6th International Conference on Earthquake Geotechnical Engineering
1-4 November 2015
Christchurch, New Zealand

# A True Zero Elastic Range Sand Plasticity Model

M. Taiebat[1] and Y.F. Dafalias[2]

## ABSTRACT

SANISAND models are a family of simple anisotropic constitutive models for sands developed over the past few years, within the framework of critical state soil mechanics and bounding surface plasticity. The present member of the SANISAND family incorporates the theory of truly zero purely elastic range in stress space, unlike other existing models which claim incorrectly to have this property. The overall performance of the model is comparable to that of classical models with very small yield surfaces from which it was derived. Elimination of the classical yield surface concept circumvents the complexity associated with satisfying the consistency condition; however the incrementally non-linear hypo-plastic nature of the new formulation requires special handling of its numerical implementation. Performance of the model in various stress paths, and under cyclic and irregular loadings are presented and discussed in this paper.

## Introduction

The idea of zero elastic range in plasticity theory where the yield surface size shrinks to zero and the surface degenerates to the current stress point in stress space was presented in Dafalias (1975), and came about as a corollary of bounding surface (BS) plasticity where such disappearance of yield surface is compensated by the still finite bounding surface that determines now the loading direction and plastic modulus, aspects which in classical plasticity are attributed to the finite yield surface and its hardening rules. The physical motivation was the effort to simulate the response of artificial graphite Dafalias (1975), Dafalias & Popov (1975, 1977), a material used in nuclear reactor technology and which exhibits zero purely elastic range in loading and unloading.

Several published models erroneously are referred to as zero elastic range models. Prominent among them are the stress-reversal models by Mroz et al. (1979), Mroz & Zienkiewicz (1984) and several subsequent publications by other authors, as well as models within generalized plasticity with or without a bounding surface formulation Aboim & Roth (1982), that can be traced back to Mroz (1966). The problem is that the structure of these models implies a neutral loading that is only elastic, thus, negating the zero elastic range character. Bottom line is that zero elastic range must be exactly what the name signifies, i.e., a null yield surface that collapses onto the stress point itself.

[1]Associate Professor, Department of Civil Engineering, University of British Columbia, Vancouver, BC, Canada, mtaiebat@civil.ubc.ca
[2]Professor, Department of Civil and Environmental Engineering, University of California, Davis, CA, USA, and Department of Mechanics, National Technical University of Athens, Zographou, Hellas, jfdafalias@ucdavis.edu

In this work a truly zero elastic range constitutive model for sands will be presented within the framework of Bounding Surface (BS) plasticity and Critical State Theory (CST) as part of the SANISAND models family Manzari & Dafalias (1997), Dafalias & Manzari (2004), and Taiebat & Dafalias (2008) to which often reference will be made. It will be named the SANISAND-Z model based on the zero elastic range feature.

## Formulation of the Constitutive Model

Based on the SANISAND model plasticity formulation, the strain rate - stress rate direct and inverse relations for a stress ratio dependent response are given as follows Dafalias & Manzari (2004):

$$\dot{\varepsilon} = \frac{1}{2G}\dot{\boldsymbol{s}} + \frac{1}{3K}\dot{p}\boldsymbol{I} + \langle L\rangle\left(\boldsymbol{R}' + \frac{1}{3}D\boldsymbol{I}\right) \tag{1}$$

$$\dot{\sigma} = 2G\dot{\boldsymbol{e}} + K\dot{\varepsilon}_v\boldsymbol{I} - \langle L\rangle(2G\boldsymbol{R}' + KD\boldsymbol{I}) \tag{2}$$

$$L = \frac{1}{K_p}\boldsymbol{n}{:}\,p\dot{\boldsymbol{r}} = \frac{2G\boldsymbol{n}{:}\,\dot{\boldsymbol{e}} - K(\boldsymbol{n}{:}\,\dot{\boldsymbol{r}})\dot{\varepsilon}_v}{K_p + 2G\boldsymbol{n}{:}\,\boldsymbol{R}' - KD(\boldsymbol{n}{:}\,\boldsymbol{r})} \tag{3}$$

where the loading index (plastic multiplier) $\langle L\rangle = L$ if $L > 0$, and $\langle L\rangle = 0$ if $L \leq 0$, the latter signifying the event of unloading. The application of the forgoing general formulation to a specific model, requires the specification of the moduli $G$ and $K$, and the plastic constitutive ingredients $\boldsymbol{n}$ (loading direction), $\boldsymbol{R}'$ (deviatoric plastic strain rate direction), $D$ (dilatancy), and $K_p$ (plastic modulus).

