# An Analysis of Size Effect Behaviour in Brittle Rock

by

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SUMMARY. Statistical and empirical methods of describing the frequently observed size-strength dependency in rock are unsatisfactory. An analysis of the energetics of fracture shows that the critical energy level required for crack propagation may vary with initial crack or discontinuity size, mode of fracture, and homogeneity of the stress distribution. Each of these factors can produce size effects. The results of punch bearing tests verify a theoretical conclusion that stress gradients can introduce apparent size effects.

## 1 INTRODUCTION

The relationship between specimen size and measured mechanical properties has long been of concern to engineers designing and building rock structures. The possibility that the results of their compression tests on small (often only 10-20 mm) cubes of rock may not have been representative of the behaviour of the larger blocks used in their structures, was certainly considered by 19th century engineers working with masonry. As early as 1851, Johnson (Ref.1) proposed an empirical law to explain the observed variation in cube crushing strength with size.

The most commonly observed size-effect is an apparent reduction in tensile or compressive strength with increasing specimen size, although the reverse relationship has been observed at very small specimen sizes (Ref.2). Brown (Ref.3) has reviewed the published data and found it to be conflicting and generally inconclusive. Experimental results have been influenced by such factors as specimen geometry, end restraint, stress gradients, surface condition, stored strain energy, and the initial degree of micro-cracking present. Brown suggested that under uniform stress fields, "hard" rock that is relatively free of pre-existing micro-cracking will not show a size effect. The strength of intensely micro-cracked rock is noticeably size dependent (Ref.4) for reasons discussed in Section 4 below.

Statistical, empirical and mechanistic theories have been used to explain size effect phenomena. In this paper, statistical and empirical theories are shown to be not entirely satisfactory, and an explanation of observed size-strength dependency is sought in terms of the mechanics of fracture. Only the behaviour of rock material is considered; the more complex question of size effects in jointed rock masses is beyond the scope of this paper.

## 2 STATISTICAL SIZE EFFECT THEORIES

In statistical theories, the material is usually represented as an assembly of like elements of uniform size. The calculation of the variation in strength with specimen size is based on the probability of failure of any element and, depending on the arrangement of elements, of the specimen itself, under a given load. The most widely applied of these theories is that due to Weibull (Ref.5). The simplest form of Weibull's extensive statistical theory of the strength of materials is based on the weakest link model in which the specimen is assumed to consist of a series arrangement of elements. Failure of any one of them causes failure of the specimen.

Weibull's basic equation for the probability of failure, S, of a specimen of volume, V, subjected to a general stress field, $\sigma$, is

$$S = 1 - \exp\left\{ - \int_V \left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m dV \right\} \qquad (1)$$

where $\sigma_u$, $\sigma_o$ and m are material constants, $\sigma_u$ being the lowest strength that any element may have.

The difficulty in applying this equation to real problems lies in performing the required integration for non-uniform stress fields. For the simple case of a uniform applied stress, $\sigma$, Equation 1 becomes

$$S = 1 - \exp\left\{ - \left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m V \right\} \qquad (2)$$

In this case, the probabilities of failure of two specimens of volumes $V_1$ and $V_2$ will be equal when

$$\frac{\sigma_1-\sigma_u}{\sigma_2-\sigma_u} = \left(\frac{V_2}{V_1}\right)^{\frac{1}{m}} \qquad (3)$$

where $\sigma_1$ and $\sigma_2$ are the uniform failure stresses associated with the chosen value of S, usually 0.5.

In the particular case in which $\sigma_u = 0$, Equation 3 becomes

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1}\right)^{\frac{1}{m}} \qquad (4)$$

It is in this form that Weibull's theory has been widely applied to the size effect behaviour of rock. Many investigators have erroneously concluded that Weibull's theory is applicable merely because a log-log plot of compressive strength against specimen volume gives an approximately straight line. Clearly, the application of the special case of Weibull's theory given by Equation 4 without an understanding of its' derivation can lead to serious error (Ref.6).

While there may be some justification for applying the weakest link model to tensile failure, it may not be applied to the fracture of rock in compression since this is a progressive mechanism in which multiple local fracturing usually precedes total collapse (Ref.7). More realistic representations of the process of compressive fracture in rock are likely to be given by other statistical theories such as the classic parallel bundle model (Ref.8) or a stochastic process approach of the type proposed by Hudson and Fairhurst (Ref.6). It should be noted, however, that the parallel bundle model predicts no size effect and may not, therefore, be regarded as satisfactory.

## 3 EMPIRICAL APPROACHES

A number of empirical methods, based largely on curve-fitting techniques, have been used to describe size effects in rock and related materials. Indeed, the determination of the material constant, m, in the form of Weibull's theory given by Equation 4, is nothing more than an exercise in curve fitting (Ref.6).

