Ultimate Load Foundation Design Using Statistically Based Factors

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SUMMARY. The results of pile load tests are presented from various sites in stiff fissured clays, with a statistical model of soil response to foundation load. The significance of some deviations in observed pile performance from conditions commonly assumed in design is tested by means of this model. It is shown that the model allows the evaluation of a material response factor for ultimate load design of foundations, and a design example is given.

1 INTRODUCTION

The uncertainties in the bearing capacity performance of foundations lie in the loads which they must carry, the strength response of the founding stratum and the adequacy of the design modelling and analysis.

For foundation loads, the uncertainties in extreme values are the same as those for the design of the supported structure.

There will be uncertainty with respect to the mean value and variability of the strength of a founding stratum, arising from limited sampling in a variable material. There will also be uncertainty with respect to the magnitude of any possible bias between the strength response of the soil at the test specimen and prototype scales. In clays, for example, such bias is often attributed to sampling disturbance or the presence of fissures or other macro-structure.

Estimates of the bearing capacity response of a soil stratum are generally based on an approximate analysis of the performance of a simplified model (eg. a homogeneous, isotropic, semi-infinite medium) and the results of these estimates may be biased with respect to the actual mean response of the stratum.

The uncertainties in bearing capacity are conventionally allowed for in foundation design by dividing the estimated nett ultimate soil resistance by a safety factor to give the maximum safe soil resistance, and hence the maximum safe working load. This procedure lumps all formal uncertainty allowance into the loading. A more rational approach would be provided by ultimate load design, in which uncertainty in loads is allowed for by a load factor and uncertainty in strength properties and due to bias is allowed for by a material response factor.

2 NOTATION

The following are the main symbols used.

- c_a ultimate shaft adhesion
- c_u undrained cohesion
- D base diameter of pile
- d shaft diameter of pile
- L length of pile
- N bearing capacity factor
- n,p number of test results
- Q load
- R soil resistance force
- s^2 variance of log property values from test results
- t Student's t value
- x log value of soil response
- y value of soil response
- α bias factor
- β material response factor
- γ load factor
- ν degrees of freedom

Subscript notation is defined as it is used in the text.

3 THE STATISTICAL MODEL

The Author has described a statistical model for the undrained strength behaviour of soils which allows the evaluation of a material response factor on the basis of the probability of satisfactory soil behaviour (McAnally, 1977). An outline of the principles of this model are given here.

- The soil zones influenced by individual foundations (referred to as significant units of influence, or sui's) are considered to be composed of a number of smaller units, within each of which strength can be considered to be homogeneous (referred to as equivalent homogeneous units, or ehu's).
- The strength properties of the ehu's are considered to have a log-normal distribution and to vary randomly spatially.
- The strength response of an sui is considered to be related to the strengths of the ehu's within it by

$$y_s = \alpha\, \bar{y}_e \qquad (1)$$

where y_s = sui strength response

$\bar{y}_e$ = geometric mean of the strength values of the ehu's in the sui

Conventional strength tests (eg. unconfined or triaxial compression tests) may be considered to be samples from the ehu population, and observations of the bearing capacity response of prototype foundations may be considered as samples from the sui population.

If a set of n_1 conventional strength test results, (log property values having mean $\bar{x}_{e1}$ and variance s_{e1}^2)+ and a set of p_1 prototype test results, (log property values having mean $\bar{x}_{s1}$ and variance s_{s1}^2) are available, then confident limits for prototype response, y_{s1}, can be found.

$$\alpha\bar{y}_{e1}\exp(-t\emptyset) < y_{s1} < \alpha\bar{y}_{e1}\exp(t\emptyset) \qquad (2)$$

where $\emptyset^2 = s_{s1}^2 + s_{e1}^2/n_1$

$\bar{y}_{e1}$ = geometric mean of the conventional (ehu) test results

t = Student's t value for the confidence coefficient chosen and degrees of freedom, $\nu_\emptyset$, given by

+ The variances are computed as the unbiased estimates of the population variances, ie.
$s^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1}$

$$\frac{\emptyset^4}{\nu_\emptyset} = \frac{(s^2_{s1})^2}{p_1 - 1} + \frac{(s^2_{e1}/n_1)^2}{n_1 - 1} \qquad (3)$$