The $G$ and $K$ are given by standard hypoelastic relations found in any SANISAND model publication. With no Lode angle dependence for simplicity, the BS and dilatancy surface (DS) are shown schematically as circles $F^b = 0$ and $F^d = 0$ in the stress ratio $\pi$-plane of Fig. 1. Their radii depend on the bounding and dilatancy stress ratios given by $M^b = M\exp(-n^b\psi)$ and $M^d = M\exp(n^d\psi)$, respectively, where $M$ is the critical state stress ratio, $\psi = e - e_c$ is the state parameter Been & Jefferies (1985), $n^b$ and $n^d$ positive model constants. At critical state where $\psi = 0$ it follows that $M^b = M^d = M$ and both BS and DS collapse to the Critical State surface (CS) $F^c = 0$, shown in Fig. 1 as a circle of radius $\sqrt{(2/3)}M$ between the BS and DS.

A stress ratio rate $\dot{\boldsymbol{r}} = |\dot{\boldsymbol{r}}|\boldsymbol{v}$ is defined at the current stress ratio $\boldsymbol{r}$ in terms of its norm $|\dot{\boldsymbol{r}}|$ and its unit norm deviatoric direction $\boldsymbol{v}$ such that $\mathrm{tr}\,\boldsymbol{v} = 0$ and $\mathrm{tr}\,\boldsymbol{v}^2 = 1$. According to the idea suggested in Dafalias (1975), the image stress ratio $\boldsymbol{r}^b$ on the BS is obtained as the intersection of the $\dot{\boldsymbol{r}}$ direction with the BS, as shown also in Fig. 1 and given analytically by

$$\boldsymbol{r}^b = \boldsymbol{r} + b\boldsymbol{v}; \quad b = -\boldsymbol{r}{:}\,\boldsymbol{v} + \left[(\boldsymbol{r}{:}\,\boldsymbol{v})^2 + \left(\tfrac{2}{3}\right)(M^b)^2 - \boldsymbol{r}{:}\,\boldsymbol{r}\right]^{1/2} \tag{4}$$

where the value of $b$ is obtained by inserting the expression $\boldsymbol{r}^b = \boldsymbol{r} + b\boldsymbol{v}$ in $F^b = 0$ and solving for $b$. The loading direction $\boldsymbol{n}$ is defined by $\boldsymbol{n} = \partial F^b / \partial \boldsymbol{r}^b = \boldsymbol{r}^b / |\boldsymbol{r}^b|$. Loading $L > 0$ takes place always since the foregoing definition of $\boldsymbol{n}$ in terms of $\dot{\boldsymbol{r}}$ is such that $\boldsymbol{n}:\dot{\boldsymbol{r}} > 0$ for convex BS with the subsequent definition of $K_p > 0$, a characteristic feature of zero elastic range hypoplasticity. For future reference the point $\boldsymbol{r}^d$ is defined as the intersection of $\boldsymbol{n}$ with the dilatancy surface $F^d = 0$ as shown in Fig. 1. It should be mentioned that the use of a stress-rate dependent mapping rule shown in Fig. 1 with or without a zero elastic range assumption, while originally proposed in Dafalias (1975), Dafalias & Popov (1975), Dafalias & Popov (1977) for artificial graphite, it was also proposed for soils in a qualitative sense only by Dafalias (1979), and applied to sands within a full constitutive modeling framework by Wang et al. (1990), Gutierrez et al. (1991), Borja (1994), Cubrinovski & Ishihara (1998), and more recently by Das (2014) and Pisano & Jeremic (2014), in settings defining dilatatancy, plastic modulus and critical state differently than in this paper.

