On the formula of Darcy's law and Seepage Force

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ABSTRACT: The formula of Darcy's law for both saturated and unsaturated soils is directly derived based on the equilibrium equation. The definition of seepage force is clarified, which is caused by the relative movement of pore fluid to soil skeleton. On this definition, according to the independent equilibrium differential equation of pore water in saturated soil, the formula of seepage force is obtained. It differs from the conventional expression by a factor of porosity *n*. Therefore, it is inferred that the definition of the seepage force or the conventional formula of seepage force needs to be modified. Moreover, considering the derivation process of the formula of seepage force in the existing soil mechanics textbook as an example, a new formula of seepage force is deduced. For an unsaturated soil, in addition to the seepage force caused by the relative movement of pore water to soil skeleton, the change of the water content may also lead to the interaction force. So, the expression of seepage force for an unsaturated soil is derived, which is in agreement with that for a saturated soil, but the coefficient of the expression is the volumetric water content, namely the porosity with respect to the water content instead of porosity *n*.

1 INTRODUCTION

Traditionally, Darcy's law (1856) and the mass conservation equation of pore fluid are necessary in the derivation of the seepage equation (Braja 2008; Lei et al. 1988; Yin et al. 2007). The formula of Darcy's law illustrates the relationship between the hydraulic potential gradient and seepage velocity, which is usually called the motion equation of pore water. For saturated soils, this relation is obtained via experiments. For unsaturated soils, this relation is derived by directly expanding the saturated soils' counterpart, which also can be obtained in experiments.

In this paper, we directly derive the formula of Darcy's law for both saturated and unsaturated soils from the equilibrium equation with the assumption that seepage resistance is proportional to seepage velocity. Then, we derive the formula of seepage force by the equilibrium equation and indicate that the traditional formula of the seepage force should be modified according to its definition.

2 DERIVATION OF THE FORMULA OF DARCY'S LAW FOR SATURATED SOIL

Water flows in pores because of gradient of energy. Since its velocity is very small in general, the kinetic energy of pore water is negligible. Only potential energy, i.e., the soil-water potential may be considered. For saturated soils, soil-water potential includes gravity potential and pressure potential. For a unit weight of water, the potential may be expressed as water head. Gravity potential is called gravity head, or position head, and pressure potential is called pressure head. The sum of these two potentials is called total water head and also called piezometer head. The pore water flows from high water head towards low water head, overcoming soil resistance (soil-water interaction).

2.1 *Soil-water interaction in saturated soils*

The pore water encounters resistance during its permeation through the soil skeleton and at the same time applies forces on the soil skeleton. The resistance that the water bears during its permeation is the force of soil-water interaction and also the seepage force. In hydraulics, the flowing status of water is classified as laminar flow and turbulent flow. Most of the pore water flow in soil is laminar flow. The moving resistance on the laminar flow water is proportional to velocity. Therefore, in unit soil mass, the interaction force applied on pore water can be expressed as:

$$f_i = \frac{a}{n_w} v_i \quad (1)$$

where, f_i is the force of soil-water interaction; v_i is the velocity of pore water at the full section; a is the index of the force between pore water and the soil skeleton in the unit volume of the soil body. The relationship between the full section average flow velocity at a point in soil v and the flow velocity at porous area v' can be determined by the following formula:

$$v = n_w v' \tag{2}$$

where, n_w is the porosity corresponding to pore water, it can also be denoted as n for saturated soils, where n is the porosity of the soil.

For the deduction of the seepage equation for unsaturated soils, we use a_w to represent the factor of soil-water interaction force for the unit volume of pore water, not the unit volume of soil body. For saturated soils, $a_w = a/n$, i.e., $a = na_w$; and for unsaturated soils, $a_w = a/n_w$, i.e., $a = n_w a_w$. According to the correlation between a and a_w, Equation (1) can be written as:

$$f_i = a_w v_i \tag{3}$$

2.2 *The formula of Darcy's law for saturated soil*

On the condition that the soil is homogeneous, and the inertia force of pore water can be ignored, we have the equilibrium differential equations (Shao 2011; Shao & Guo 2014) of pore water as:

$$(n_w u_w)_{,i} + f_i + X_i = 0 \tag{4}$$

where, $X_x = X_y = 0$ and $X_z = n_w \rho_w g$.

Let:

$$H = \frac{u_w}{\rho_w g} + z \tag{5}$$

where u_w is the pore water pressure; z is the position head and H is the piezometer head or total head.