If a set of n_2 conventional strength test results (log property values having mean $\bar{x}_{e2}$ and variance s_{e2}^2) and a set of p_2 prototype response results (log property values having mean $\bar{x}_{s2}$ and variance s_{s2}^2) are available, then confidence limits for α can be found.

$$a_2\exp(-t\psi) < \alpha < a_2\exp(t\psi) \qquad (4)$$

where $\psi^2 = s_{s2}^2/p_2 + s_{e2}^2/n_2$

$\bar{y}_{s2}$ = geometric mean of prototype response (sui) results

$a_2 = \bar{y}_{s2}/\bar{y}_{e2}$

t = Student's t value for the confidence coefficient chosen and degrees of freedom, ν_ψ, given by

$$\frac{\psi^4}{\nu_\psi} + \frac{(s_{s2}^2/p_2)^2}{p_2 - 1} + \frac{(s_{e2}^2/n_2)^2}{n_2 - 1} \qquad (5)$$

If the bias factor, α, is the same for both pairs of samples, then(2) and (4) may be combined to give confidence limits for y_{s1}, independant of α.

$$a_2\bar{y}_{e1}\exp(-t\omega) < y_{s1} < a_2\bar{y}_{e1}\exp(t\omega) \qquad (6)$$

where $\omega^2 = \phi^2 + \psi^2$

t = Student's t value for the confidence coefficient chosen and degrees of freedom, ν_ω, given by

$$\frac{\omega^4}{\nu_\omega} = \frac{\phi^4}{\nu_\phi} + \frac{\psi^4}{\nu_\psi} \qquad (7)$$

If the geometric mean of the results of a set of conventional strength measurements, $\bar{y}_{e1}$, is chosen as the shear strength design parameter, then a material response factor, β, can be estimated from (6) to cover uncertainties in soil properties and bias in prototype response.

$$\beta = a_2\exp(-t\omega) \qquad (8)$$

Comparison of design on this basis with conventional design on the basis of a safety factor indicates that a confidence coefficient of 0.99 on the lower confidence limit will give a similar probability of failure as a safety factor of 3. (The probability of failure will be much less than 0.01, as both the material response factor and the load factor must be concurrently exhausted for failure to occur).

4 PILE LOAD TEST RESULTS

The results of load tests on a number of cast-in-situ piers and piles in fissured clays have been collected to illustrate the use of this model and provide data for design. The results of site investigation work on these sites are summarised in Table 1. Sites A to F were located in south-eastern Queensland. Sites G and H are the results of deep plate load tests and cast-in-situ pier performance in London Clays reported by Marsland (1971) and Whitaker and Cooke (1965) respectively, which have been included for comparison purposes.

Values of ultimate shaft resistance, R_{u_s}, and nett ultimate base resistance, R_{nu_b}, were determined from the load-deflection plots for the piles from sites A to F, by the method outlined by Whitaker (1970). An example of this determination is shown in Fig. 1.

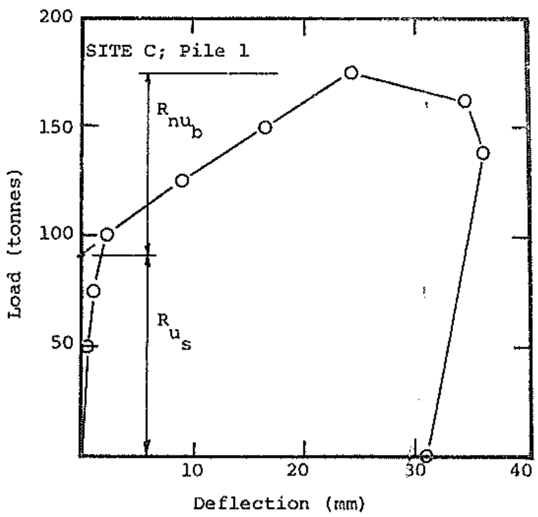


Fig. 1. Typical Pile Load-Deflection Plot Showing Estimation of R_{u_s} and R_{nu_b}

