A Case Study of Ground Water Levels in Relation to a Flooding Stream

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SUMMARY. During the flood period of a stream, ground water levels are temporarily raised near the channel by inflow from the stream. The water so stored and released after the flood is referred to as bank storage. In this study, analytical solutions are developed considering the bank storage and its rate of inflow and outflow, and fluctuation of ground water levels subjected to various boundary conditions, and an idealized flood hydrograph. The flood hydrograph is assumed to be approximated by a triangle. The solutions are based on the Dupuit assumptions, and the assumption that the magnitude of the ground water fluctuation is small compared with the saturation thickness of the aquifer. The solutions are obtained by using the Laplace transformation technique and are expressed in terms of infinite series. Experiments to verify the theoretical solutions are reported. It is concluded that the deviation between the theoretical and experimental results arise from some of the assumptions. However, the trends of the two results are very similar, so that the theory can be used in practice to give an approximate solution.

1 INTRODUCTION

Where a stream channel or reservoir is in direct contact with an unconfined aquifer, the stream or reservoir may recharge the ground water or receives discharge from the ground water, depending on the relative water levels. The water so stored after a flood and subsequently released is referred to as bank storage. In this study, bank storage will be analyzed in response to an idealized triangular stream flood hydrograph.

The problem is to determine the variation of the ground water table in response to a triangular flood hydrograph in a stream. The initial water table is assumed to be linear. A sketch defining the problem is shown in Fig.1. In the figure, 0 is the origin of the coordinate system, L is the length from the bank to the point where the elevation of the water table can be treated as a constant denoted by H. The water table elevation above the impervious boundary at a distance x from the bank is h and D is the depth of the initial saturated thickness.

The differential equation governing this case of ground water flow in an unconfined aquifer is

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{a}\frac{\partial h}{\partial t} \qquad (1)$$

where a is called the coefficient of diffusivity $a = \frac{KD}{S_y}$ and S_y is the specific yield of the aquifer. Eq.(1) was derived using the Dupuit assumptions and the condition that the variation of water table level is small compared with the average depth of the flow. The initial and boundary conditions of this problem are:

$$h(x,o) = HX/L \qquad (2)$$

$$h(o,t) = \frac{h_o t}{t_A} - \frac{h_o t}{t_A} U(t-t_A) + h_o \frac{(t_A+t_B-t)}{t_B} U(t-t_A) - h_o \frac{(t_A+t_B-t)}{t_B} U(t-t_A-t_B) \qquad (3)$$

$$h(L,t) = H \qquad (4)$$

where U(t) is the unit step function.

Eq.(2) states that the water table profile is assumed to be linear. Eq.(3) states that the flood hydrograph in the stream is approximately represented by a triangle (Fig.1) as suggested by the United States Bureau of Reclamation (2) and Eq.(4) states that the water table level at distance X = L is kept constant.

2 SOLUTIONS OF THE PROBLEM

Eq.(1) is a linear homogeneous differential equation. The solution of this boundary value problem which processes two non-homogeneous boundary conditions can therefore be written as the summation of the solutions of the problems each of which contains only one non-homogeneous boundary condition. In the analysis each of the problems will be solved independently using the Laplace transformation technique.

Let h = u + v be the solution of the problem where u and v are respectively the solutions of the two boundary value problems: containing only one non-homogeneous condition;

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial x}{\partial t} \qquad (5)$$

with conditions,

$$u(x,o) = HX/L \qquad (6)$$

$$u(o,t) = 0 \qquad (7)$$

$$u(L,t) = H \qquad (8)$$

and

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a}\frac{\partial v}{\partial t} \qquad (9)$$

with conditions,

$$v(x,0) = 0 \qquad (10)$$

$$v(o,t) = \frac{h_o t}{t_A} - \frac{h_o t}{t_A} U(t-t_A) + h_o \frac{(t_A+t_B-t)}{t_B} U(t-t_A) - h_o \frac{(t_A+t_B-t)}{t_B} U(t-t_A-t_B) \quad (11)$$

