Non-Linear Consolidation and the Effect of Layer Depth

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SUMMARY.- This paper generalises the previous theory published by Davis and Raymond which only dealt with a relatively thin layer and showed that, although the rate of pore pressure dissipation is slower than that of the classical Terzaghi linear theory, the degree of settlement is the same. This theory and its generalisation are based on the assumption that the coefficient of consolidation is constant but that the coefficients of volume decrease and of permeability are not.

The generalisation of the original theory takes into account the effects of non-linear behaviour on the rate of consolidation of deep layers. For such situations it is shown that the rate of pore pressure dissipation may be faster or slower than that of the classical Terzaghi linear theory depending on the ratio of the initial to the final effective stress at the top of the layer and on the ratio of the initial effective stress at the top to that at the bottom of the layer. The rate of settlement on the other hand is virtually always faster than that given by the classical theory and in the case of very deep layers subject to small consolidation pressuresthe increase in rate may have practical significance.

I.- INTRODUCTION

In the classical consolidation theory of Terzaghi it is assumed that the coefficient of volume decrease m_v and the coefficient of permeability k are individually constant both with time and throughout the depth of the consolidating layer, that is the theory is a linear one. For real soils both m_v and k decrease with decrease in void ratio so that in problems where the consolidation pressure is high enough to eventually cause a significant change in void ratio, or where the soil layer is deep enough for the void ratio at the top to be significantly higher than at the bottom, it may be necessary to take the non-linear behaviour of the soil into account.

The effect of variation of m_v and k with depth but not with time has been studied by Schiffman and Gibson (Ref.1). Davis and Raymond (Ref.2) on the other hand developed a theory which allowed for the variation of m_v and k with time but not with depth and was therefore only applicable to oedometer tests and to relatively thin layers in the field. A similar approach was developed independently by Mikasa (Ref.3).

Non-linear effects are likely to be most often of practical importance in considering the settlements produced by sand reclamation filling on deep beds of soft normally consolidated clay. For such problems the depth of the clay is likely to be a significant factor in the divergence of the consolidation behaviour from that of the classical linear theory.

It is therefore the purpose of this paper to extend the theory given in Ref.2 to include the depth effect. The earlier theory will then appear as a special limiting case of the more general theory.

II.- FORMULATION OF THEORY

The analysis is restricted to vertical one-dimensional strain conditions with pore pressure dissipation occurring only in the vertical direction. It is assumed that the relation between void ratio e and vertical effective pressure p_e follows the usually accepted logarithmic relation:

$$e = e_1 - C_c \text{ Log } p_e \qquad (1)$$

where C_c is the Compression Index, a constant,
and e_1 is the void ratio corresponding to $p_e = 1$.

It follows that:

$$m_v = -(de/dp_e)/(1+e) = 0.434C_c/(1+e)p_e \qquad (2)$$

Since (1+e) varies, both throughout the soil mass and with time, much less than p_e, m_v may be taken to be inversely proportioned to p_e, and the resulting mathematical analysis is rendered much easier. Thus it is assumed that:

$$m_v = A/p_e \qquad (3)$$

where $A = 0.434C_c/(1+e)$ = constant.

Typical tests results showing the reasonable nature of this assumption are given in Fig.1.