The soil response values for ultimate shaft resistance and nett ultimate base resistance were estimated, for sites A to F, from the common design expressions

$$c_a = R_{u_s}/(\pi dL) \qquad (9)$$

$$c_u = R_{nu_b}/(\frac{9\pi D^2}{4}) \qquad (10)$$

Details of the pile dimensions, and the estimated values of soil response are given in Table 2, for sites A to F. The estimated values of soil res-

TABLE 1

SUMMARY OF SITE INVESTIGATION RESULTS						
SITE	SOIL DESCRIPTION	CLASSIFICATION			STRENGTH	
		w %	L.L. %	P.I. %	Type of Test	Shear Strength (kPa)
A	Stiff mottled grey & brown fissured CLAY (Shaft Zone)	29 - 61	-	-	U(48)	34 - 99+
	Stiff mottled grey & brown fissured CLAY (Base Zone)	43 - 57	-	-	U(48)	46 - 125
B	Stiff to very stiff mottled red, grey & brown fissured CLAY	28 - 49	88 - 91	47 - 48	UC(48)	29 - 99
C	Stiff to very stiff mottled red grey & brown fissured CLAY (Shaft Zone) Stiff to very stiff mottled red grey & brown fissured CLAY (Base Zone)	21 - 34	80 - 96	59 - 72	UC(48)	35 - 265
					UC(48)	45 - 195
D	Stiff mottled grey & brown fissured CLAY	32 - 43	-	-	UC(48)	25 - 103
E	Stiff mottled grey & brown fissured CLAY	15 - 31	-	-	U (48)	34 - 105
F	Very stiff to hard grey brown fissured CLAY	14 - 27	42 - 80	26 - 63	U(48)	150 - 335
G	Very stiff grey fissured CLAY	-	-	-	U(38)	104 - 240
H	Stiff to very stiff grey to brown fissured CLAY (Shaft Zone)	-	-	-	U(38)	58 - 149
	Stiff to very stiff grey fissured CLAY (Base Zone)	-	-	-	U(38)	75 - 150

* U( ) denotes undrained triaxial test on ( ) mm dia. specimen

UC( ) denotes unconfined compression test on ( ) mm dia. specimen

\+ Residual strengths in sensitive clay

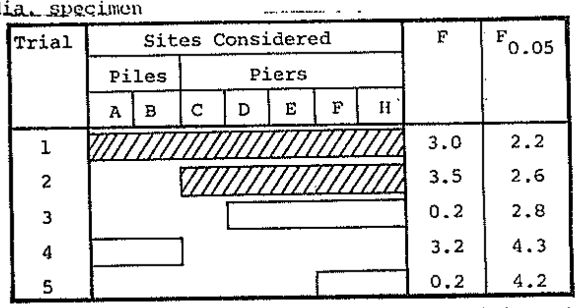
ponse, determined by the methods of 9 and 10, were taken directly from the references for sites G and H.

The statistics for the ehu and sui sample sets for each site are given in Table 3.

5 ANALYSIS OF VARIANCE

Valid use of data from more than one site, in (6), depends upon the condition that the same bias factor applies for all the sites from which the data is drawn. An analysis of variance expression may be developed, as described in Appendix I, to test this hypothesis.

Analysis of variance was made on the ehu and sui statistics from various combinations of sites. The results of this analysis is given in Fig. 2 for shaft adhesion, and in Fig. 3 for nett end bearing resistance.



Shaded trials indicate estimated probability of less than 0.05 that α is the same for all sites considered.

Fig. 2. Results of Analysis of Variance for Shaft Adhesion Bias.

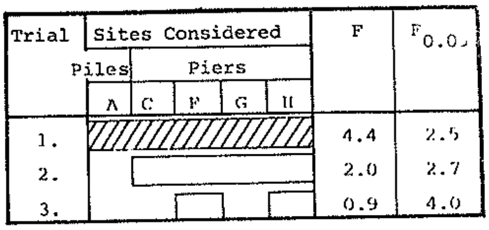


Fig. 3. Results of Analysis of Variance for Nett Base Resistance Bias.