Perhaps the best known of the empirical formulae is that due to Protodyakonov (Ref.2) who was concerned with the relationship between the compressive strength of a rock mass broken up by discontinuities and that of a cylindrical laboratory specimen.

The considerable amount of experimental work that has been carried out on the crushing strength, $\sigma_c$, of cubes of coal with a view to predicting the strength of coal pillars has yielded a law of the form

$$\sigma_c = K\,a^{-\alpha} \qquad (5)$$

where a is the side length of the cube and K and $\alpha$ are constants for the material. There can be little doubt that the results of most of the tests of this type reported in the literature are strongly influenced by end effects. Although the original application of Equation 5 was essentially empirical, it can be shown to have considerable theoretical basis (see Section 4 below). The extended form of Equation 6 usually applied to non-cubical coal mine pillars is (Ref.9)

$$\sigma_c = k\,\frac{w^{\beta}}{h^{\alpha}} \qquad (6)$$

where h is the height of the pillar, w is its least width, and k, $\alpha$ and $\beta$ are constants.

Clearly, these empirical or curve-fitting methods, while being of some practical use, suffer from the fundamental disadvantage that they have been developed without reference to the mechanics of fracture. Because of this, they do not help provide insight into the causes of size effects in rock.

## 4 THE ENERGETICS OF BRITTLE FRACTURE

Although the total application of Griffith's theory of rupture to the fracture of rock is a matter of some controversy, there can be little doubt about the applicability of Griffith's basic energy instability concept (Ref.10). This is essentially a statement of the theory of minimum potential energy applied to the extension of a pre-existing crack. A crack will extend only when the potential energy of the system of applied forces and the specimen decreases with an increase in crack length, i.e. when

$$\frac{\partial(\Delta P)}{\partial c} \leqslant 0 \qquad (7)$$

where c is an initial crack length parameter, and $\Delta P$ is the change in potential energy of the system accompanying an increase in crack length. Generally, $\Delta P$ may be expressed as

$$\Delta P = W_d - W_s \qquad (8)$$

where $W_d$ is the energy used for crack extension or the work of fracture, and $W_s$ is the strain energy released from the system as a result of crack extension (Ref.11). Equation 8 then becomes

$$\frac{\partial}{\partial c}\,(W_d - W_s) \leqslant 0 \qquad (9)$$

In applying the energy instability approach to the extension of a crack of initial length 2c at right angles to the direction of loading in an elastic plate subjected to uniaxial tension, Griffith (Ref.10) found that

$$\Delta P = 4c\gamma - \frac{\pi c^2 \sigma^2}{E} \qquad (10)$$

where $\sigma$ is the applied stress at which crack extension occurs, $\gamma$ is the specific surface energy of the material, and E is Young's modulus.

Substitution of $\Delta P$ in Equation 7 gave the well known criterion for crack extension,

$$\sigma \geqslant \sqrt{\frac{2E\gamma}{\pi c}} \qquad (11)$$

Cook (Ref.12) has extended the energy instability analysis to the case of plane compressive stress assuming collapse in shear. The resulting criterion is of the same form as Equation 11. In fact, Fairhurst (Ref.11) has shown that any criterion based on an elastic energy instability analysis can be expressed in the general form

$$(\text{stress})^2.(\text{crack length}) = \text{constant} \qquad (12)$$

By considering the variation in the crack length parameter, c, with specimen size, Fairhurst (Ref.11) has drawn some important conclusions regarding size-strength dependency. Two extreme situations may be postulated - that in which maximum crack size increases in proportion to the size of the specimen, and that in which crack size is constant irrespective of specimen size. If in the first case, specimen size is characterized by a length, L, then $c \alpha L$ and Equation 12 may be written as

$$\sigma = K_1.L^{-0.5} \qquad (13)$$

where K is a constant. Millard, Newman and Phillips (Ref.13) first derived this equation in considering size effects in coal.

In the second extreme case in which the specimen is assumed to contain a number of identical flaws, an increase in specimen size will not affect the strength so that

$$\sigma = K_2.L^{0} \qquad (14)$$

Thus an estimate of size-strength dependency may be made from a knowledge of the degree to which larger flaws are introduced with increases in specimen size. Obviously, a range of behaviour between the two extremes is possible in real materials so that, in the general case, the relationship becomes