$$v(L,t) = 0 \quad (12)$$

The solution of the first boundary value problem can be readily obtained as,

$$u(x,t) = \frac{XH}{L} \quad (13)$$

To obtain v, let $\bar{v}(x,s)$ be the Laplace transformation of $v(x,t)$;

$$\bar{v}(x,s) = \int_o^\infty e^{st} v(x,t)dt \quad (14)$$

Transformation of Eq.(9) and using the transformed initial condition of Eq.(10), one obtains the subsidiary equation as,

$$\frac{d^2\bar{v}}{dx^2} - \frac{s}{a}\bar{v}(x,s) = 0 \quad (15)$$

The solution of Eq.(15) after substituting the transformed conditions of Eqs.(11) and (12), becomes

$$\bar{v}(x,s) = h_o\left[\frac{1}{t_A s^2} - \frac{e^{-t_A s}}{t_A s^2} - \frac{e^{-t_A s}}{t_B s^2} + \frac{e^{-(t_A+t_B)s}}{t_B s^2_q}\right] [e^{-qx} - e^{-q(2L-x)}] \sum_{n=0}^{\infty} e^{-2nqL} \quad (16)$$

Inverse Laplace Transform of Eq.(16) yields the solution $v(x,t)$.

$$V = 4h_o\Bigg[\frac{t}{t_A}\sum_{n=0}^{\infty}(i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{at}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{at}}) - (\frac{t}{t_A} + \frac{t}{t_B} - \frac{t_A}{t_B} - 1)U(t-t_A)\sum_{n=0}^{\infty}(i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{a(t-t_A)}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{a(t-t_A)}}) + (\frac{t}{t_B} - \frac{t_A}{t_B} - 1)U(t-t_A-t_B)\sum_{n=0}^{\infty}\{(i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{a(t-t_A-t_B)}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{a(t-t_A-t_B)}}\}\Bigg] \quad (17)$$

The summation of U and V in Eqs.(13) and (17) yields the solution after simplification as,

$$\frac{h}{H} = \frac{x}{L} + 4\frac{h_o}{H}\Bigg[\frac{t}{t_A}\sum_{n=0}^{\infty}\{i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{at}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{at}}\} - \{\frac{t}{t_A} + \frac{t}{t_B} - \frac{t_A}{t_B} - 1\}U(t-t_A)\sum_{n=0}^{\infty}\{i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{a(t-t_A)}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{a(t-t_A)}}\} + \{\frac{t}{t_B} - \frac{t_A}{t_B} - 1\}U(t-t_A-t_B)\sum_{n=0}^{\infty}\{i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{a(t-t_A-t_B)}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{a(t-t_A-t_B)}}\}\Bigg] \quad (18)$$

The rate of the flow into the reservoir is given by the equation,

$$q = KD\frac{\partial h}{\partial x}\Big|_{x=0} ,$$

thus

$$\frac{q}{q_o} = 1-2\frac{h_o}{H}\Bigg[\frac{t}{t_A}\cdot\frac{L}{\sqrt{at}}\sum_{n=0}^{\infty}(\mathrm{ierfc}\frac{nL}{\sqrt{at}} + \mathrm{ierfc}\frac{(n+1)L}{\sqrt{at}}) - (\frac{t}{t_A} + \frac{t}{t_B} - \frac{t_A}{t_B} - 1)U(t-t_A)\frac{L}{\sqrt{a(t-t_A)}}\sum_{n=0}^{\infty}(\mathrm{ierfc}\frac{nL}{\sqrt{a(t-t_A)}} + \mathrm{ierfc}\frac{(n+1)L}{\sqrt{a(t-t_A)}}) + (\frac{t}{t_B} - \frac{t_A}{t_B} - 1)U(t-t_A-t_B)\frac{L}{\sqrt{a(t-t_A-t_B)}}\sum_{n=0}^{\infty}(\mathrm{ierfc}\frac{nL}{\sqrt{a(t-t_A-t_B)}} + \mathrm{ierfc}\frac{(n+1)L}{\sqrt{a(t-t_A-t_B)}})\Bigg] \quad (19)$$

where q_o is the initial rate of the steady flow, $q_o = KDH/L$.