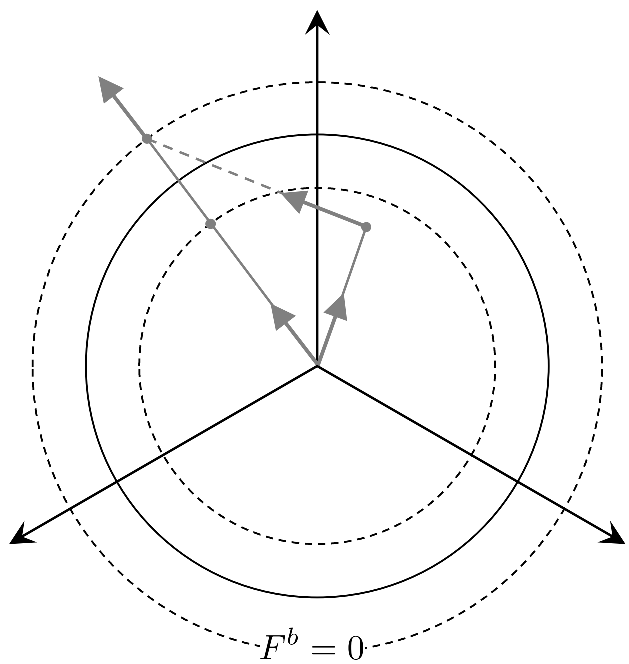


Figure 1: Illustration of model mechanism in deviatoric stress ratio space.

Assuming an associative deviatoric flow rule one can set $\boldsymbol{R}' = \boldsymbol{n}$. It follows that $\boldsymbol{n}:\boldsymbol{R}' = \boldsymbol{n}:\boldsymbol{n} = 1$ in the last member of Eq. (3) for $L$. One can use also a non-associative deviatoric flow rule as presented in Dafalias and Manzari (2004) for the SANISAND models for greater accuracy, but this will not be pursued here.

Following the original suggestion in Manzari & Dafalias (1997) and its modifications by Li & Dafalias (2000), Dafalias & Manzari (2004) and Taiebat & Dafalias (2008), the dilatancy will depend on the distance $\boldsymbol{r}^d - \boldsymbol{r}$ of the current stress ratio $\boldsymbol{r}$ from its image $\boldsymbol{r}^d$ on the DS, projected on the unit direction $\boldsymbol{n}$, Fig. 1, thus, the following expression holds

$$D = A_d(\boldsymbol{r}^d - \boldsymbol{r}):\boldsymbol{n} = A_d(\sqrt{(2/3)}M^d - |\boldsymbol{r}|\boldsymbol{n}^r:\boldsymbol{n}) \tag{5}$$

with $\boldsymbol{n}^r$ a unit norm tensor along $\boldsymbol{r}$. The $A_d$ in the simplest case is constant, but it was found beneficial for the simulation of liquefaction to render it a function of a fabric dilatancy tensor $\boldsymbol{z}$ as $A_d$=$A_0(1 + \sqrt{(3/2)}\langle \boldsymbol{z}:\boldsymbol{n}\rangle)$, where $\boldsymbol{z}$ evolves according to $\dot{\boldsymbol{z}} = -c_z\langle -\dot{\varepsilon}_v^p\rangle(\sqrt{(2/3)}\mathrm{z}_{\max}\boldsymbol{n} + \boldsymbol{z})$, Dafalias & Manzari (2004).

Consistent with a BS formulation the value of the plastic modulus $K_p$ will depend on the "distance" $\boldsymbol{r}^b - \boldsymbol{r}$ of the current stress ratio $\boldsymbol{r}$ from its image $\boldsymbol{r}^b$ on the BS projected on the unit direction $\boldsymbol{n}$. The proposition made in Dafalias & Manzari (2004) will be adopted with the observation that what was for the foregoing reference the back-stress ratio $\boldsymbol{\alpha}$ is now the stress ratio $\boldsymbol{r}$ (recall that for zero elastic range one obtains $\boldsymbol{r} = \boldsymbol{\alpha}$ as the yield surface shrinks to zero), and which reads:

$$K_p = \left(\frac{2}{3}\right)ph[(\boldsymbol{r}^b - \boldsymbol{r}):\boldsymbol{n}]/[(\boldsymbol{r} - \boldsymbol{r}_{\text{in}}):\boldsymbol{n}] \tag{6}$$