Substituting Equation (3) into Equation (4), we can obtain that:

$$n_w \rho_w g H_{,i} + a_w v_i = 0 \tag{6}$$

And let:

$$k = \frac{n_w \rho_w g}{a_w} \tag{7}$$

$$i_i = H_{,i} \tag{8}$$

where, k is the coefficient of permeability of the soil in saturated state and i is hydraulic gradient or water potential gradient.

Consequently, Equation (6) can be rewritten as:

$$v_i = -\ k\ i_i \tag{9}$$

Equation (9) is the motion equation for seepage in saturated soils. It reflects the correlation between the seepage velocity of pore water and soil-water potential gradient, namely, the velocity of pore water in seepage is proportional to water potential gradient. Darcy obtained this equation via experiment in 1852, and it was known as the famous Darcy's law.

3 DERIVATION OF THE FORMULA OF DARCY'S LAW FOR UNSATURATED SOIL

3.1 *Soil-water interaction force of unsaturated soils*

The soil-water interaction force in saturated soils only includes seepage force, while there is no interaction force between the soil skeleton and pore water when seepage does not exist; for unsaturated soils, except the forces induced by seepage (i.e., the force induced by the relative motion between pore water and the soil skeleton), the non-uniform distribution of pore water will also induce the interaction force between the soil skeleton and pore water, even though there is no water motion.

As shown in Figure 1, assuming that the volumetric water content distribution in the soil layers above the ground water level in static equilibrium condition is:

$$n_{w0(z_0)} = nS_{0(z_0)} \tag{10}$$

where, $n_{w0(z_0)}$ is the function of porosity corresponded to the pore water (volumetric water content function) along the vertical direction in static equilibrium condition; n is the porosity; $nS_{0(z_0)}$ is the function of the saturation degree along vertical direction in static equilibrium condition and z_0 is the vertical height from the ground water level.

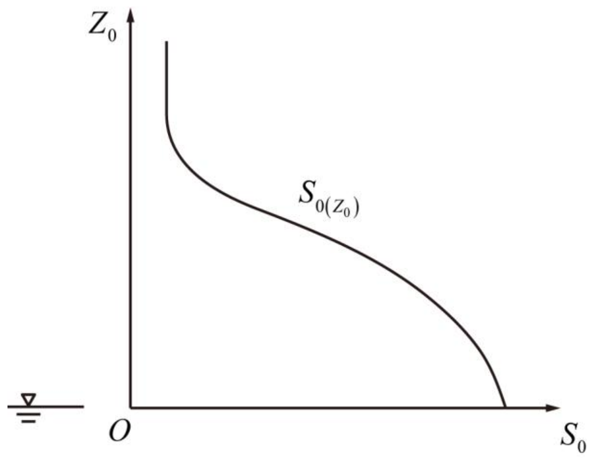

Figure 1. Water content distribution of the soil layer above ground water level.

By analysing the pressure distribution of static pore water in the unsaturated and saturated soil layers above ground water level, it can be known that pore water pressure presents a continuously linear distribution along the height above and below ground water level, i.e.,

$$u_w = \psi_m = -\rho_w g z_0 \tag{11}$$

Substituting Equation (11) into the equilibrium differential equation of pore water, the soil-water interaction force in the unit volume of the soil body in saturated and unsaturated soil layers under static equilibrium conditions can be obtained. In saturated soil, the interaction force between the soil skeleton and pore water is equal to zero; while in unsaturated soils:

$$f_{sw0i} = n\rho_w g z_0 S_{0,i} \tag{12}$$

The components of soil-water interaction force in the *X* and *Z* axis are denoted with f_{sw0x} and f_{sw0z}. Equation (12) illustrates that the interaction force between the soil skeleton and pore water in the unit volume of soil mass is equal to the variation ratio of pore water weight of the water column at the unit area above ground water level when the pore water is in static equilibrium state. This ratio is introduced by the change of water content, which indicates the variation of the suction force on pore water by soils.

When the seepage of pore water appears in unsaturated soils, we still assume that the resistance force on pore water's motion is proportional to the velocity of pore water. In addition, the seepage resistance force on the pore water in saturated soils and that in unsaturated soils are the same, as far as the water in these two states has the same volume. Therefore, the seepage resistance force on the pore water in the unit volume of unsaturated soil is:

$$f_{swfi} = a_{wi} v_i \tag{13}$$

where, a_w is the seepage force factor of the unit volume of pore water (rather than the pore water in the unit volume of soil mass); v is the velocity of pore water and f_{swf} is the soil-water interaction force due to seepage.