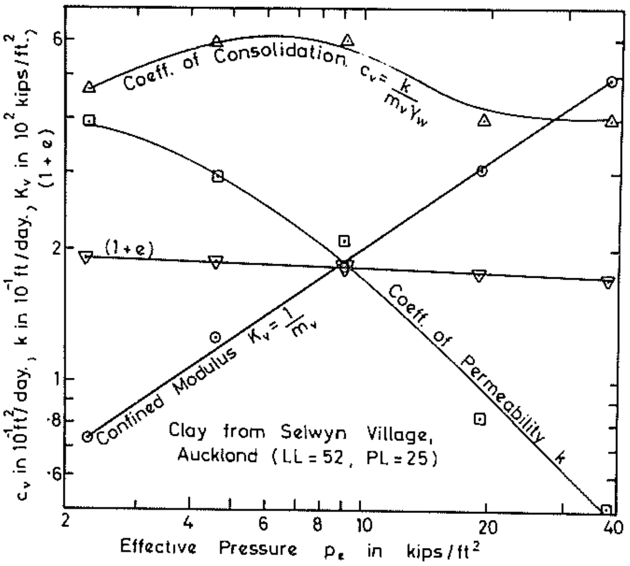


Fig.1 Typical Oedometer Results on Normally ConsoL Clay

It is found with most normally consolidated clays that the coefficient of consolidation ($c_v = k/m_v\gamma_w$ where k is the coefficient of permeability) varies much less with p_e than either m_v or k. As the void ratio decreases, the consequent decrease in the coefficient of permeability is approximately proportional to the decrease in the coefficient of volume decrease. For the purposes of the present theory it is assumed that c_v is actually constant. The reasonable nature of this assumption is also illustrated in Fig.1. It therefore follows that:

$$k = c_v\gamma_w m_v = c_v\gamma_w A/p_e \qquad (4)$$

Assuming that the soil is saturated, that the pore water and soil particles are incompressible relative to the soil skeleton and that the movement of water obeys Darcy's Law, the rate of water loss from an elemental layer may be equated to the rate of volume decrease:

$$\partial(k\partial u/\partial z)/\partial z = c_v\gamma_w(\partial e/\partial T)/(1+e) \qquad (5)$$

where u = excess pore pressure,
zD = depth below top of consolidating stratum of total depth D,
$T = c_v t/D^2$,
and t = time from start of consolidation.

If T_v is the Time Factor as normally defined in one-dimensional consolidation theory, T_v = T for one way drainage and T_v = 4T for two way drainage.

Using eqs.(1) and (4) eq.(5) becomes:

$$-\partial p_e/\partial T = \partial^2u/\partial z^2 - (\partial p_e/\partial z)(\partial u/\partial z)/p_e \qquad (6)$$

If the stress causing consolidation remains constant during the consolidation process, eq.(6) can be written as:

$$\partial u/\partial T = \partial^2u/\partial z^2 - (\partial p_e/\partial z)(\partial u/\partial z)/p_e \qquad (7)$$

Without the second term on the right-hand-side, eq.(7) reduces to the usual equation of Terzaghi's linear theory of consolidation in which m_v and k are both constant. This term is therefore a modifying term which introduces the effects of non-linear behaviour of soil into the theory.

In passing, it can be remarked that, if (1+e) in eq.(2) is not regarded as constant, the equation corresponding to eq.(7) is:

$$\partial u/\partial T = \partial^2u/\partial z^2 - (1-A)(\partial p_e/\partial z)(\partial u/\partial z)/p_e \qquad (8)$$

The only difference between this equation and eq.(7) is the factor (1-A). Although this factor changes during the consolidation process, for the majority of clays and situations it is greater than about 0.90 and is always less than one. In view of the fact that the second term on the right-hand-side of eq.(7)& (8) is only a modifying term, it can be seen that the use of the simpler eq.(7) should not introduce serious errors.

In order to facilitate comparison between the linear and non-linear theories, the boundary conditions selected in this analysis correspond to those of the Terzaghi linear theory.

Throughout, it is assumed that the total stress causing consolidation remains constant during the consolidation process. The conditions and their code letters are set out in Table 1.

Thus for conditions A_1 and A_2:

$$p_e = p_{eo} + \gamma_s Dz + q - u \qquad (9)$$

and for conditions B_1 and B_2

$$p_e = p_{eo} + \gamma_s Dz + q(1-z) - u \qquad (10)$$

where γ_s = the average submerged density of the soil.

Introducing the dimensionless parameters:

$$\beta = p_{eo}/\gamma_s D \qquad (11)$$

$$\alpha = (p_{eo}+q)/p_{eo} \qquad (12)$$

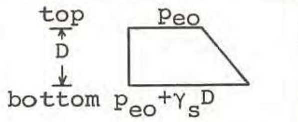
and changing the definition of u so that the actual excess pore pressure is uq, substitution of eq.(9) or (10) into the general eq. (7) gives:

$$\partial u/\partial T = \partial^2u/\partial z^2 + (\partial u/\partial z)(b\partial u/\partial z-a)/(1+az-bu) \qquad (13)$$

where $b = (\alpha-1)/\alpha$
and $a = 1/\alpha\beta$ for conditions A_1 and A_2
or $a = (1+\beta-\alpha\beta)/\alpha\beta$ for conditions B_1 and B_2.