TABLE 2

DETAILS OF PILES AND LOAD TEST RESULTS

Site	Pile No.	Length (m)	Shaft Diam. (m)	Base Diam.* (m)	Soil Response c_a (kPa)	c_u (kPa)
Cast-in-situ Piles (Hammered base and shaft)						
A	1	9.0	0.55	0.75	41.5	108.6
	2	9.0	0.40	0.75	49.4	108.3
	3	7.3	0.40	0.75	53.4	147.9
	4	9.0	0.40	0.86	52.9	110.6
	5	7.0	0.40	0.75	40.0	160.2
B	1	10.0	0.50	0.64	62.4	-
	2	8.7	0.50	0.64	71.7	-
	3	7.0	0.40	0.75	85.3	193.4
Cast-in-situ Piers						
C	1	4.0	0.75	1.0	89.3	123.4
	2	4.0	0.75	1.0	112.3	92.9
D	1	9.0	0.50	0.50	46.1	-
	2	9.0	0.50	0.50	41.6	-
E	1	10.0	0.60	0.60	39.0	115.5
	2	8.4	0.40	0.47	65.0	-
F	1	5.0	0.64	0.64	66.7	114.0
	2	5.0	0.64	0.64	65.7	152.0
	3	5.0	0.64	0.64	60.4	-

* Base diameter of cast-in-situ piles estimated from volume of concrete displaced and assumed spherical shape.

\- Indicates ultimate end bearing resistance not fully developed.

In the design of cast-in-situ piers or piles, the design value of c_a is generally obtained by multiplying the average measured value of undrained shear strength by a factor (less than or equal to one) according to the magnitude of that average value (SAA Piling Code - 1978). The values of $\bar{y}_e$ (=c_a) for shaft adhesion were obtained by reducing the mean of the measured values of c_u by the recommended factors from the SAA Piling Code (Fig. 4). Hence, the bias factor, α, for shaft adhesion will represent bias with respect to the design value of c_a which would be chosen on the basis of this code recommendation. The values of $\bar{y}_e$ (= c_u) for end bearing have been obtained from the actual measured values of undrained cohesion, and hence the bias factor, α, for base resistance will represent bias with respect to the bearing capacity factor of 9 from (10).

A number of observations may be made, in passing, on the results of these analyses.

- The ultimate shaft adhesion developed on bored piers can be significantly higher than the design values which would be chosen from the SAA Piling Code (Fig. 2; Tials 2 and 3, and Fig. 4, with respect to Site C). This is of particular significance in the design of piers in expansive soil, where the development of high values of shaft adhesion can result in high tension stresses in the pier shafts.

- The ultimate shaft adhesion developed on cast-in-situ piles, in which the shaft concrete is compacted by hammering, can be significantly greater than that predicted by the SAA Piling Code and developed on bored pier shafts in many instances (Fig. 2; Trials 1 and 3, and Fig. 4: Site B).

- The estimated nett ultimate bearing pressure for enlarged base cast-in-situ piles, for which the bases have been formed by hammering the concrete to displace the soil, can be significantly higher than the nett ultimate bearing pressure for bored piers (Fig. 3; Trials 1 and 2, and Fig. 4: Site A).

