The Primary/Secondary Transition during the Consolidation of Clay

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Pore pressure gradient affects rate of strain.

Steeper than primary

SUMMARY.- The question of a transition from the primary to the secondary stages of consolidation occurring at different times at different depths within a stratum is examined. The particular case of one-dimensional consolidation in saturated clay is considered in detail. The types of depth-time loci of transition implied by different theories and different primary/secondary criteria are discussed. The influence of stratum thickness on the transition, and in particular the limited relevance of the t/H^2 similarity rule, is emphasised.

I.- INTRODUCTION

Although it is common practice to assume that the transition from the primary to the secondary stage of the consolidation process takes place at the same instant throughout the depth of a stratum, it is evident that, whatever criterion is used to distinguish between primary and secondary processes, the transition must take place at the drainage surface(s) at $t = 0$ (the instant of application of an increment of load) and progress through the stratum towards the mid-plane as consolidation proceeds. Soil at a drainage surface is always in the secondary stage.

Although presented in the context of one-dimensional consolidation in saturated clays the ideas developed in this paper are not restricted to that particular case.

The adjective 'local' will be used in this paper to distinguish the value which a variable may have at a particular point within a stratum from the average value which the same variable may have when the stratum is considered as a whole. A simple instance occurs in connection with the variable 'strain'. Average strain within a stratum (at any instant) is closely related to 'settlement'. A consideration of local rather than average (or maximum) values of variables is the more rational procedure insofar as elements of soil need not then be assumed to 'know' where they lie in relation to a drainage face, and in particular whether they form parts of thin laboratory samples or thick field strata.

II.- NOTATION

e	Voids ratio
e_o	Voids ratio for an effective stress σ_o'
e_i	Initial voids ratio
e_f	Final voids ratio
σ'	Effective stress
σ_i'	Initial effective stress
σ_f'	Final effective stress
u	Excess pore pressure
z	Length
a	Length measured on an independent (fixed) reference space
H	Maximum drainage path length
γ_w	Density of water
k	Permeability
t	Time
T_v	Dimensionless time factor $c_v\, t/H^2$
T_o	" " " $c_F\, t/H^2$
c	Constant
c_v	Coefficient of consolidation
c_F	Coefficient equal to $c_v\,(1+e_o)/(1+e)$
a_v	Coefficient of compressibility
C_c	Compression index

III.- THE TRANSITION LOCUS FOR AN IDEAL PRIMARY SOIL

In this section the behaviour of an ideal primary soil will be considered. Such a soil obeys a primary (e.g. Terzaghi, Ref. 1) theory by occupying at all times, states which correspond with points on a unique voids ratio (e) - effective stress (σ') curve. Strictly such a soil would approach the end of consolidation asymptotically but an end of the primary stage in finite time may be calculated according to a criterion such as either of the two following:

(A) the attainment by each element of soil within the stratum of a specified fraction X (say 0.99) of ultimate strain (as calculated from the unique $e - \sigma'$ relationship obeyed by this ideal soil).

(B) the reduction of excess pore pressure in each element within the stratum to a specified fraction x (say 0.01) of the load increment.

The loci in depth-time coordinates of the end of the primary stage have been plotted in Fig. 1 for each of the above two criteria. Three primary theories have been considered; those of Terzaghi (Ref. 1), Davis and Raymond (Ref. 2) and Gibson, England and Hussey (Ref. 3). In terms of voids ratio all three of these theories can be made to yield equations of the form

$$\frac{\partial e}{\partial t} = c\,\frac{\partial^2 e}{\partial z^2} \qquad 1$$

where z is a length term and t is time.