TABLE 1

BOUNDARY CONDITIONS

	Condition	Code
Original effective stresses	 top P_{eo} D bottom $p_{eo}+\gamma_s D$	$A_1, A_2,$ B_1 & B_2
Final effective stresses	Original plus  q q	A_1 & A_2
	Original plus  q	B_1 & B_2
Drainage at boundaries	permeable $D=H \quad T_v = \frac{c_v t}{H^2} = T$ impermeable	A_1 & B_1
	permeable $D=2H \quad T_v = \frac{c_v t}{H^2} = 4T$ permeable	A_2 & B_2

By making the substitution $-w = \ln(1+az-bu)$, a form of eq. (13) which has some computational advantages may be obtained:

$$\partial w/\partial T = \partial^2 w/\partial z^2 + a\,\exp.(w)\partial w/\partial z \qquad (14)$$

From eq. (1) it can be shown that:

$$\text{Log}\alpha = (e_{ot}-e_{ft})/C_c \qquad (15)$$

$$\text{and} \quad \text{Log}((\beta+1)/\beta) = (e_{ot}-e_{ob})/C_c \qquad (16)$$

where the suffixes o,f,t and b to the void ratios e indicate original, final, top and bottom respectively.

In determiining the values of α and β appropriate to particular field problems it is probably more reliable to substitute the directly measured values of e_{ot} and e_{ob} and the calculated value of e_{ft} into eqs. (15) and (16) than to use eqs. (11) and (12).

III.- LIMITING CASE I

For shallow strata and for the oedometer test, the original effective stress at the bottom is negligibly higher than that at the top so that the parameter $\beta\to\infty$. This is referred to as limiting case I. For conditions A_1 and A_2, $a = 0$ and eq. (13) becomes:

$$\partial u/\partial T = \partial^2 u/\partial z^2 + (\partial u/\partial z)^2 b/(1-bu) \qquad (17)$$

and eq. (14) becomes:

$$\partial w/\partial T = \partial^2 w/\partial z^2 \qquad (18)$$

It was shown in Ref.2 that, because eq. (18) is identical with the Terzaghi equation except for the substitution of w for u, and because the boundary conditions for w are similar to those for u in Terzaghi's theory, the degree of settlement U_s for limiting case I (for conditions A_1 and A_2) is exactly equal to that given by the Terzaghi theory. However the pore pressures of the non-linear theory are only equal to those of the Terzaghi theory when $\alpha = 1$. As α increases the pore pressures of the non-linear theory became increasingly greater than the Terzaghi values. The pore pressures of the non-linear theory are given by:

$$u = \alpha(1-\alpha^{-u^*})/(\alpha-1) \qquad (19)$$

where u^* = pore pressure given by Terzaghi linear theory with $u_o = 1$. The distributions of pore pressure with depth given by eq. (19) at three values of T_v are plotted in Fig.2

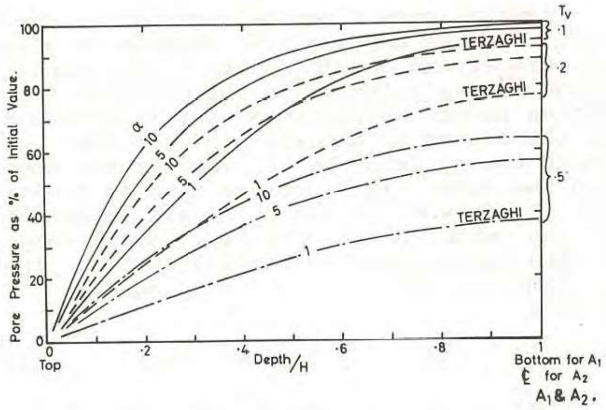


Fig.2 Pore Pressures, Case I

It can easily be shown that, provided c_v is constant, the conclusion that U_s is the same as for the classical linear theory, still holds whatever the relation between stress and strain, not only when it is that implied by eq. (1). It is also clear that, in contrast to the linear theory, the degree of settlement U_s in any non-linear theory is not identical with the average degree of pore pressure dissipation $\overline{U}_p$.