Eqs.(18) and (19) are the equations representing the water table profile and the rate of flow into the reservoir respectively. These equations can be reduced to be the solutions of the problem with the conditions of linear rise and sudden rise of water level in the downstream reservoir. For the linear rise downstream level $o < t_A < \infty$ and $t_B \to \infty$ Eqs.(18) and (19) respectively become,

$$\frac{h}{H} = \frac{x}{L} + 4\frac{h_o}{H}\frac{t}{t_o}\sum_{n=0}^{\infty}(i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{at}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{at}}) - 4\frac{h_o}{H}\frac{(t-t_o)}{t_o}U(t-t_o)\sum_{n=0}^{\infty}(i^2\mathrm{erfc}\frac{2nL+x}{2\sqrt{a(t-t_A)}} - i^2\mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{a(t-t_A)}}) \quad (20)$$

$$\frac{q}{q_o} = 1-2\frac{h_o}{H}\frac{L}{\sqrt{at}}\cdot\frac{t}{t_o}\sum_{n=0}^{\infty}(\mathrm{ierfc}\frac{nL}{\sqrt{at}} + \mathrm{ierfc}\frac{(n+1)L}{\sqrt{at}}) + 2\frac{h_o}{H}\frac{(t-t_o)}{t_o}\frac{L}{\sqrt{a(t-t_o)}}U(t-t_o)\sum_{n=0}^{\infty}(\mathrm{ierfc}\frac{nL}{\sqrt{a(t-t_A)}} + \mathrm{ierfc}\frac{(n+1)L}{\sqrt{a(t-t_A)}}) \quad (21)$$

Substituting $t_A \to o$ and $t_B \to \infty$, one obtains the solutions for sudden rise of downstream level as,

$$\frac{h}{H} = \frac{x}{L} + \frac{h_o}{H}\sum_{n=0}^{\infty}(\mathrm{erfc}\frac{2nL+x}{2\sqrt{at}} - \mathrm{erfc}\frac{2(n+1)L-x}{2\sqrt{at}}) \quad (22)$$

and

$$\frac{q}{q_o} = 1 - \frac{1}{\sqrt{\pi}}\frac{h_o}{H}\frac{L}{\sqrt{at}}\sum_{n=0}^{\infty}(e^{-\frac{n^2L^2}{at}} + e^{-\frac{(n+1)^2L^2}{at}}) \quad (23)$$

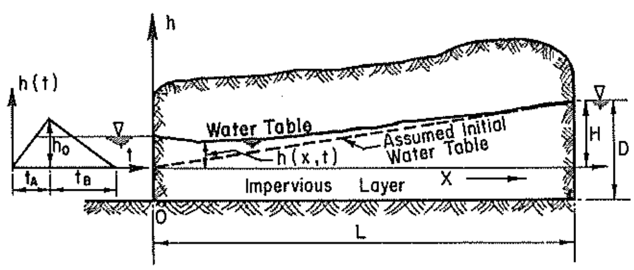

Fig.1 Definition sketch of the problem

The solution for the case of linear and sudden drop downstream level can be readily obtained by replacing h_o in Eqs.(20) through (23) by $(-h_o)$. It is obvious that the solutions for the cases of horizontal initial water table, H=0 and infinite extended aquifer $L=\infty$ can be easily obtained from Eqs.(18) to (23).

3 EXPERIMENTAL INVESTIGATION

Experiments were conducted to verify the theoretical solutions in a Hele-Shaw model. A schematic diagram of the apparatus and detailed description of the Hele-Shaw model are shown in Fig.2.

