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Analytical solution of longitudinal behaviour of tunnel lining

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ABSTRACT: Staged construction of segmented tunnels result in permanent and constant bending moment in the longitudinal direction. This fact has been confirmed empirically, and analytical solutions for the longitudinal behaviour of a bored tunnel lining have been presented. This paper summarizes published analytical solutions for simple loading conditions, and includes corrections for the solutions where necessary. Solutions for additional loading conditions relevant to TBM tunnel construction are also presented. The analytical solutions have been built in a powerful Excel spreadsheet for rapid analyses. The results have been validated with staged construction FEM calculations in PLAXIS 2D. For final validation of the staged construction behaviour, tunnelling data from the Groene Hart tunnel in the Netherlands were analysed. Segments were instrumented with axial strain gauges, and the results have been analysed and converted to longitudinal bending moments. It has been demonstrated that the measured behaviour is reproducible with the analytical model, although selection of input parameters is complex. The presented model is well suited for quick analyses of TBM back-up train lay out, grouting conditions and moments from jacking forces with respect to longitudinal lining behaviour.

1 INTRODUCTION

TBM tunnelling is characterised by phased construction of the segmented lining. Excavation by the tunnel boring machine is followed by erection of a single ring of lining segments. The complete structure can be regarded as a beam on elastic foundation according to classic structural engineering theory. In each phase of construction, a load free member is introduced, which contributes to the structural system. In the meanwhile, loading progresses simultaneously with the TBM. Examples are jacking forces on the first ring, buoyancy forces within the grouting zone and self weight of the back-up train.

Previous theoretical analyses of the structural system have shown that the distribution of bending moment and shear forces is fundamentally different from a wished-in-place beam on an elastic foundation. Measurements from the Groene Hart tunnel clearly confirm this difference.

2 ANALYTICAL SOLUTIONS

2.1 *General procedure*

The progressive loading conditions are:

1 uniform loading
2 shear force at front end of the beam
3 bending moment at front end of the beam
4 local uniformly distributed load

Bogaards & Bakker (1999) have given analytical solutions for the section forces of a beam on an elastic foundation with progressive extension of the beam under a uniform load. Bakker (2000) published solutions for items 1 to 3.

Section forces can be derived in several ways. Bakker's (2000) derivation is explained and the corrected results are given below.

2.2 *Uniform load*

A staged extension of a uniformly loaded beam can be solved as the sum of the analytical solutions of a partially loaded beam (Figure 1, left side). The mechanical

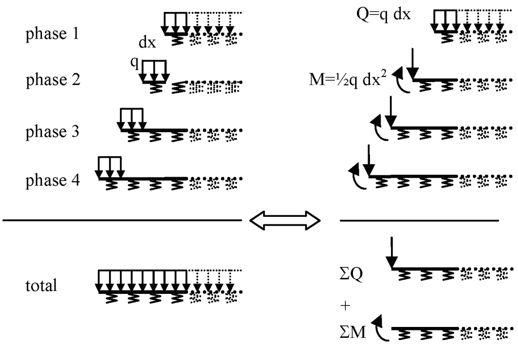
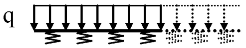


Figure 1. Staged construction uniform load.

scheme is equivalent to the scheme given on the right side of Figure 1.

The section forces $M_q(x)$ and $D_q(x)$ (Equation 1 to 4) can easily be summed from the analytical solution of a beam loaded with a shear force and the bending moment at the beginning of the beam. These simple solutions have been given by Hetényi (1946), Bouma (1993) and Young et al. (2002).