It was shown in Ref.2 that oedometer tests on three normally consolidated clays with one-way drainage and pore pressure measurement at the impermeable boundary gave satisfactory confirmation of eq. (19), at least in the later stages of pore pressure dissipation. A summary of this experimental evidence is given in Fig.3. Experimental confirmation has also been reported by Barden (Ref.4) and Burland and Roscoe (Ref.5). Further evidence is shown in Fig.7 for hydrostatic triaxial tests on a Kaolin with axial drainage for which the same analysis should hold.

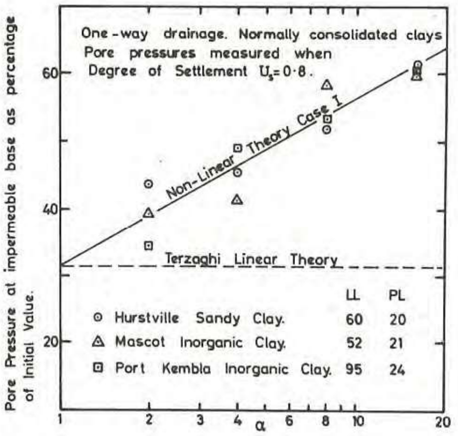


Fig.3 Experimental Confirmation of Case I (Ref.2)

IV.- LIMITING CASE II

Limiting case I can be regarded as the limit when the depth of the stratum is of no importance, and the magnitude of the consolidation stress relative to the original effective stress is all important in determining the extent of departure of the non-linear theory from the Terzaghi linear theory. Then the other limit, referred to as limiting case II, is when the depth is all important and the consolidation stress is small compared with the original effective stress. Thus for limiting case II $\alpha = 1$, and eq.(13) becomes:

$$\partial u/\partial T = \partial^2 u/\partial z^2 - (\partial u/\partial z)/(\beta+z) \qquad (20)$$

From eq.(20) it can be seen that as $\beta \rightarrow \infty$ Case II tends to the classical linear theory. Equation (20) can be solved analytically to give:

$$u = \sum_{n=1}^{\infty} a_n x \, \Psi_1(\lambda_n x) \, \exp.(-\lambda_n{}^2 T) \qquad (21)$$

where $x = \beta + z$,

$$\Psi_m(\lambda_n x) = J_m(\lambda_n x) + b_n Y_m(\lambda_n x),$$

and J and Y are Bessel Functions of the 1st and 2nd kind respectively.

The values of a_n, b_n and λ_n are given by equations determined by the boundary conditions. Thus λ_n and b_n are given by the following:

For one-way drainage (conditions A_1 and B_1):

$$\left.\begin{array}{l} u = 0 \text{ when } z = 0 \text{ and hence } \Psi_1(\lambda_n\beta)=0 \\ \partial u/\partial z=0 \text{ when } z=1 \text{ and hence } \Psi_0(\lambda_n(\beta+1))=0 \end{array}\right\} (22)$$

For two-way drainage, (conditions A_2 and B_2):

$$\left.\begin{array}{l} u = 0 \text{ when } z = 0 \text{ and hence } \Psi_1(\lambda_n\beta)=0 \\ u = 0 \text{ when } z = 0 \text{ and hence } \Psi_1(\lambda_n(\beta+1))=0 \end{array}\right\} (23)$$

The values of a_n are then determined by the distribution of the initial excess pore pressure with depth.