The $\boldsymbol{r}_{\text{in}}$ is the value of $\boldsymbol{r}$ at the initiation of a plastic loading event and $h$ is a model parameter which is function of the void ratio $e$ and pressure $p$ according to $h = G_0h_0(1 - c_he)(p/p_{\text{at}})^{-0.5}$, with $h_0$ and $c_h$ model parameters. The $\boldsymbol{r}_{\text{in}}$ must be updated to a new value at initiation of a new plastic loading event in order to obtain the infinite value of the plastic modulus resulting from Eq. (6) when $\boldsymbol{r} = \boldsymbol{r}_{\text{in}}$, expected in such event. Given that $L$ never becomes negative based on the mapping rule definition, a new plastic loading event is defined as follows. When $(\boldsymbol{r} - \boldsymbol{r}_{\text{in}}):\boldsymbol{n} \leq 0$, it means that the current loading direction $\boldsymbol{n}$ forms an angle greater than 90° in the generalized stress ratio space with the tensor $\boldsymbol{r} - \boldsymbol{r}_{\text{in}}$, which is a measure of the overall direction of ongoing stress loading path, and this is tantamount to unloading and the beginning of a new loading process. Hence the $\boldsymbol{r}_{\text{in}}$ is updated to the current $\boldsymbol{r}$ value when the denominator of Eq. (6) becomes negative, and a new loading process begins with an initially infinite value of the plastic modulus. In other words the unloading event is followed immediately by a new loading event, without any purely elastic response taking place.

The presence of $\boldsymbol{r}_{\text{in}}$ in Eq. (6) is beneficial because it renders $K_p \to \infty$ at the initiation of a loading event when $\boldsymbol{r} - \boldsymbol{r}_{\text{in}} = \boldsymbol{0}$, thus, inducing what is called a smooth elasto-plastic transition, but also creates the so-called overshooting response upon reverse loading/immediate reloading, caused by its ensuing updating, known since the time of its inception Dafalias (1975). Overshooting implies a stress-strain curve which unrealistically overshoots the continuation of a previous curve had the event of small reverse loading/reloading not taken place. A remedy of overshooting was proposed in Dafalias (1986) and is applied here. The reader is referred to the foregoing reference for details and suffices to mention that a threshold of previous cumulative plastic strain is used to properly weight the updating of $\boldsymbol{r}_{\text{in}}$ between its value in past and current loading processes.

## Model Performance

Figure 2 presents performance of the SANISAND-Z model in simulation of selected undrained triaxial loading and unloading paths for which the experimental data is available in the literature. For the sake of consistency, the same experimental data of Toyoura sand by Verdugo & Ishihara (1996) that was studied for the SANISAND model by Dafalias & Manzari (2004), Taiebat & Dafalias (2008), and Taiebat et al. (2010) is used here as the benchmark data to study the performance of SANISAND-Z.