$$M_q(x) = 0 \quad for \quad x = 0 \tag{1}$$

$$M_q(x) = \frac{-q\,dx}{\beta}\sum_{n=0}^{n=x/dx} e^{-\beta n dx}\cdot\sin(\beta\, n\, dx) + \ldots$$

$$\ldots + \frac{1}{2}\sqrt{2}\, q\, dx^2 \sum_{n=0}^{n=x/dx} e^{-\beta n dx}\cdot\sin(\beta\, n\, dx + \frac{\pi}{4}) \quad for \quad x > 0 \tag{2}$$

$$D_q(x) = 0 \quad for \quad x = 0 \tag{3}$$

$$D_q(x) = \frac{-q}{\beta} e^{-\beta x}\cdot\sin(\beta x) + \ldots$$

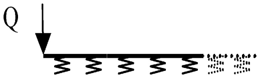
$$\ldots + \frac{1}{2} q\, dx \sqrt{2}\cdot e^{-\beta x}\cdot\sin(\beta x + \frac{\pi}{4}) \quad for \quad x > 0 \tag{4}$$

$$\beta = \left(\frac{k}{4\,EI}\right)^{0.25} \tag{5}$$

with x = coordinate starting on the left side; dx = length of beam increment; k = modulus of subgrade reaction; E = Young's modulus; I = moment of inertia.

2.3 *Shear force at front end of the beam*

Figure 2 shows the mechanical scheme of a shear force at the front end of a stage-constructed beam. As in Figure 1, the analytical solution of a staged (beam) extension with shear force at the front end can be found as the sum of the analytical solution of a partially loaded beam as explained on the left side of Figure 2. The mechanical scheme is equivalent to the scheme given on the right side of Figure 2. The resulting section forces $M_Q(x)$ and $D_Q(x)$ (Equation 6 to 8) have been derived from the right side of Figure 2.



$$M_Q(x) = 0 \quad for \quad x = 0 \tag{6}$$

$$M_Q(x) = -Q\, d\, x\sqrt{2}\sum_{n=1}^{n=x/dx} e^{-\beta n dx}\cdot\sin(\beta\, n\, dx + \frac{\pi}{4})$$

$$for \quad x > 0 \tag{7}$$

$$D_Q(x) = -\sqrt{2}\, Q e^{-\beta x}\cdot\sin(\beta x + \frac{\pi}{4}) \tag{8}$$

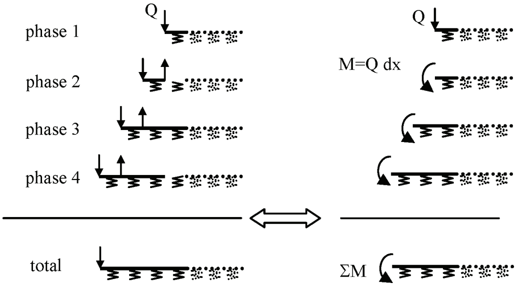


Figure 2. Staged construction shear force.

2.4 *Bending moment at front end of the beam*

Progressive extension of the bending moment at the front end of the beam results in the simple solutions $M_M(x)$ and $D_M(x)$ in Equations 9 and 10.

$$M_M(x) = -M \tag{9}$$

$$D_M(x) = 0 \tag{10}$$

2.5 *Local uniformly distributed load*

The analytical solution for a progressive local uniform load can be derived as shown in the mechanical scheme in Figure 3. The resulting section forces are complex and are not presented here.

3 SPREADSHEET-MODEL

The mechanical scheme for the design of a tunnel lining as a beam on an elastic foundation is given in Figure 4. An unsupported part of tunnel lining is present within the TBM (l_i) and also between the TBM and stable grout (l_u). The following loads are present:

- Bending moment from jack forces (M_{jack})
- Shear force from jacking forces (D_{jack})
- Shear force from steel brushes (D_{br})
- Weight of lining segments (q_w)
- Uniformly distributed load (q) starting behind the TBM at distance (l) with a length (l_q).

An Excel-spreadsheet model was built for this mechanical scheme with the basic solutions given in paragraph 2. The model is equipped with 6 local, uniformly distributed loads to account for buoyancy forces and load configurations of the back-up train, backfill and permanent structure within the tunnel. Therefore the model is a powerful tool to calculate section forces for a wide variety of load conditions simply by entering the necessary parameters.