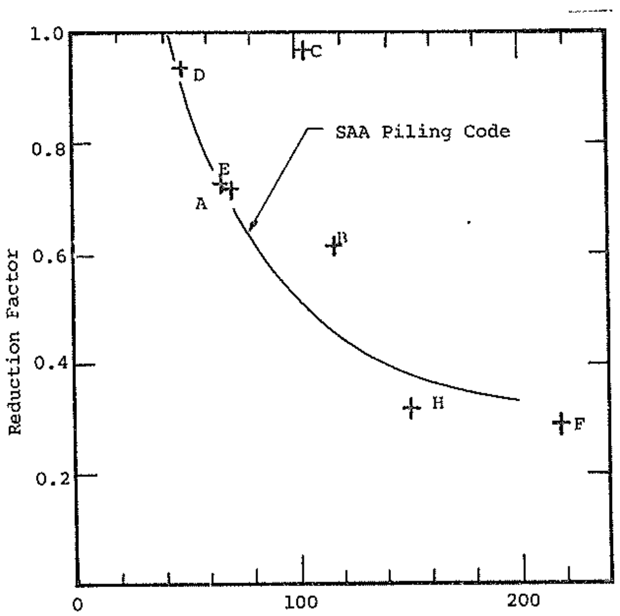


Fig. 4. Adhesion Reduction Factor for Piles in Clays

TABLE 3

STATISTICS FROM PILE LOAD TEST SITES

Site	Investigation (ehu) Statistics n	$\bar{y}_e$ (kPa)	s_e	Load Test (sui) Statistics p	$\bar{y}_s$ (kPa)	s_s
Shaft Adhesion						
A	13	65.7+	0.368+	5	47.1	0.136
B	17	117.9	0.341	3	72.5	0.157
C	21	104.1	0.549	2	100.1	0.162
D	5	47.2	0.605	2	43.8	0.073
E	18	70.9	0.339	2	50.3	0.361
F	7	219.1	0.307	3	64.2	0.053
H	18	148.2	0.278	10	47.7	0.202
Base Resistance						
A	13	95.3	0.224	5	125.3	0.191
C	6	99.2	0.621	2	107.1	0.201
G	18	148.3	0.290	6	98.6	0.063
H	18	138.7	0.356	10	110.4	0.186
F	7	219.1	0.307	2	131.6	0.203

+ Residual strengths in sensitive clay

6 EXAMPLE OF ULTIMATE LOAD DESIGN

The form of the ultimate load design equation for cast-in-situ piers or piles in clays will be

$$\gamma Q + Q_F = N_a \beta_a y_{da} A_a + (N_b \beta_b y_{db} + p_o) A_b \qquad (11)$$

where Q = working load
γ = load factor
Q_F = gravity load of the foundation
N_a = adhesion factor ($c_a/c_{u_{av}}$ from SAA Piling Code in this instance)
β_a = material response factor for shaft adhesion
y_{da} = design value of shear strength of soil around shaft ($\bar{y}_e$: geometric mean of shear strength measurements in this instance)
A_a = area of soil contact on shaft
N_b = bearing capacity factor (9 in this instance)
β_b = material response factor for end bearing capacity
y_{db} = design value of shear strength for soil at base ($\bar{y}_e$: geometric mean of shear strength measurements in this instance)
A_b = area of base

A method for the estimation of material response factors for shaft adhesion and end bearing, where the two act concurrently, is developed in Appendix II.

Suppose that the following shear strength measurements have been obtained by triaxial testing from a Site, J, in the same soil type as Site F.

Shaft Zone: 220 kPa, 110 kPa, 145 kPa, 180 kPa, 130 kPa
Base Zone: 135 kPa, 155 kPa, 230 kPa, 190 kPa

This data, together with the sui data from Site F, can be used to estimate the effect of soil variability on foundation performance. If the ehu and sui data from Site H is available, this can be used to estimate the effect of bias on foundation performance. (The probability that the bias factors for these two soils are different has been shown to be very low). The statistics for these tests are summarised in Table 4.

TABLE 4

SUMMARY OF STATISTICS FOR EXAMPLE

SITE INPUT	Investigation (ehu) Statistics			Load Test (sui) Statistics		
	n	$\bar{y}_e$ (kPa)	s_e	p	$\bar{y}_s$ (kPa)	s_s
Shaft Adhesion						
F & J Variability	5	152.4	0.272	3	64.2	0.053
H Bias	18	148.2	0.278	10	47.7	0.202
Base Resistance						
F & J Variability	4	173.9	0.233	2	131.6	0.203
H Bias	18	138.7	0.356	10	110.4	0.186

This data will be used to estimate the acceptable working load on a 500mm diameter bored pier, with 6m shaft length and base enlargement to 850mm diameter (founding depth 6.3m), for a load factor of 1.6 and a material response factor calculated for a lower confidence limit with confidence coefficient of 0.99. The average bulk density of the soil is 2.0 t/m^3.