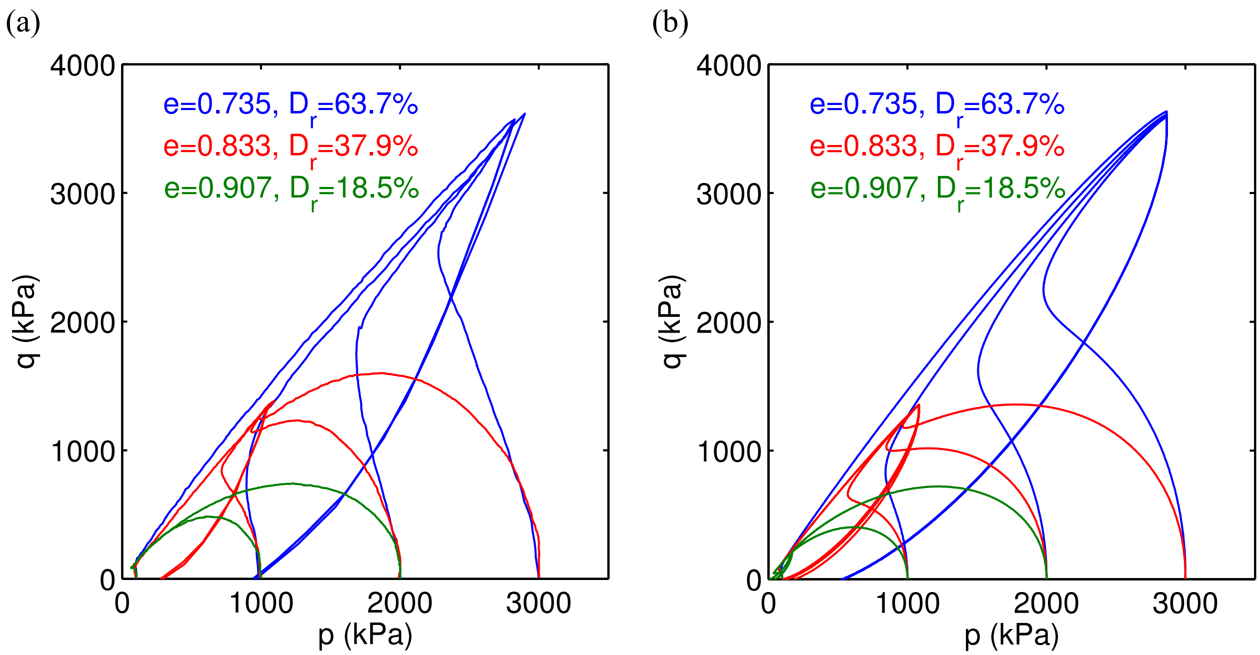


Figure 2: Comparison of (a) data of Verdugo & Ishihara (1996) and (b) SANISAND-Z simulations for undrained triaxial compression and unloading on isotropically consolidated samples of Toyoura sand ($e_{in}$ = 0.735, 0.833, 0.907, and $p_{in}$ = 100, 1000, 2000, 3000 kPa).

Additional simulation under a particular rotational shear path comprising a circular stress path in the π-plane is presented in Fig. 3, because it exhibits in a very illustrative way the advantages of SANISAND-Z for unorthodox stress paths. Similar response was obtained by Choi & Arduino (2004) and Choi (2004) using a small but finite size yield surface SANISAND model.

## Conclusion

The present work presents the SANISAND-Z model as a member of the family of SANISAND models, with the distinctive feature of zero (vanishing) purely elastic range that implies the shrinking of the yield surface size to zero and its degeneration into the stress point itself. The constitutive mechanism of the model follows the original suggestion of zero elastic range bounding surface plasticity formulations Dafalias (1975, 1986), adjusted to reflect the basic properties of SANISAND models within Critical State Theory.