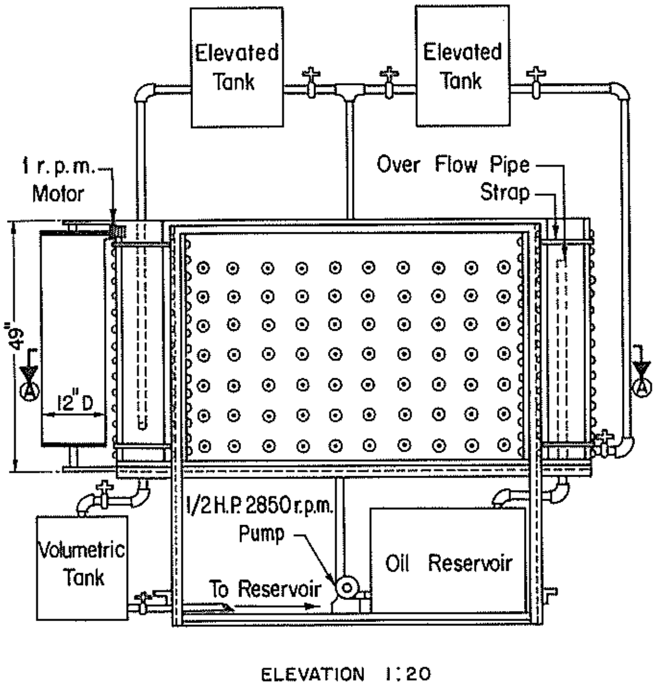

Fig.2 Definition sketch of the Hele-Shaw model apparatus

The Hele-Shaw model was made of 6 ft. by 4 ft. by 1/2 plexi-glass plates placed vertically with a spacing of 1/16 in. Two reservoirs of 8 in. by 8 in. cross section were constructed at the two ends of the model. The upstream reservoir level was controlled by an outflow tube.

A 1 rpm. motor was used to drive a drum of 12 in. diameter to which was attached a plot of the hydrograph. This served as an indicator whereby the oil level in the downstream reservoir could be adjusted manually.

Before the unsteady experiment was carried out, the properties of the Hele-Shaw model (i.e. the spacing b and permeability k) were obtained by operating the model at steady state conditions for various values of $H/D < 0.2$ so that the Dupuit Formula could be accurately applied. For each experimental run at steady flow, the free surface profile and discharge Q were recorded. The values of b and k were then calculated using the equation derived from the continuity equation and Darcy's law as reported by Todd (1),viz:

$$k = \frac{b^2 \rho g}{12\mu} \qquad (24)$$

and the Dupuit formula:

$$Q = kb\,\frac{(h_1^2 - h_2^2)}{\Delta L} \qquad (25)$$

In these equations ρ is fluid density, μ is dynamic viscosity, g is gravity and h_1 and h_2 are the elevations of the free surface at distance ΔL apart. To satisfy the Dupuit assumptions, the values of h_1 and h_2 were recorded where the free surface profiles were relatively horizontal.

The unsteady flow experiment was conducted by allowing the oil to flow from the constant head upstream reservoir to the downstream reservoir. The valve connected to the downstream reservoir was adjusted until the flow became steady, corresponding to the initial condition. The 1 rpm. motor was then switched on to rotate the graph-paper drum. The oil level downstream was then regulated manually corresponding to the hydrograph plotted on the graph paper. The profile of the free surface was photographed at every interval of $0.25(t_A+t_B)$ until the system became nearly steady. The temperature of the oil was also recorded to determine its viscosity.

4 RESULTS AND DISCUSSION

Comparisons between theoretical and experimental plots of h/H and the dimensionless time parameter at/L^2 for three different values of H/D are shown in Figs.3 to 5.

At the starting point $t \to o$, the experimental levels were higher than those predicted by the theory. This effect is due to the assumptions that H is small compared with D, that the initial free surface is treated as a straight line and that the head loss across the edge of the downstream reservoir is negligible (i.e. $h(-o,t) = h(+o,t)$).

The discrepancy between the assumed and experimental initial profiles for various value of H/D are shown in Fig.6. The experimental initial flow profile was close to the assumed profile for the value of H/D less than 0.2; the precision improved as H/D decreased. It is also interesting to note that the variation of the experimental water table shown in Figs.3 to 5 was not affected by H/D when $H/D < 0.2$.