$A_a = 9.42\ m^2 \qquad A_b = 0.567\ m^2$

$N_a = 0.38$ for $y_{da} = 152.4$ kPa (SAA Piling Code)

$N_b = 9$

$$a_{2a} = y_{s2a}/y_{e2a} = 47.7/(0.38 \times 148.2) = 0.847 \qquad \text{(Eqn. II.2)}$$

$$a_{2b} = y_{s2b}/y_{e2b} = 110.4/138.7 = 0.796 \qquad \text{(Eqn. II.2)}$$

$$\omega_a^2 + \omega_b^2 = 0.053^2 + 0.272^2/5 + 0.202^2/10 + 0.278^2/18 + 0.203^2 + 0.233^2/4 + 0.186^2/10 + 0.356^2/18 = 0.0914 \qquad \text{(Eqn. II.2)}$$

$$\frac{(\omega_a^2 + \omega_b^2)^2}{\nu} = \frac{(0.053^2)^2}{2} + \frac{(0.272^2/5)^2}{4} + \frac{(0.202^2/10)^2}{9} + \frac{(0.278^2/18)^2}{17} + \frac{(0.203^2)^2}{1} + \frac{(0.233^2/4)^2}{3} + \frac{(0.186^2/10)^2}{9} + \frac{(0.356^2/18)^2}{17}$$

giving $\nu = 4.6$ (Eqn. II.3)

From tables of Student's t for P = 0.01; t = 3.52

$$\beta = 0.847 \times 0.796 \exp(-3.52\sqrt{0.0914}) = 0.233 \qquad \text{(Eqn. II.4)}$$

$$\beta_a = \sqrt{\frac{9 \times 173.9 \times 0.567}{0.38 \times 152.4 \times 9.42}} \times 0.233 = 0.616 \qquad \text{(Eqn. II.7}$$

$$\beta_b = \beta/\beta_a = 0.233/0.62 = 0.378 \qquad \text{(Eqn. II.7)}$$

$$Q_F = 9.8 \times 2.4 \times \frac{\pi}{4} \times 0.5^2 \times 6.3 = 29.1 \text{ kN}$$

$$1.6Q + 29.1 = (0.38 \times 0.616 \times 152.4 \times 9.42) + (9 \times 0.378 \times 173.9 + 9.8 \times 2.0 \times 6.3)\ 0.567 \qquad \text{(Eqn. 11)}$$

giving $Q = 445$ kN

7 CONCLUSIONS

Foundation design is carried out using limited information and inexact methods. A bias will often exist between the predicted and the actual foundation performance. The designer must steer a course between undue and costly conservatism and an unacceptably high probability of failure.

Conventional design requires judgement to be exercised in the sensitive and subjective area of choice of design strength and safety factors, to select this course. In the proposed method, the designer's judgement decisions are removed from this area to the less sensitive and more objective decision on the compatibility of his concept of the soil properties on the site with his experience of similar soils. In addition, he is able to make quantitative use of data from previous experience in his judgement decisions and design.

The collection, correlation and dissemination of data from engineering experience (eg. the results of investigation and prototype tests) is of advantage to the profession, whatever design method is used. The prototype tests which yield information on the ultimate soil resistance are often those which have failed to meet the designer's expectations, and there is an understandable reluctance to publish this information. However, the value of such information lies in the comparison of actual and predicted performance, and provided that this is presented in a manner similar to the sites in this paper, sensitive information, such as the actual site location and the design loads, would not be relevant. Therefore, it would seem both feasible and advantageous to the profession to have such information processed by reliable, independant bodies on this basis.

8 ACKNOWLEDGEMENTS

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10 APPENDIX I - ANALYSIS OF VARIANCE FORMULAE

Equation 4 is derived from its logarithmic form

$$T = \frac{\ln.a - \ln.\alpha}{\psi} \qquad \text{I.1}$$

where ln.a is a normal variable

T is an approximate Student's t variable with degrees of freedom, given by (5).

Equation I.1 is of the form

$$T = \frac{\ln.a - \ln.\alpha}{s/\sqrt{\nu + 1}} \qquad \text{I.2}$$

where s = the unbiased estimate of the variance, σ^2, of ln.a

$\nu s^2/\sigma^2$ will be χ^2 variable with ν degrees of freedom, and hence $\nu(\nu + 1)\ \psi^2/\sigma^2$ will be an approximate χ^2 variable with ν degrees of freedom. Thus, the sum of k such variables from independant samples.

$\Sigma_1^k\ \nu_j(\nu_j + 1)\ \psi_j^2/\sigma^2$ will be a χ^2 variable with $\Sigma_1^k\ \nu_j$ degrees of freedom.