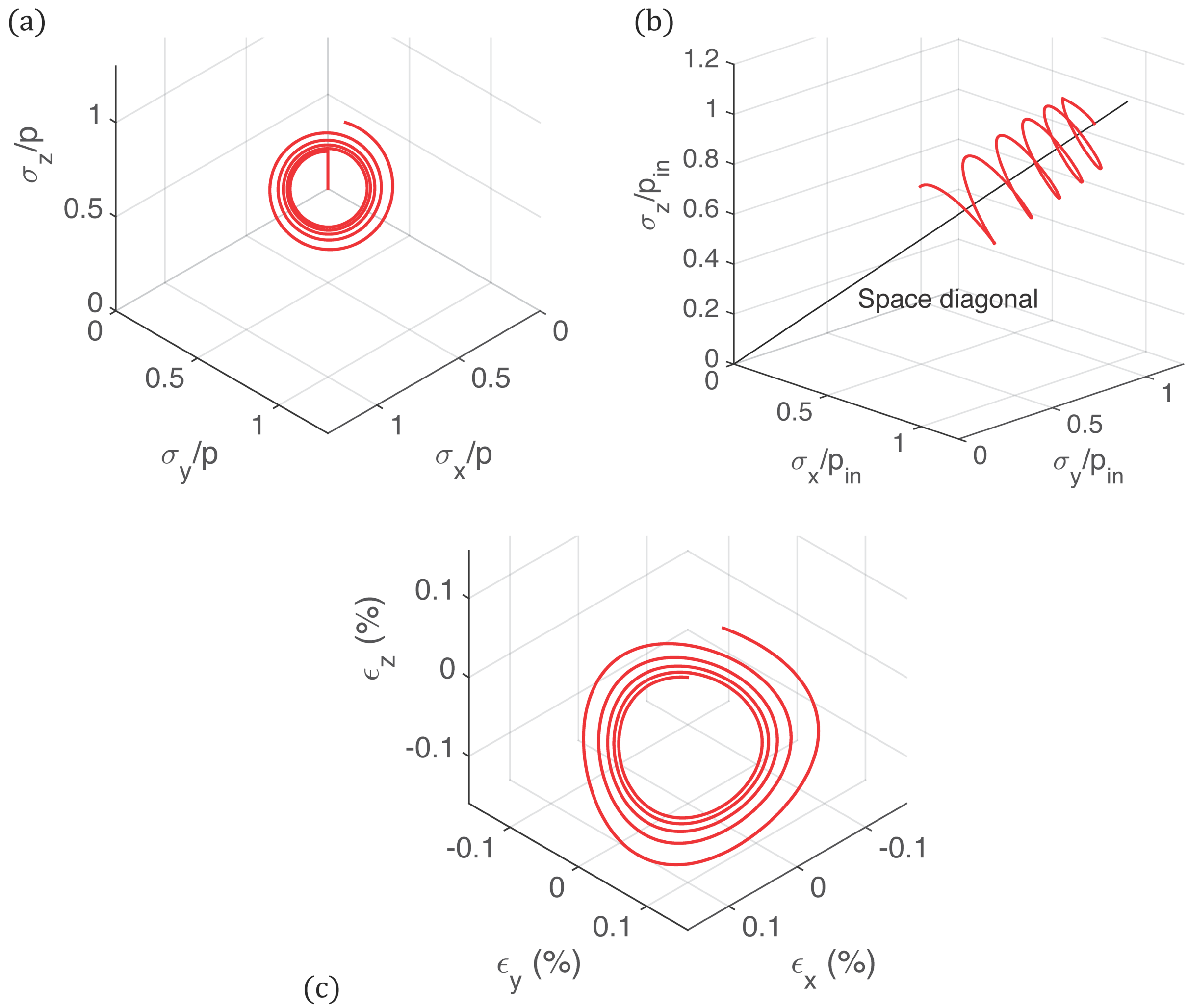


Figure 3: SANISAND-Z simulations for undrained circular stress path in the π-plane following undrained triaxial compression until $\tau_{oct}$=45 kPa on an isotropically consolidated sample of Toyoura sand ($e_{in}$ = 0.880, and $p_{in}$ = 500 kPa)

The model as expected has shown similar simulation capabilities for triaxial loading as its dual model with a very small yield surface by Dafalias & Manzari (2004), and it was also shown qualitatively that it can simulate rotational shear. In general one must expect that a zero elastic range model has the same capabilities with a corresponding model that uses a very small size yield surface, because the former is obtained from the latter as the yield surface shrinks to zero in the limit. The plastic loading occurrence for any stress rate direction and the dependence of the plastic strain rate direction on the stress rate direction (hypoplasticity feature), renders the SANISAND-Z model appropriate for consideration of bifurcation and other instability phenomena. However these traits of incremental non-linearity, also render its numerical implementation a demanding task that will be presented in future works.

## Acknowledgments

The research leading to these results received funding from the European Research Council under the European Union’s Seventh Framework Program FP7-ERC-IDEAS Advanced Grant Agreement no. 290963 (SOMEF), and partial support from the Natural Sciences and Engineering Research Council of Canada (NSERC), and National Science Foundation (NSF) project CMMI-1162096.

## References

Aboim, C. A. & Roth, W. H. (1982). Bounding-surface-plasticity theory applied to cyclic loading of sand. In Dungar, R., Pande, G. N., & Studer, J. A. (Eds.), *International Symposium on Numerical Models in Geomechanics*, (pp. 65–72)., Zurich, Switzerland. A. A. Balkema.

Been, K. & Jefferies, M. G. (1985). A state parameter for sands. *Geotechnique*, **35**(2), 99–112.