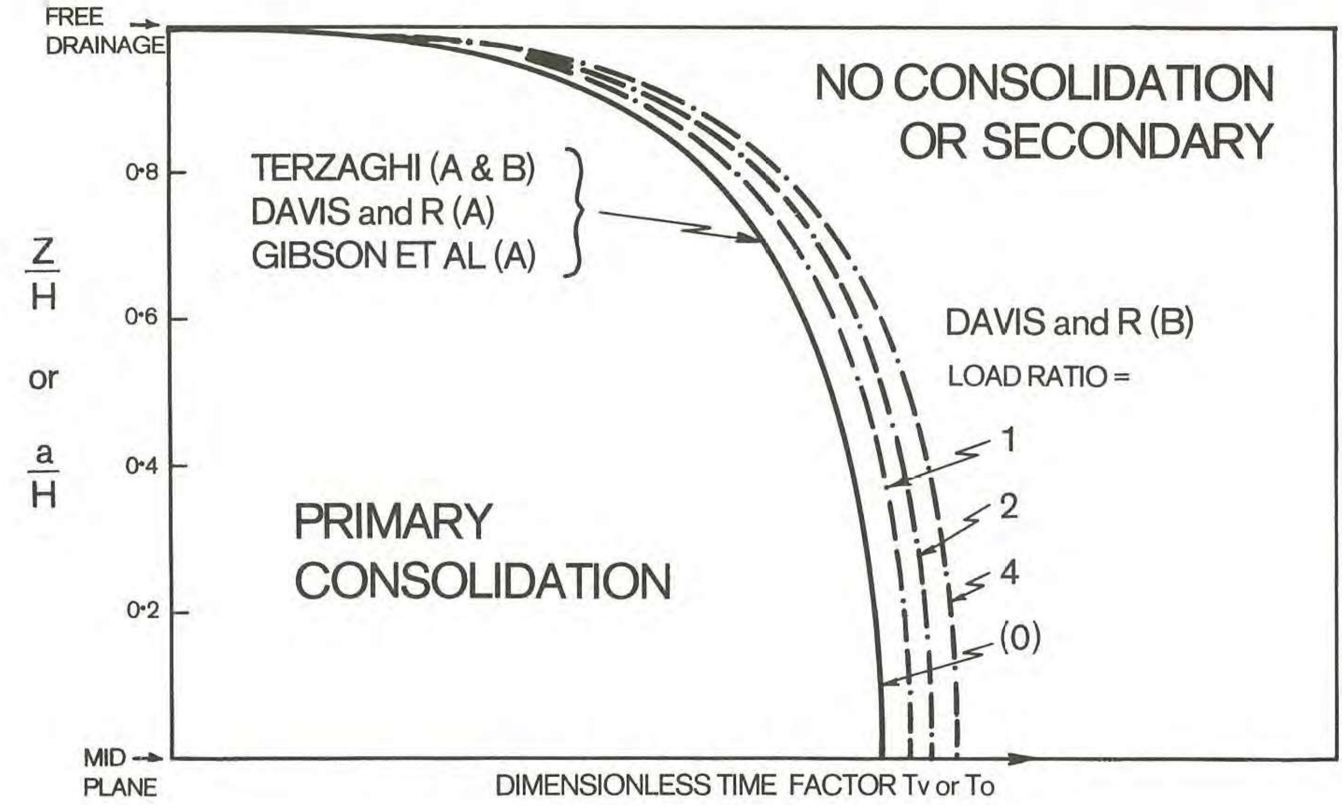

Fig. 1. Depth-Time Loci of the End of the Primary Stage for an Ideal 'Primary' Soil.

Terzaghi derives an equation of the form of equation 1 but in terms of excess pore pressure rather than voids ratio,

$$\text{i.e.} \quad \frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \qquad 2$$

Equation 1 may be derived using only the equation of continuity, Darcy's law and the effective stress equation, if self weight effects (i.e. effects due to the difference between the densities of the soil particles and the pore water) are neglected and if the product $k\, d\sigma'/de$ is assumed constant. The derivation of equation 2 involves, in addition, a restriction to step-loading conditions and the assumptions that k and $\frac{d\sigma'}{de}$ are each constant for individual increments of load.

In the Terzaghi and the Davis and Raymond theories the constant c in equation 1 is the coefficient of consolidation c_v defined as

$$c_v = - \frac{k\,(1+e_o)}{\gamma w} \frac{d\sigma'}{de} \qquad 3$$

where k is the permeability and
γw is the density of water.

In these two theories no distinction is made between lengths measured at time 0 and lengths measured on the (moving) soil skeleton at time t. These two theories are therefore small strain theories.