Also, $\sqrt{\nu_j + 1}\ (\ln.a_j - \ln.a)/\sigma$ will be a standard normal variable, where $\ln.a = \Sigma_1^k\ (\nu_j + 1)\ln.a_j / \Sigma_1^k(\nu_j + 1)$.

Therefore, $\Sigma_1^k\ (\nu_j + 1)\ (\ln.a_j - \ln.a)^2/\sigma^2$

$$= \{\Sigma_1^k\ (\nu_j + 1)(\ln.a_j)^2 - (\ln.a)^2\ \Sigma_1^k\ (\nu_j + 1)\}/\sigma^2 .$$

will be a χ^2 variable with (k-1) degrees of freedom if α is the same for all the populations from which the k sample sets have been selected.

Under the hypothesis $H_o : \alpha_1 = \alpha_2 = \ldots$ the ratio of the two χ^2 variables, divided by their respective degrees of freedom

$$F = \frac{\Sigma_1^k\ \nu_j\{\Sigma_1^k\ (\nu_j+1)(\ln.a_j)^2 - (\ln.a)^2\Sigma_1^k\ (\nu_j + 1)\}}{(k - 1)\Sigma_1^k\ \nu_j(\nu_j + 1)\ \psi_j^2}$$

will have an f distribution and provides a form of the conventional one way analysis of variance expression.

11 APPENDIX II - MATERIAL RESPONSE FACTORS FOR JOINT ACTION OF BASE AND SHAFT RESISTANCE

The ultimate soil resistance of a deep foundation will be given by

$$R_u = N_a y_{sa} A_a + (N_b y_{sb} + p_o) A_b \qquad \text{II.1}$$

It can be shown from the statistical model that

$$a_{2a} a_{2b} \exp\left(-t\sqrt{\omega_a^2 + \omega_b^2}\right) < y_{sa} y_{sb}/\bar{y}_{ela}\bar{y}_{elb} \ldots \qquad \text{II.2}$$

where a_{2a}, a_{2b} are values of $\bar{y}_{s2}/\bar{y}_{e2}$ from (6) for shaft and base respectively

ω_a^2, ω_b^2 are values of ω from (6) for shaft and base respectively

y_{sa}, y_{sb} are values of sui response for shaft and base respectively

$\bar{y}_{ela}$, $\bar{y}_{elb}$ are values of $\bar{y}_{el}$ for shaft and base respectively

t is the Student's t value for the confidence level chosen and degrees of freedom, ν, given by

$$\frac{(\omega_a^2 + \omega_b^2)^2}{\nu} = \frac{\omega_a^4}{\nu_a} + \frac{\omega_b^4}{\nu_b}, \qquad \text{II.3}$$

If $\beta = a_{2a}\ a_{2b} \exp\left(-t\sqrt{\omega_a^2 + \omega_b^2}\right) = y_{sa} y_{sb}/\bar{y}_{ela}\bar{y}_{elb}$ II.4

Also, the values of y_{sa} and y_{sb} giving the minimum value of R_u can be found from

$$\partial R_u/\partial y_{sa} = 0 \qquad \partial R_u/\partial y_{sb} = 0 \qquad \text{II.5}$$

The values of y_{sa} and y_{sb} giving the minimum value of R_u at the chosen lower confidence limit can be found from II.1, II.4 and II.5

$$y_{sa}^2 = \frac{N_b A_b}{N_a A_a}\ \bar{y}_{ela}\bar{y}_{elb}\beta \qquad y_{sb}^2 = \frac{N_a A_a}{N_b A_b}\ \bar{y}_{ela}\bar{y}_{elb}\beta \qquad \text{II.6}$$

If the material response factors for shaft and base response are defined in terms of the values of y_{sa} and y_{sb} from 6 as $y_{sa} = \beta_a\bar{y}_{ela}$ and $y_{sb} = \beta_b\bar{y}_{elb}$ respectively, then from II.6 where $\beta = \beta_a\beta_b$

$$\beta_a = \sqrt{\frac{N_b\bar{y}_{elb}A_b\ \beta}{N_a y_{ela} A_a}} \qquad \beta_b = \sqrt{\frac{N_a\bar{y}_{ela}A_a\ \beta}{N_b y_{elb} A_b}} \qquad \text{II.7}$$