In the rising state $(\frac{dh}{dt} > 0)$ the rate of rise of the experimental elevation is slower than the theoretical, the experimental elevation later dropping below the theoretical curve before reaching its peak. This effect was due to the boundary condition at the edge of the downstream reservoir (i.e. $x = +0$). When the level of the downstream reservoir (i.e. $x = -0$) begins to rise the elevation of the free surface at the edge of the model (i.e. $x = +0$) cannot be instantaneously raised to the same level as the reservoir ($x = -0$) as treated in the theoretical analysis. This is because the reverse discharge from the reservoir during the rising stage provides a certain head loss across the edge of the reservoir causing $h(-o,t) > h(+o,t)$.

This above effect increased in the dropping stage case $(\frac{dh}{dt} < 0)$. When the level in the downstream reservoir dropped, the discharge flowing into the reservoir increased rapidly. A larger thickness of seepage face $(h(+o,t)-h(-o,t))$ therefore occurred to provide the head loss for the amount of discharge.

5 CONCLUSION

Although the experimental results seem to deviate from the theoretical the trend was very similar and the agreement was sufficient good for practical use.

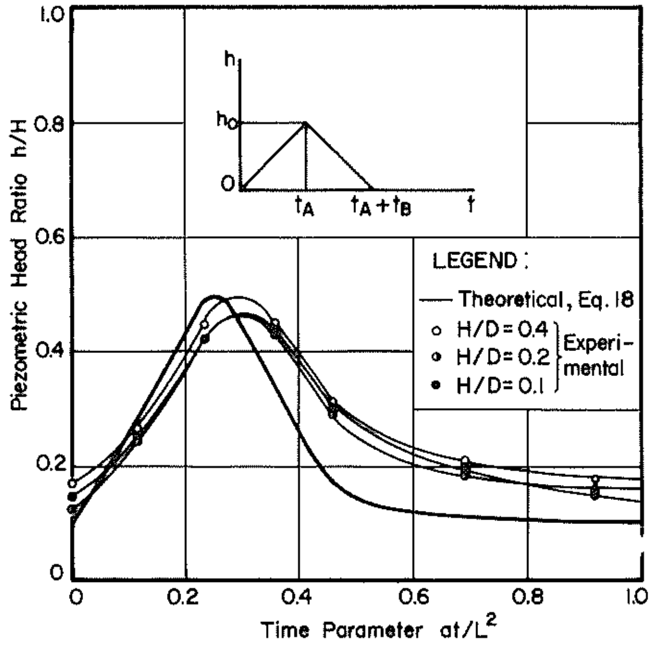

Fig.3 Piezometric head distribution due to modified hydrograph at 10% distance x/L for $at_A/L^2 = at_B/L^2 = 0.230$, $h_o/H = 0.5$

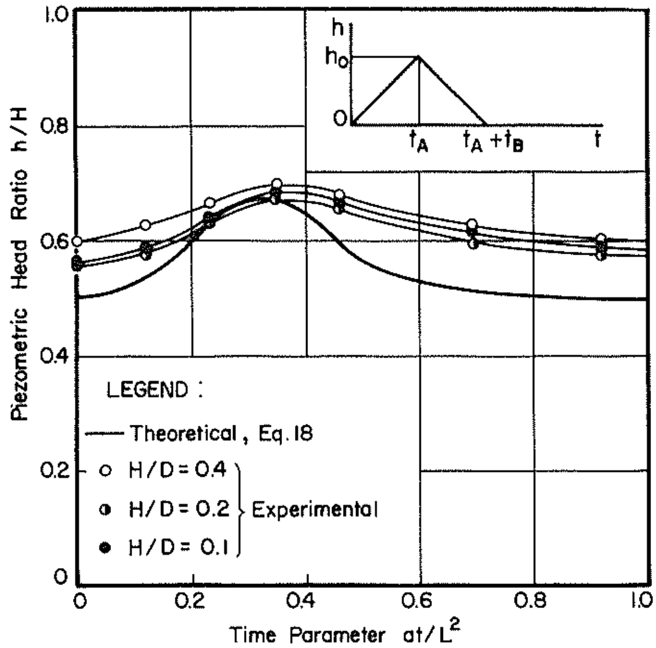

Fig.5 Piezometric head distribution due to modified hydrograph at 50% distance x/L for $at_A/L^2 = at_B/L^2 = 0.230$, $h_o/H = 0.5$