Gibson et al. following McNabb (Ref. 4) in this respect, have removed the limitation to small strains by making careful definition of length (and velocity) terms. They show that equation 1 may be written more precisely as

$$\frac{\partial e}{\partial t} = c_F \frac{\partial^2 e}{\partial a^2} \qquad 4$$

where lengths 'a' are measured at t = 0 (i.e. on an independent, fixed, reference space) and are not therefore functions of time. c_F is given by

$$c_F = - \frac{k}{\gamma w} \frac{(1+e_o)^2}{(1+e)} \frac{d\sigma'}{de} \qquad 5$$

The loci in Fig. 1 have been plotted in dimensionless form; the vertical axis in terms of $^z/H$ or $^a/H$ where H is the maximum drainage path length (at t = 0 in the case of $^a/H$). The horizontal axis has been plotted in terms of the dimensionless time factor

$$T_v = c_v\, {}^t/H^2 \qquad 6$$

for the first two theories

$$\text{or} \quad T_o = c_F\, {}^t/H^2 \qquad 7$$

for the Gibson et al. theory.

For the first two theories under conditions of step-loading, the local voids ratios are given by

$$e = e_f + \frac{4}{\pi}(e_i - e_f)\sum_{n=0}^{n=\infty}\frac{(-1)^n}{(2n+1)}\cos\left[\frac{(2n+1)\pi z}{2H}\right]\exp\left[\frac{-(2n+1)^2\pi^2 c_v t}{4H^2}\right] \quad 8$$

where e_i and e_f are the initial and final voids ratios corresponding to the initial and final values of total applied stress, and z = 0 has been taken as the mid-plane of the stratum. For the Gibson et al. theory the same expression for voids ratio applies, with the modification that the z term becomes an 'a' term and c_v becomes c_F.

For criterion (A) above, the end of the primary stage is given by

$$X = \frac{e_i - e}{e_i - e_f} \quad 9$$

and the depth-time locus for the end of the primary stage of consolidation has the same shape for all three theories. This has been plotted as the continuous curve in Fig. 1. The position of the curve relative to numbers which might be put on the time factor axis) depends on the value of X considered appropriate to signify the end of the primary stage.

For criterion (B) above, the end of the primary stage is given by

$$x = \frac{u}{\Delta\sigma} \quad 10$$

where u is the excess pore water pressure, and $\Delta\sigma$ is the applied increment of load.

Because equations 1 and 2 have the same form it is apparent that, for the Terzaghi theory, the same shape of transition locus is obtained for criterion (B) as was obtained for criterion (A). The same position of the locus is obtained if x = 1 - X. The continuous line in Fig. 1 represents therefore the locus of the end of the primary stage derived from Terzaghi theory for criterion (B).

The Terzaghi assumption (in the derivation of equation 2) that $\frac{d\sigma'}{de}$ is constant appears in the form

$$de = -a_v \, d\sigma' \quad 11$$

where a_v is assumed constant for individual increments of step loading. Davis and Raymond have shown that the assumptions of a constant coefficient of consolidation c_v and an e-σ' relationship of the form

$$e = e_o - C_c \log_{10}\left(\frac{\sigma'}{\sigma_o'}\right)$$

$$\text{i.e. } de = -.434 \, C_c \frac{d\sigma'}{\sigma'} \quad 12$$

rather than the Terzaghi assumptions of constant permeability and the e-σ' relationship given as equation 11 above, lead to the expression for excess pore pressure

$$u = \sigma_f'\left[1 - \left(\frac{\sigma_i'}{\sigma_f'}\right)^B\right] \quad 13$$

where e_o is the voids ratio corresponding to an effective stress σ_o'

C_c is the compression index, assumed constant

σ_i' and σ_f' are the initial and final effective stresses, and B is given by

$$B = \frac{4}{\pi}\sum_{n=0}^{n=\infty}\frac{(-1)^n}{(2n+1)}\cos\left[\frac{(2n+1)\pi z}{2H}\right]\exp\left[\frac{-(2n+1)^2\pi^2 c_v t}{4H^2}\right] \quad 14$$

The transition locus implied by criterion (B) using the Davis and Raymond theory is therefore given by

$$x = \frac{u}{\Delta\sigma} = \frac{\sigma_f'}{\Delta\sigma}\left[1 - \left(\frac{\sigma_i'}{\sigma_f'}\right)^B\right] \quad 15$$

For any given value of x the position of this transition locus relative to the T_v axis is dependent on the load ratio $\frac{\sigma_f'}{\sigma_i'}$. The transition would occur later (at all values of $^z/H$ for a higher load ratio, see Fig. 1.