For conditions A_1 and A_2

$$a_n = -2\Big[\Psi_o(\lambda_n x)\Big]_\beta^{\beta+1} \Big/ \lambda_n\Big[x^2\Psi_o{}^2(\lambda_n x) + x^2\Psi_1{}^2(\lambda_n x)\Big]_\beta^{\beta+1} \qquad (24)$$

For conditions B_1 and B_2

$$a_n = -2\Big[(1+\beta)\Psi_o(\lambda_n x) + 0.5\pi\ x\ \Psi_1(\lambda_n x)H_o(\lambda_n x) - 0.5\pi\ x\ \Psi_o(\lambda_n x)H_1(\lambda_n x)\Big]_\beta^{\beta+1} \Big/ \lambda_n\Big[x^2\Psi_o{}^2(\lambda_n x) + x^2\Psi_1{}^2(\lambda_n x)\Big]_\beta^{\beta+1} \qquad (25)$$

where H is Struve's Function.

Distributions of pore pressure calculated from eq.(21) at three values of T_v are shown in Fig.4 for condition A_2. From this figure it can be seen that, when the pore pressures of case II are significantly different from those of the Terzaghi linear theory, they are smaller than the Terzaghi values. This is the reverse of case I. The discrepancy can be seen to increase as β decreases, that is as the depth of the stratum increases. The same trend was found to be even more marked for the conditions A_1. This can be seen from the vertical axis values in Fig.7. Similar general trends were found for the conditions B_1 and B_2 although for B_1, the non-linear theory gave somewhat higher pore pressures at the impermeable base than the Terzaghi values during the earlier stages of consolidation.

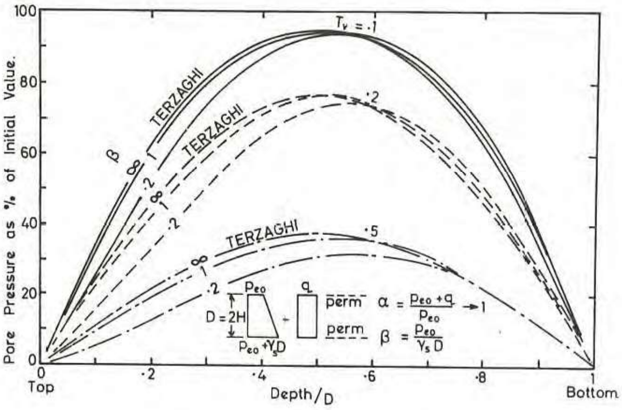


Fig.4 Pore Pressures, Case II, Condition A_2

The degree of settlement U_s is given by the following:

For conditions A_1 and A_2

$$U_s = \int_o^1 m_v(1-u)dz \Big/ \int_o^1 m_v dz \qquad (26)$$

and for conditions B_1 and B_2

$$U_s = \int_o^1 m_v(1-z-u)dz \Big/ \int_o^1 m_v(1-z)dz \qquad (27)$$

After some manipulation eqs. (26) and (27) give:

$$(1-U_s)\ L = \sum_{n=1}^{\infty} G_n \qquad (28)$$

$$\left.\begin{array}{l} \text{where } G_n = \dfrac{-a_n}{\lambda_n}\Big[\Psi_o(\lambda_n x)\Big]_\beta^{\beta+1} \exp.(-\lambda_n{}^2T) \\ \quad \text{for all conditions } A_1,\ A_2,\ B_1 \text{ and } B_2 \\ \text{and } L = \ell n((\beta+1)/\beta) \text{ for conditions } A_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } A_2 \\ \text{or} \quad L = (1+\beta)\ell n((\beta+1)/\beta)-1 \\ \qquad\qquad \text{for conditions } B_1 \text{ and } B_2 \end{array}\right\} (29)$$

The relationship between U_s and T_v has been determined from eq.(28) and for condition A_1 is plotted for the range β = 0.1 to ∞ in Fig.5. From this figure it can be seen that settlement proceeds significantly faster as β decreases. There is also some change in the general shape of the settlement/log time curve with change in β.