Choi, C. H. (2004). *Physical and Mathematical Modeling of Coarse-Grained Soils.* Doctor of philosophy, University of Washington , Seattle, WA, USA.

Choi, C. H. & Arduino, P. (2004). Behavioral characteristics of gravelly soils under general cyclic loading conditions. In *Cyclic behaviour of soils and liquefaction phenomena*, Proc. of CBS04, (pp. 115–122)., Bochum, Marz. Balkema.

Cubrinovski, M. & Ishihara, K. (1998). State concept and modified elastoplasticity for sand modelling. *Soils and Foundations*, **38** (4), 213–225.

Dafalias, Y. (1975). On cyclic and anisotropic plasticity. PhD thesis, Department of Civil Engineering, University of California, Berkeley, CA, USA.

Dafalias, Y. F. (1979). A model for soil behavior under monotonic and cyclic loading conditions. In Transactions, *5th International Conference, on SMiRT*, volume K, (pp. paper No. K 1/8)., West Berlin, Germany.

Dafalias, Y. F. (1986). Bounding surface plasticity. I: Mathematical foundations and hy-poplasticity. *ASCE Journal of Engineering Mechanics*, **112** (9), 966–987.

Dafalias, Y. F. & Manzari, M. T. (2004). Simple plasticity sand model accounting for fabric change effects. *ASCE Journal of Engineering Mechanics*, **130** (6), 622–634.

Dafalias, Y. F. & Popov, E. P. (1975). A simple constitutive law for artificial graphite- like materials. In Transactions, *3rd International Conference on Structural Mechanics in Reactor Technology*, volume 1, paper C 1/5, London, U.K.

Dafalias, Y. F. & Popov, E. P. (1977). Cyclic loading for materials with a vanishing elastic region. *Nuclear Engineering and Design*, **41** (2), 293–302.

Das, S. (2014). *Three dimensional formulation for the stress-strain-dilatancy elasto-plastic constitutive model for sand under cyclic behavior*. Master’s thesis, Department of Civil and National Resource Engineering, University of Canterbury, Christchurch, New Zealand.

Gutierrez, M., Ishihara, K., & Towhata, I. (1991). Flow theory for sand during rotation of principal stress direction. *Soils and Foundations*, **31** (4), 121–132.

Li, X. S. & Dafalias, Y. F. (2000). Dilatancy for cohesionless soils. Geotechnique, 54(4), 449–460. Manzari, M. T. & Dafalias, Y. F. (1997). A critical state two–surface plasticity model for sands. *Geotechnique*, **47** (2), 255–272.

Mroz, Z. (1966). On forms of constitutive laws for elastic-plastic solids. *Arch. Mech. Stos*., **18**, 3–35.

Mroz, Z., Morris, V. A., & Zienkiewicz, O. C. (1979). Application of an aniso- tropic hardening model in the analysis of elastoplastic deformation of soils. *Geotechnique*, **29,** 1–34.

Mroz, Z. & Zienkiewicz, O. (1984). Uniform formulation of constitutive equations for clays and sands. In Desai, C.

S. & Gallagher, R. H. (Eds.), *Mechanics of engineering materials*, (pp. 415–449). John Wiley and Sons, Chichester, UK.

Pisano, F. & Jeremic, B. (2014). Simulating stiffness degradation and damping in soils via a simple visco-elastic–plastic model. *Soil Dynamics and Earthquake Engineering*, **63**, 98–109.

Taiebat, M. & Dafalias, Y. F. (2008). SANISAND: simple anisotropic sand plasticity model. *International Journal for Numerical and Analytical Methods in Geomechanics*, **32** (8), 915– 948.

Taiebat, M., Jeremic, B., Dafalias, Y. F., Kaynia, A. M., & Cheng, Z. (2010). Propagation of seismic waves through liquefied soils. *Soil Dynamics and Earthquake Engineering*, **30** (4), 236–257.

Verdugo, R. & Ishihara, K. (1996). The steady state of sandy soils. *Soils and Foundations*, 36 (2), 81–91.

Wang, Z.-L., Dafalias, Y. F., & Shen, C.-K. (1990). Bounding surface hypoplasticity model for sand. *Journal of Engineering Mechanics*, **116** (5), 983–1001.