Because they do not derive a solution for excess pore pressure, Gibson et al. do not need to assume any particular e-σ' relationship. They do assume however (in their derivation of equation 4 above) that there exists some unique relationship between e and σ'. The position of the transition locus according to criterion (B) and Gibson et al. theory will be dependent upon the form of the e-σ' relationship used to extend their theory. If e-σ' relationships of the form given by equation 11 or 12 were assumed then the position of the transition locus would be respectively independent of or dependent on the load ratio in the manner already described for the Terzaghi and the Davis and Raymond theories.

IV.- THE TRANSITION LOCUS FOR A REAL SOIL

Common experience indicates that real soils exhibit secondary as well as primary consolidation behaviour. Attempts to derive transition loci based on primary (e.g. Terzaghi, Ref. 1) and secondary (e.g. Buisman, Ref. 5) equations are destined to be unsound because in order for a primary solution (such as equation 8) to be applicable, the whole stratum must obey the primary theory. Before a primary/secondary transition locus may be derived it is necessary that

(a) the terms 'primary' and 'secondary' be precisely defined, and

(b) the primary and secondary stages be united under the control of a single set of equations.

With regard to (a) the author (Ref. 6) has pointed out that elements of soil can have no knowledge of the origin for the measurement of the excess component of the local pore pressure at any instant. Rather it is the effective stress borne by, and the strain rate suffered by each element of soil, which determines the behaviour of (i.e. the path in e - σ' space followed by) that element of soil. Secondary consolidation is shown to be essentially "small strain rate" consolidation: primary consolidation is shown to be "that during which all elements of soil within a stratum follow the same path in e - σ' space, under conditions of high strain rate". Such primary consolidation is commonly observed in the early stages of routine laboratory tests in which thin