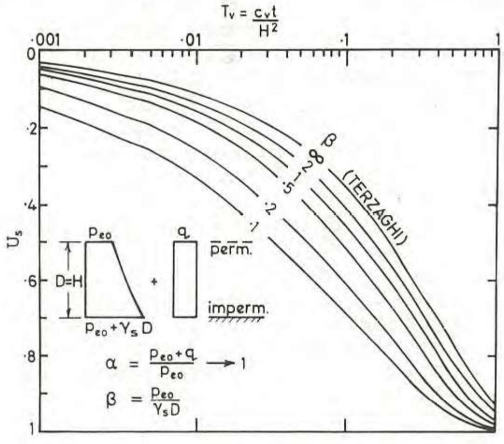


Fig.5 Rate of Settlement, Case II, Condition A_1

A similar change in shape of the settlement/log time curve was found for the other conditions A_2, B_1 and B_2 but again the most significant feature of the results was the increase in general rate of settlement as β decreased. This is clearly shown in Fig.6 in terms of the value of T_v for 50% settlement. The effect is more marked for one-way than for two-way drainage.

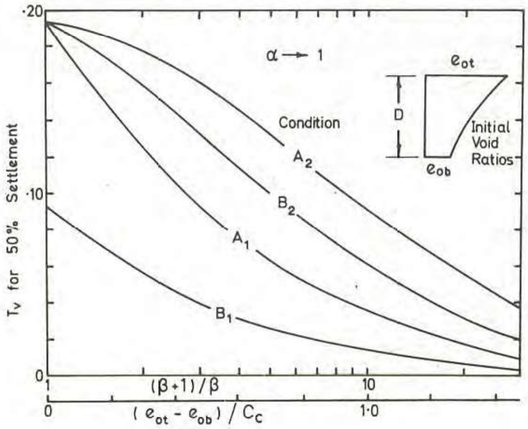


Fig.6 Case II, T_v Values for U_s = 0.5

V.- GENERAL SOLUTION

The general solution of the differential eqs.(13) or (14) to give the distribution of excess pore pressure at all times, and from this the relation between U_s and T_v, for cases when neither α = 1 nor $\beta = \infty$, does not appear to be feasible by analytical methods. Solutions covering this general field between limiting cases I and II have however been obtained by numerical computer methods.

A selection of the numerical results for pore pressure are presented in Fig.7 for the condition A_1. This figure demonstrates the transition from Limiting Case I ($\beta\to\infty$), which always gives pore pressures higher than the linear theory, to Limiting Case II ($\alpha\to1$), which normally gives pore pressures lower than the linear theory.

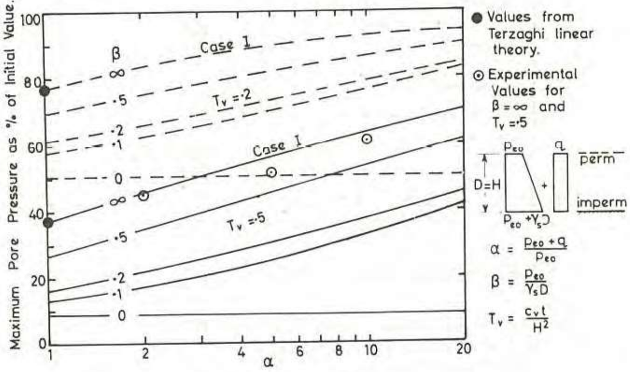


Fig.7 Maximum (Base) Pore Pressures, Condition A_1

For β = 0.1 and Condition A_1, the relationship between U_s and T_v is shown in Fig.8 from which it can be seen that the accelerated rate of settlement produced by a small value of β with case II ($\alpha\to1$) is counterbalanced by the effect of having valuesof $\alpha>1$ until the linear Terzaghi answer is once again approached as $\alpha\to\infty$. A similar trend was found for condition A_2. For conditions B_1 and B_2, although increasing α while keeping β constant slowed the rate of settlement, the linear solution no longer provided the limit as $\alpha \to \infty$. Fig.9 shows the general effects for all conditions A_1, A_2, B_1, and B_2 in terms of the value of T_v for $U_s = 0.5$ and for $\beta = 0.1$. For values of β other than 0.1 and for $\alpha > 1$ it should be possible to obtain for practical purposes a sufficiently accurate estimate of the U_s/T_v relationship from the information provided by Figs.5,6,8 and 9.