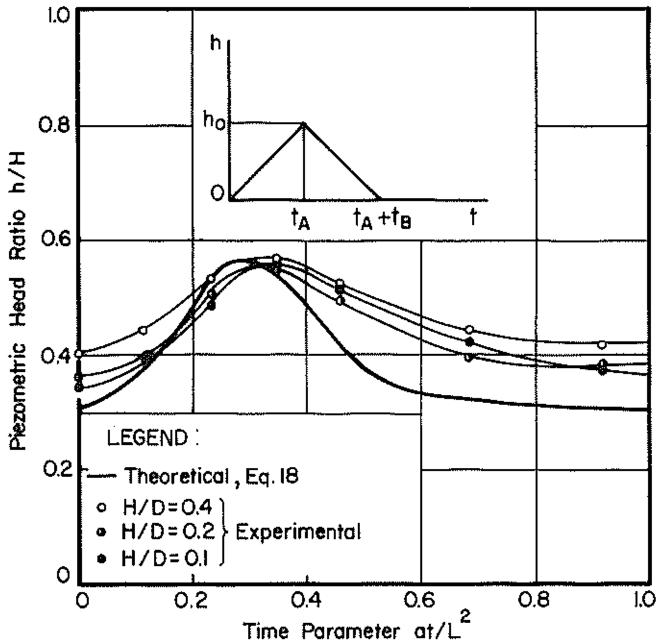

Fig.4 Piezometric head distribution due to modified hydrograph at 30% distance x/L for $at_A/L^2 = at_B/L^2 = 0.230$, $h_o/H = 0.5$

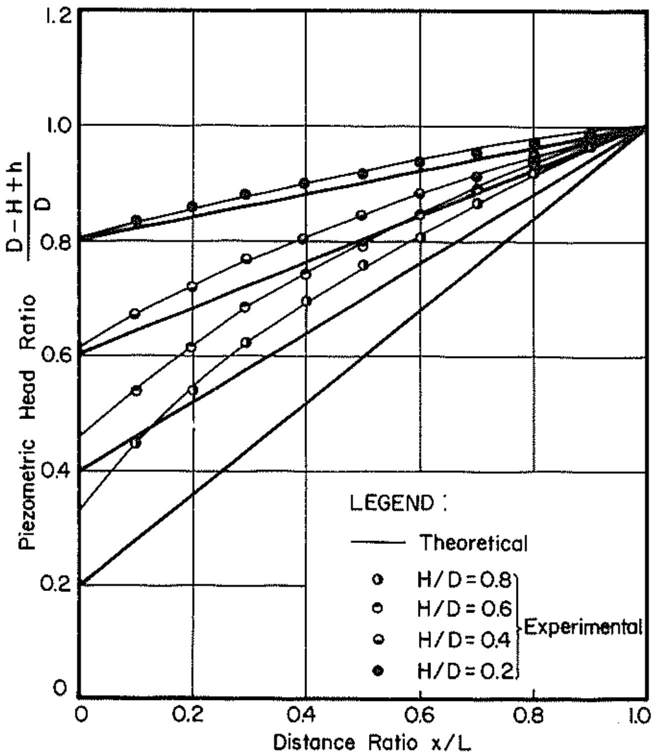

Fig.6 Comparison between experimental and theoretical flow profile (steady state experiment)

The effect of the head loss across the edge of downstream reservoir was not taken into account in the theoretical analysis. Care should be made in correcting for the effect, especially in the dropping stage.

The error caused by neglecting changes in transmissivity is small enough to be neglected when $H/D < 0.2$.

6 REFERENCES

1. TODD, D.K. Ground Water Hydrology. John Wiley and Son, Inc., New York, p.315, 1959.

2. U.S. BUREAU OF RECLAMATION. Design of Small Dam. United States Government Printing Office, Washington, 1960.