The soil-water interaction force in the equilibrium equation of unsaturated soils includes the forces due to pore water's motion and the non-uniform distribution of pore water, which can be expressed as:

$$f_{swi} = f_{swfi} + f_{sw0i} \tag{14}$$

where, a_w is the soil-water interaction factor of the unit volume of pore water in unsaturated soils and $\check{f}_{sw0}$ represents the soil-water interaction due to the non-uniform distribution of pore water, of which the components were given in Equation (12). Consequently, the total soil-water interaction forces in *x* and *z* directions are:

$$f_{swi} = a_{wi} v_i + \rho_w g n z_0 S_{e,i} \tag{15}$$

3.2 *The formula of Darcy's law and the coefficient of permeability for unsaturated soils*

For the seepage in unsaturated soils, the motion equation is the formula of Darcy's law. Different from saturated soils, for unsaturated soils, the velocity of pore water is not proportional to the soil-water potential gradient. Therefore, the formula of Darcy's law for unsaturated soils is only a formal expression.

As previously mentioned, the total water head of the pore water in unsaturated soils consists of gravity potential Ψ_g and matrix potential Ψ_m. The soil-water potential expressed with the potential energy of the unit weight of pore water can be written as:

$$H = z + \psi_m \tag{16}$$

or

$$H = z + \frac{u_w}{\rho_w g} \tag{17}$$

where, *H* is the total head; z is the gravity head; u_w is the pore water pressure of unsaturated soils, which is the matric potential of the unit volume of pore water; ρ_w is the density of water and *g* is the gravitational acceleration.

Substituting the expression of soil-water interaction forces for unsaturated soils, Equation (15) into the equilibrium Equation (4), can obtain the following expressions after reorganization:

$$v_i = -k\left[(S_e H)_{,i} - (z - z_0) S_{e,i}\right] \tag{18}$$

where, $k_i = \dfrac{n\rho_w g}{a_{wi}}$ is supposed to be the permeability coefficient like that of saturated soils; S_e is the effective degree of saturation and *H* is the total head. The effective degree of saturation:

$$S_e = \frac{n_w}{n} = \frac{S - S_r}{1 - S_r} \tag{19}$$

Furthermore, Equation (18) can be written as:

$$v_i = -k_{ui} H_{,i} \tag{20}$$

where, k_u is the permeability coefficient function for unsaturated soils, and

$$k_u = k\left[S_e + (H - z + z_0)\frac{\partial S_e}{\partial H}\right] \tag{21}$$

where, *k* has the same expression and physical meaning as the permeability coefficient for saturated soils. That said, if the respective resistance force due to motion on the unit volume of pore water is the same for saturated and unsaturated soils, the factor *k* in Equation (20) is the same as the permeability coefficient for saturated soils. The relationship between the permeability coefficient for unsaturated soils and that for saturated soils is therefore established by Equation (20). Apparently, besides its correlation to the permeability coefficient for saturated soils, the permeability coefficient for unsaturated soils is also related to the water content and soil-water potential.

4 FORMULA OF SEEPAGE FORCE

In soil mechanics, the seepage force is generally defined as the interaction force on the soil skeleton by pore water in the unit volume of soil mass. Seepage force is a type of body force with the same dimension as the volume-weight of water and the application direction consistent with the seepage velocity. The expression of seepage force of saturated soil given in almost all the soil mechanics textbooks is:

$$\boldsymbol{J} = \gamma_w \boldsymbol{i} \tag{22}$$

where, $\boldsymbol{J}$ represents the seepage force; γ_w is the volume-weight of water and $\boldsymbol{i}$ is the soil-water potential gradient.

According to the deduction procedure of the equilibrium differential equation of the soil skeleton and pore water, the seepage force defined in Equation (22) is the same as the interaction force between the soil skeleton and pore water in the equilibrium equation of pore water (Equation 4). The interaction force between the soil skeleton and pore water in the unit volume of soil mass obtained via the equilibrium differential equation of pore water is:

$$\boldsymbol{f} = n\gamma_w \boldsymbol{i} \tag{23}$$

where, n is the porosity of soils; γ_w is the unit weight of water and $\boldsymbol{i}$ is the water potential gradient.

As $\boldsymbol{J}$ and $\boldsymbol{f}$ represents the same meaning, why is there a factor n in the expression of $\boldsymbol{f}$ but not in that of $\boldsymbol{J}$? This is due to the derivation procedure of $\boldsymbol{J}$ in soil mechanics textbooks, which implied that the definition of $\boldsymbol{J}$ is the seepage force on the unit volume of pore water, not on the pore water in the unit volume of soil mass.

