification, irrespective of the footing ductility demand. The proposed equations account more accurately for the effect of plastic energy dissipation on hysteretic damping.

REFERENCES

ASCE 2017. *Seismic Evaluation and Retrofit of Existing Buildings*, ASCE/SEI 41-17, American Society of Civil Engineers, Reston, Virginia.

Deng, L., Kutter, B.L., and Kunnath, S.K. 2014. Seismic design of rocking shallow foundations: displacement-based methodology. *Journal of Bridge Engineering*, ASCE 19 (11).

Gajan, S., and Kutter, B.L. 2008. Capacity, settlement, and energy dissipation of shallow footings subjected to rocking, *Journal Geotechnical and Geoenvironmental Engineering*, 134(8): 1129–1141.

Givens, M.J., Mylonakis, G., and Stewart, J.P. 2016. Modular analytical solutions for foundation damping in soil-structure interaction applications, *Earthquake Spectra*, 32(3): 1749–1768.

Hill, R. 1950. *The Mathematical Theory of Plasticity*, Oxford University Press.

NIST 2012. *Soil-Structure Interaction for Building Structures*, NIST GCR 12-917-21, National Institute of Standards and Technology.

Wang, S., Kutter, B.L., Chacko, J.M., Wilson, D.W., Boulanger, R.W., and Abghari, A. 1998. Nonlinear Seismic Soil-Pile-Structure Interaction. EERI Earthquake Spectra, 377–396.

Wolf, J.P. 1985. *Dynamic Soil-Structure Interaction*, Prentice-Hall, Upper Saddle River, New Jersey.