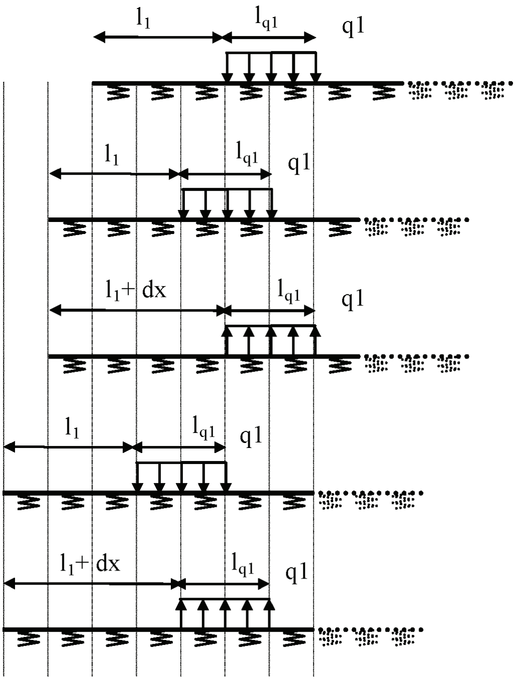


Figure 3. Staged construction local uniform load.

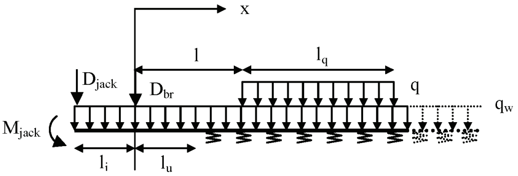


Figure 4. Load scheme and subgrade reaction tunnel beam.

Determining the angular and vertical deformation of the tunnel is not straight-forward. According to classical structural engineering theory, the angular and vertical deformation can be found by integration of the bending moment.

$$\frac{d}{dx}\varphi(x) = \frac{-M(x)}{EI} \tag{11}$$

$$\frac{d}{dx}w(x) = \varphi(x) \tag{12}$$

The solution of a stage-constructed tunnel results in a constant bending moment at great distance from the TBM. Direct integration of this constant value results in a linear increase of angular deformation, and a quadratic increase of vertical displacement. Indeed, finite element calculations of a stage-constructed tunnel with strain-less extension of the beam show this type of deformation. Since the final tunnel must be positioned according to its design alignment, tunnel rings must be installed with an inclination onto the previous ring. Figure 5 shows the basic idea of ring installation.

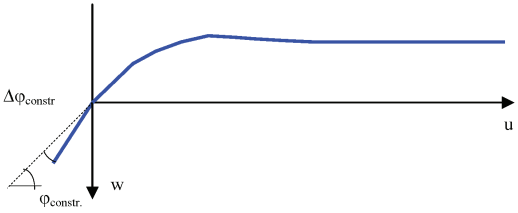


Figure 5. Angular offset installation new ring.

The spreadsheet-model is equipped with the calculation of angular and vertical deformation based on this principle. Part of the result is the required installation offset angle relative to the vertical and relative to the previous ring.

The model has been verified using finite element modeling with PLAXIS 2D version 8.4. For each of the basic loading conditions given in paragraph 2.1, a verification has been performed. In PLAXIS, a structural beam on an elastic soil mass was modeled under the applicable loading conditions. Phased calculations have been performed with each phase extending the beam and soil support, and moving the load one step ahead. Results form the analytical spreadsheet-model and finite element model show good to excellent agreement.

4 GROENE HART TUNNEL

For the High Speed Railway link between Amsterdam and Brussels, a 7.2 km long TBM tunnel was recently constructed. The tunnel's outer diameter is 14.5 m and the lining thickness is 0.6 m. The Dutch research committee COB (Center for Underground Construction) organized an extensive measuring campaign to study the longitudinal behavior of the lining. Strain gauges, tilt sensors and two systems to measure vertical deformation were placed on some of the tunnel rings.

4.1 *Strain gauges*

Strain gauges were used to measure axial and tangential strain during construction. Pairs of gauges were placed in axial direction, both on the inside and on the outside of the segments.

After 4 days, the signals reach general equilibrium, with minor fluctuations due to jacking forces. Complete results from axial gauges for ring number 2117 are shown in Figure 6. The average of the strain measurements from the inside and the outside of the segment are given in Figure 7.