In the textbooks of soil mechanics, the expression of seepage force is generally obtained by conducting equilibrium analysis on the free body of pore water in a vertical direction or diagonal direction. This can be illustrated by taking the relevant description in Soil Mechanics written by Chen et al. (1994) as an example. Figure 2 shows a vertical placed soil column in a stable seepage test. The forces of the free bodies of the soil skeleton and pore water are presented. When the force analysis was respectively conducted on free body of the soil skeleton and pore water that were taken separately, the seepage force is expressed as an external force, as show in Figure 2. The following is the original text that describes the force and equilibrium analysis procedure on the free body of pore water.

The forces applied on the free body of pore water in soil column (as presented in Figure 2(c)) include:

(1) The sum of the gravity of pore water and the buoyancy of soil particles. The later should be equal to the gravity of the water with the same volume as the soil particles. This can be expressed:

$$W_w = V_v\gamma_w + V_s\gamma_w = V\gamma_w = L\gamma_w \tag{B1}$$

It can be known that W_w is the weight of the water column with the length of L.

(2) The water pressure at the boundary of the top and bottom sides of water column, $\gamma_w h_w$ and $\gamma_w h_1$, and

(3) The resistance force on the water flow applied by the soil particles in soil column. It has the same amount but opposite direction as the seepage force. If the resistance force on the water flow applied by the soil particles in the unit of soil body is assumed to be j', the total resistance force is $J'=j'L=J$, of which the direction is vertically downward.

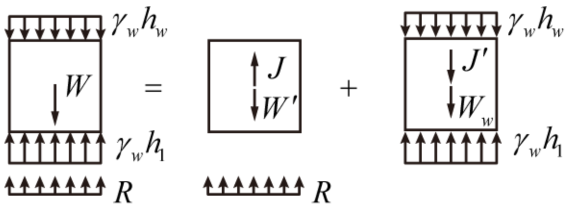

Figure 2. Two isolation approaches types of the free body in seepage.

Considering the equilibrium conditions of the free body of pore water (as shown in Figure 2(c)) can obtain:

$$\gamma_w h_w + W_w + J' = \gamma_w h_1 \tag{B2}$$

$$\gamma_w h_w + L\gamma_w + j'L = \gamma_w h_1 \tag{B3}$$

$$j' = \frac{\gamma_w (h_1 - h_w - L)}{L} = \frac{\gamma_w \Delta h}{L} = \gamma_w i \tag{B4}$$

Therefore, the seepage force is $j = j' = \gamma_w i$.

It can be found that the area of the free body of pore water is assumed to be a unit area in this force and equilibrium analysis.

If the seepage force is defined as the force on the soil skeleton applied by pore water within the unit of soil body, i.e., the force on the pore water by the soil skeleton in the unit of soil body (the unit volume of soil rather than the unit volume of pore water), the force analysis on the aforementioned free body of pore water in soil column should be:

(1) The gravity of pore water, which is the weight of the pore water in the soil column: $W_w = \gamma_w V_v = n\gamma_w V = n\gamma_w L$;

(2) The water pressure at the boundaries on the top and bottom sides of water column, $\gamma_w h_w$ and $\gamma_w h_1$

(when the area of the soil column is *1*, the area of pore water column is *n*);

(3) The resistance force on the flowing water applied by the soil skeleton in the soil column, which has the same amount but opposite direction as the seepage force. If the resistance force on the water by the skeleton in the unit of soil body is assumed to be j', the total resistance force is $J' = j'L = J$, of which the direction is vertically downward.

Considering the equilibrium conditions for the free body of water (as shown in Figure 2(c)), we have:

$$n\gamma_w h_w + W_w + J' = n\gamma_w h_1 \quad (24)$$

$$n\gamma_w h_w + nL\gamma_w + j'L = n\gamma_w h_1 \quad (25)$$

$$j' = \frac{n\gamma_w \left(h_1 - h_w - L\right)}{L} = \frac{n\gamma_w \Delta h}{L} = n\gamma_w i \quad (26)$$

Therefore, the seepage force is $\boldsymbol{j} = \boldsymbol{j}' = n\gamma_w i$, which is the same as the result derived by applying the equilibrium differential equation of pore water.