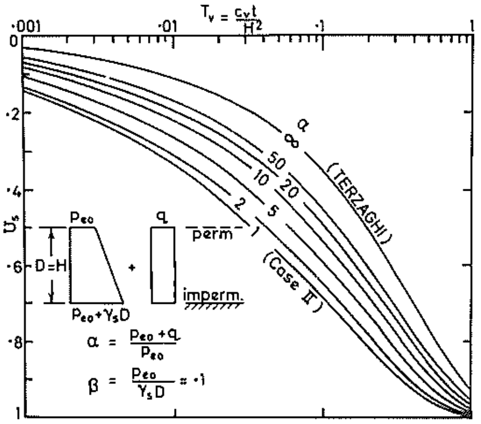


Fig.8 Rate of Settlement, Condition A_1, Varying α

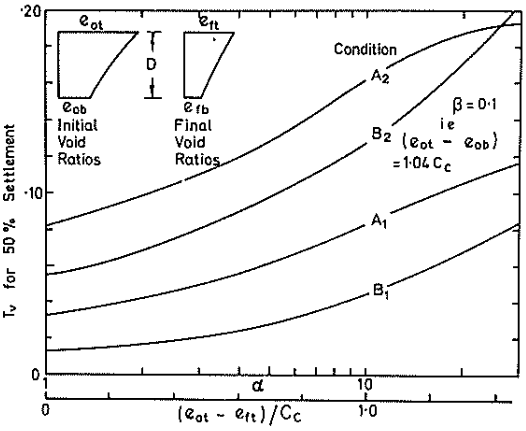


Fig.9 T_v Values for $U_s = 0.5$, $\beta = 0.1$, Varying α

VI. - DISCUSSION AND CONCLUSIONS

The non-linear theory developed in this paper shows that the pore pressures, at intermediate stages of consolidation of a deep bed of soft clay subject to relatively minor consolidation pressure, may be considerably less than those predicted by classical linear theory. On the other hand, reducing the thickness of the clay layer and increasing the consolidation pressure both have the effect of, in general, increasing the pore pressure at a given T_v and relative depth, so that, with thin layers and high consolidation pressures, the pore pressures may be considerably higher than those predicted by classical theory. Unless these effects are taken into account, the use of piezometers to monitor the progress of consolidation in reclamation and similar work may lead to an inaccurate interpretation of the field measurements. In this connection it is worth recording that a reformulation of the non-linear theory for the axi-symmetric conditions applicable to sand drains can be readily done and was successfully employed in interpreting piezometer readings for reclamation work with sand drains in W.Australia. When drainage is only radial, it is easily shown that for Limiting Case I, as with the one-dimensional vertical problem of this paper, the degree of settlement U_s (averaged over the inner to outer radii) is identical to that calculated from linear theor but that the pore pressures are higher as given by eq.(19). u^* must now be obtained from standard sand drain theory, see Barron (Ref.6) and earlier references.

The non-linear theory developed in this paper also shows that, although for relatively thin beds of clay the rate of settlement should be given sufficiently accurately by classical linear theory no matter how high the consolidation pressure, the rate for deep beds with low consolidation pressures may be considerably faster than that indicated by the classical theory.

It will be clear from its formulation that the theory presented in this paper is a small strain theory. This may appear to be an anomaly since the theory attempts to allow for relatively large changes in void ratio. However, the use of the average of the initial and final thicknesses of the clay layer in calculating the real time from the time factor T_v should remove the effects of the anomaly to a large extent and the experimental evidence from several sources supports this. For extreme values of the governing parameters it may be advisable to resort to finite strain theory, see Gibson et al. (Ref. 7). Further work to obtain guides, such as that the small strain theory is sufficiently accurate for practical purposes as long as the final settlement is no more than 10 or 20% of the total thickness, is desirable.

Although the non-linear theory presented in this paper is only one-dimensional, it seems reasonable to use it in approximate fashion to obtain a correction factor to the rate of settlement for three dimensional situations given by linear theory such as that in Refs.8 and 9. The solutions for the conditions B_1 and B_2 in which the consolidation pressure decreases with depth may be of assistance for this purpose. These solutions are of course also directly applicable to one-dimensional problems in which a steady vertical flow of water is either the initial or final equilibrium state.

VII.- ACKNOWLEDGMENTS

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