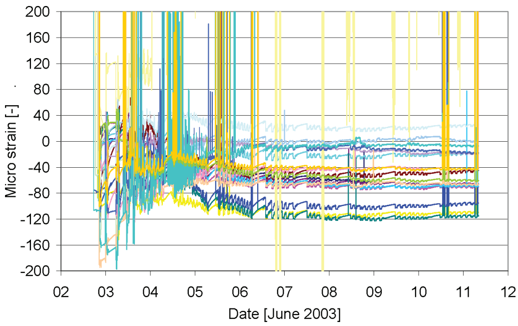


Figure 6. Results of all strain gauges ring 2117.

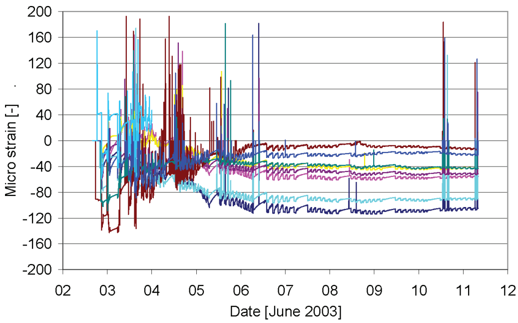


Figure 7. Average results strain gauges ring 2117.

In Figure 8 typical results of the strain distribution have been plotted against the vertical position, shortly after having passed through the TBM. The same has been done at the end of the measuring campaign (Figure 9). The slope of the trend line through these measurements changes of sign between the beginning and end of the measurement program. This axial gradient is determined by the bending moment in the cross section.

4.2 *Bending moment and normal force*

Using the axial strain at tunnel axis and the gradient of the trend line, the average normal force and bending moment can be determined with Equations 13 and 14.

$$N = \sigma_{av} A = EA \cdot \varepsilon_{av} \tag{13}$$

$$M = EI \cdot \kappa \tag{14}$$

with N = average normal force; σ_{av} = average axial stress; ε_{av} = average axial strain; κ = axial strain gradient; E = Young's modulus = 38500 MPa; A = section area = 26.2 m^2; I = moment of inertia = 634 m^4.

In Figure 10, the derived axial normal force and bending moment from strain gauge measurements for the entire logging period are shown. Results of normal force and bending moment at time of leaving the TBM have been compared with TBM data of jacking forces. A reduction factor has been included in the results of Figure 10. When a reduction factor of 0.9 is applied to the axial strain, the back-calculated jack forces match the directly measured jack forces. The reduction factor takes into account the influence of non-uniform strain distribution within the segments.

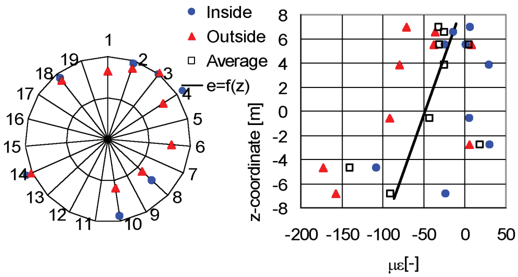


Figure 8. Axial strain ring 2117, just after having passed through the TBM, June 3 2003, 1:00 hrs.

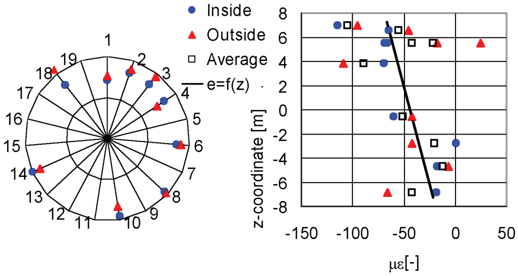


Figure 9. Axial strain ring 2117, at the end of measuring campaign, June 11 2003, 0:00 hrs.