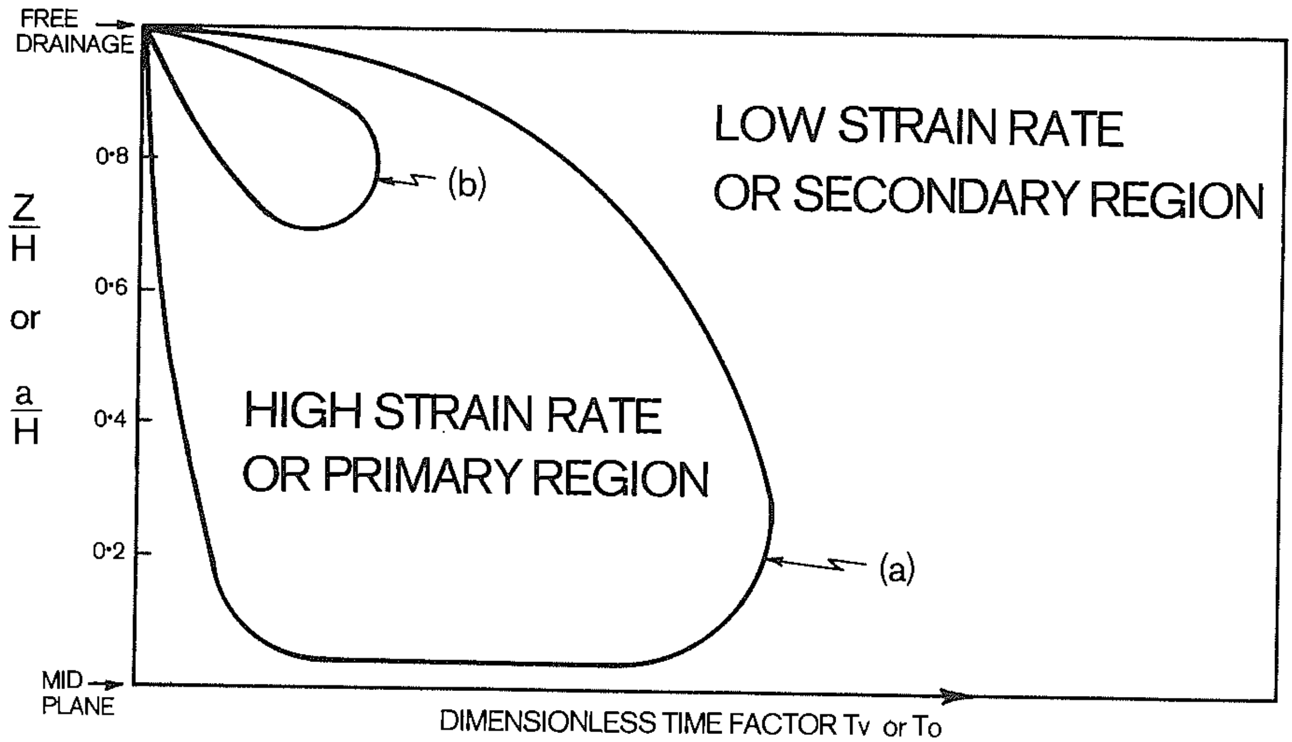

Fig. 2. Depth-Time Loci for the Primary/Secondary Transition for a 'Real' Soil, according to Strain Rate.

Curve (a) Typical locus for thin sample subjected to large load ratio.
" (b) " " " thicker stratum and/or smaller load ratio.

($H \doteqdot 1$ cm) samples are subjected to large (generally > 1.5) load ratios.

The high strain rate region in depth - time space lies within a 'bulb' such as is shown in Fig. 2. For thicker strata or smaller load ratios the high strain rate 'bulb' penetrates to a smaller fraction of the depth of the stratum. Under such conditions primary solutions for the whole stratum cannot be applied.

With regard to (b) it has been shown (Ref. 7) that by modifying primary theory to incorporate a unique $e - \sigma' - \frac{\partial e}{\partial t}$ relationship rather than a unique $e - \sigma'$ relationship, the laboratory primary and secondary stages may be united under the control of a single set of six equations, namely

i Continuity
ii Darcy's Law
iii Stress equilibrium (self weight)
iv The effective stress equation
v A permeability - voids ratio (k-e) relationship
vi An $e - \sigma' - \frac{\partial e}{\partial t}$ relationship.

This last equation constitutes the volumetric stress-strain-strain rate relationship for the soil. In general there will be no explicit solution to these six equations and the implicit solutions for particular problems must be calculated numerically. (Sensitivity and other physico-chemical effects, such as time-hardening which may be of importance over long times at low strain rates, are not included in this theory).

The well-known t/H^2 similarity rule follows from the form of equations such as equation 1. In a derivation of equation 1 it must be assumed that

(1) self-weight effects may be ignored (equation iii), and

(2) there exists some unique $e - \sigma'$ relationship - independent of $\frac{\partial e}{\partial t}$ (equation vi).

Furthermore if solutions to equation 1 are to be calculated it must be assumed that

(3) the coefficient c (c_v or c_F) is constant or a function of voids ratio alone.

This last condition is known to be violated in routine laboratory testing when, at large times after the application of increments of step loading, settlement is commonly observed to continue at sensibly constant effective stress. Under such conditions $\frac{de}{d\sigma'}$ tends to $-\infty$ and c_v (or c_F) tends to zero. In thick field strata all of these three conditions will commonly be violated. The t/H^2 similarity rule can not therefore be expected to hold for thick strata.

There is a need for more experimental investigation of the form of, and the constants in, both the $e - \sigma' - \frac{\partial e}{\partial t}$ and the k - e relationships for different soils. Only with the aid of such information and the acceptance (without modification for mathematical expediency) of all of equations i to vi above, may the behaviour of thick strata be predicted from the results of tests on thin laboratory samples.

V.- CONCLUSIONS

The assumption of a transition from the primary to the secondary stages of consolidation occurring at the same instant of time throughout the depth of a stratum is unsound. A consideration of the values of variables at points within a stratum rather than average values for the stratum considered as a whole, reveals a need for a theory which unites the primary and secondary stages under a common set of equations. Only by means of such a theory may the results of laboratory tests on (thin) laboratory samples be used to predict the rate of consolidation of (thick) field strata. In general the t/H^2 similarity rule can not be expected to hold for thick strata.

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