Certainly, if the seepage force is defined as the resistance force on a unit volume of pore water by the soil skeleton, the expression of seepage force is:

$$\boldsymbol{j} = \boldsymbol{j}' = \gamma_w i \quad (27)$$

where $\boldsymbol{j}$ is the seepage force on a unit volume of water. If $\boldsymbol{f}$ is the seepage force on the pore water in a unit volume of soil, then:

$$\boldsymbol{f} = n\,\boldsymbol{j} \quad (28)$$

This correlation can also be obtained according to the fact that the seepage forces on the same section of soil body are the same. Since $f \cdot A = j \cdot A_w$ and $A_w = nA$ (*A* is the sectional area), then $fA = jA_w = jnA$, i.e. $f = jn$.

Another question is: should the counterforce of the buoyancy of the soil skeleton be included in the gravity (force) of pore water? The answer is NO. The reasons are as follows:

In static equilibrium conditions, the self-weight of pore water induces the variation of pressure; the action of pressure results in the buoyancy of soil skeleton particles, whose value is equal to the weight of the water replaced by the soil skeleton particles. The counterforce to the buoyancy of soil particles is actually water pressure, which does not vary the weight of water nor the pressure of water. No matter how the soil particles interact with pore water, the pore water pressure on the surface of the free body of pore water stays constant. Consequently, when the pore water is designated as the analysis subject, its gravity should not include the counterforce of buoyancy force.

The seepage in unsaturated soils includes the seepage of pore water and pore air. Both the forces induced by the seepage of pore water and that of pore air are called the seepage force of unsaturated soils. Similar to saturated soil, the seepage force of the pore water in unsaturated soils is defined as the force on the soil skeleton by the pore water in a unit volume of the soil body; the counterpart of the pore air in unsaturated soils is defined as the force on the soil skeleton by the pore air in a unit volume of the soil body. They are equal to the seepage resistance on the pore water and pore air applied by the soil skeleton in a unit volume of the soil body, respectively. According to these definitions, we can use the equilibrium differential equations to derive the seepage force formula of unsaturated soils.

The interaction force between pore water and the soil skeleton is considered for example. Assuming the water content of soils is uniform, the equilibrium differential equation of the pore water in unsaturated soils is:

$$n_w u_{w,i} + f_{swi} + X_i = 0 \quad (29)$$

When the atmosphere is open, and the pore air pressure and solute potential can both be neglected, the expression of total water potential (total water head) is Equation (17). Consequently, the seepage force of the unsaturated soils with uniform water content can be obtained based on Equation (29), as:

$$\boldsymbol{f} = n_w \gamma_w \boldsymbol{i} \quad (30)$$

where, n_w is the porosity corresponding to the pore water, i.e., the volumetric water content.

It can be seen that, the expression of seepage force for unsaturated soils in the condition of uniform water content is the same as that for saturated soils, where only the n_w is the porosity corresponding to the pore water.

Now the question is briefly discussed herein whether the interaction force corresponding to the change of water content should be included in the seepage force. It can be understood through the designation that the seepage force is induced by the percolation of pore fluid in the soil skeleton. In the condition of laminar seepage, the seepage force is proportional to the seepage velocity (i.e., the velocity of the relative motion between the pore fluid and the soil skeleton). There is no seepage force while the relative motion between the pore fluid and the soil skeleton, i.e. the seepage, does not exist. However, it is known based on the equilibrium equation that there is an interaction force induced by the change of water content for unsaturated soils even in the static equilibrium condition, i.e., where seepage does not exist. Therefore, we suggest that the definition of seepage force should not include the interaction force corresponding to the change of water content. In this case, the interaction force would not be equal to the seepage force, which includes the interaction force induced by the change of water content.

5 CONCLUSIONS

The formula of Darcy's law for both saturated soils and unsaturated soils can be derived directly from the equilibrium equation of pore fluid. For saturated soils, the soil-water interaction force is only the seepage force; and there is no interaction force between the soil skeleton and pore water when seepage does not exist. For unsaturated soils, except for the forces induced by seepage, there is another interaction force between soil skeleton and pore water which is induced by the non-uniform distribution of pore water. If the seepage force is defined as the resistance force on a unit volume of pore water by the soil skeleton, which is exactly the same as that in the equilibrium equation of pore water, the formula of seepage force is $\boldsymbol{f} = n_w \gamma_w \boldsymbol{i}$, rather than $\boldsymbol{f} = \gamma_w \boldsymbol{i}$ as in the existing textbook of Soil Mechanics. However, if the seepage force is defined as the resistance force on a unit volume of pore water, the expression of seepage force is $\boldsymbol{f} = \gamma_w \boldsymbol{i}$.

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