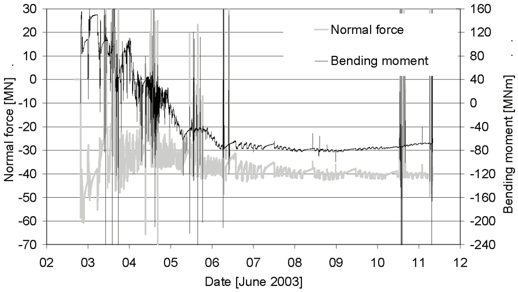


Figure 10. Normal force and bending moment from axial strain measurement ring 2117.

Also the section forces as functions of distance from the tail end of the TBM are presented in Figure 11. These lines can only be regarded as normal force and

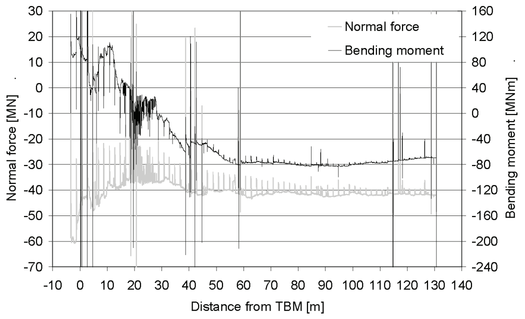


Figure 11. Normal force and bending moment ring 2117.

Table 1. Input parameter.

Description	Identification Fig. 4	Value unit
Outside diameter		14.50 m
Wall thickness		0.60 m
Young's modulus		3.85E+07 kN/m^2
Reduction factor stiffness		0.650
Modulus of subgrade reaction		367,000 kN/m^2
Length segments in tunnel	l_i	6 m
Unsupported length	l_u	2 m
Moment jack forces	M_{jack}	79,000 kNm
Shear force jack	D_{jack}	0 kN
Shear force brushes	D_{br}	0 kN

bending moment distribution along the tunnel axis during constant tunnelling. From Figure 11 it is evident that the bending moment reaches a constant value at approximately 60 m behind the TBM. This remaining bending moment is a consequence of the staged construction of tunnelling.

4.3 *Back analyses*

The analytical model has been applied to calculate the distribution of the bending moment within the lining behind the TBM. The input parameters have been given in Tables 1 and 2. They consist of simple geometric and material parameters and loading conditions. Two parameters need further explanation. First, a reduction factor for the bending stiffness has been applied to account for the structural behaviour of joints between segments. Secondly, the external grout load requires further analyses. Refer to the paper on this subject by Talmon et al. (2008).

The calculated and measured bending moments are in Figure 12. There is good agreement between measurements and calculation results. It is noted that alternate combinations of input parameters may also result in fair agreement. However, the analytical model clearly shows the sensitivity of different loading conditions. Therefore the model is well suited to analysing loading conditions like position of back-up train, backfill, jacking forces and grouting conditions.

Table 2. Input parameter; uniformly distributed load.

Load identification Fig. 4	Load [kN/m]	Position behind TBM [m]
q_w	629	−6 to 1000
q_1	−2064	0 to 1000
q_2	437.5	2 to 26
q_3	70	30 to 1000
q_4	300	52 to 1000
q_5	187	82 to 108

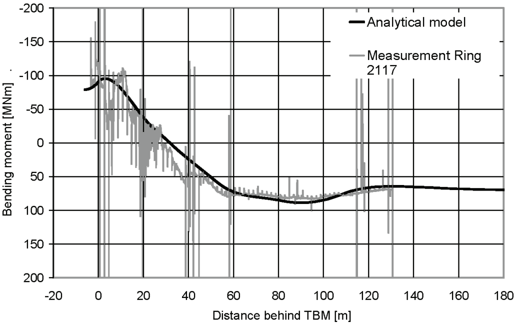


Figure 12. Analytical model and measurement results ring 2117.

5 CONCLUSIONS

The mechanical behaviour of a stage-constructed TBM tunnel is fundamentally different from a wished-in-place tunnel. Analytical solutions for bending moment, shear force, angular deflection and vertical displacement have been used to create a spreadsheet-model. Measurements from the Groene Hart Tunnel clearly show a residual bending moment far behind the TBM, a result which is in good agreement with results of the analytical model. The longitudinal behaviour of the tunnel as a consequence of the staged construction needs to be accounted for in tunnel design.

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