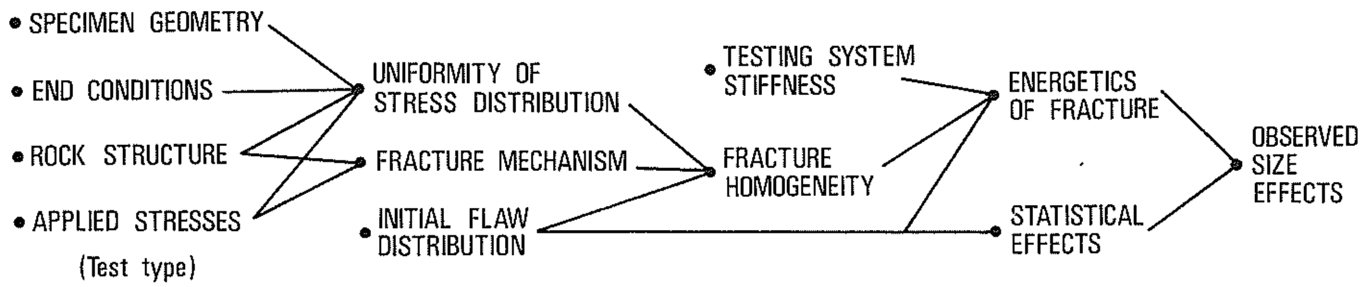


Fig.1 General relationship between causative factors and observed size effects.

$$\sigma = K.L^{-\alpha} \tag{15}$$

where $0 \leq \alpha \leq 0.5$, with the upper extreme of 0.5 being approached for extensively cracked material such as coal (Refs.9,13). Laws of the form of Equation 15 have also been fitted to the results of tensile and compressive strength tests on concrete and rock (Refs.14,15).

A consideration of the nature of the terms $W_d$ and $W_s$ in the basic statement of the energy instability criterion (Equation 9) shows that factors other than the initial state of cracking can influence size effect behaviour. In the case of crack extension in an ideal Griffith material (Equation 10), the energy demand term, $W_d$, represents the energy absorbed as surface energy of the extended crack surfaces. Recent investigations of the nature of crack propagation in rock have shown the classical Griffith material concept to be too simplified for this application (Refs.7,11,16-18). It has been demonstrated that a zone of micro-cracking is created around the tip of a crack extending in rock under tension. With increasing deformation, the damage in the micro-cracked zone becomes more intense until the macro-crack extends. A crack may extend along grain boundaries or by cleavage of individual grains. Thus the energy required to propagate a tension crack in rock will depend on the mineralogy, grain size and extent of initial micro-cracking. If multiple macro-fracture surfaces form, the energy absorbed will be different from that in the single macro-crack case.

It is well established that in compression, multiple local inter-and intra-granular cracking occurs before peak load is reached (Refs.7,11). The extent of this cracking varies with rock type, being less marked for dense, fine-grained rocks (Refs.7,19). Failure of these dense, fine-grained rocks (such as the Solenhofen limestone) is generally more brittle than that of coarser grained rocks, and is quite difficult to control even with servo-controlled testing machines (Ref.19). Furthermore, fracture in such cases is localized on one or two major fracture planes and not distributed throughout the specimen as in other rock types. Even in those rock types that normally exhibit multiple pre-peak fracturing distributed throughout the specimen, failure on a limited number of major planes can result from the presence of zones of weakness in the specimen or non-uniform application of load. It must be expected that energy demand will vary with the fracture distribution.

A comparison of the energetics of fracture in these various cases is difficult because of its' complexity. However, a conceptual evaluation may be made by considering two extremes of fracture development - type A in which cracking is developed uniformly throughout the specimen, and type B in which fracture occurs on a single plane traversing the specimen. In type A situations, energy demand will vary as the cube of a specimen length parameter, and energy demand per unit volume will be constant irrespective of specimen size. In type B situations, energy demand will vary as the square of the length parameter, and energy demand per unit volume will decrease with increasing specimen size. Since the energy supply term, $W_s$, can be expected to vary with the cube of the length parameter in both cases, it follows that a size effect will exist for type B but not for type A. It is suggested that many of the size effect observations made have resulted from the fact that testing techniques used pre-determined that type B failures would occur. Contributory factors include the use of spherical seats, excessive end restraint, and inadequate tolerance on the flatness and parallelism of loaded surfaces.

The energy supply term, $W_s$, is the change in strain energy of the specimen accompanying crack growth. For the ideal Griffith material, it is a function of Young's modulus, crack length and the applied stress (Equation 10). In the more general case, it will depend on the constitutive relations of the material, specimen size, and distribution of stress within the specimen. Using the finite element method of stress analysis, the strain energy stored in a cracked material can be calculated for a number of test configurations taking account of non-linearities such as variations in modulus with crack length and stress level (Ref.17). It will be shown in Section 5, that under stress gradients, higher applied stresses are required to produce the same energy release rate as under homogeneous stresses, and that this gives rise to an apparent size effect because stress gradients generally increase with decreasing specimen size.

Glucklich and Cohen (Ref.20) recognized the importance of stored strain energy in brittle fracture and argued that the stability of cracks or flaws of any size depends on the total machine-specimen system strain energy. It is important to note, however, that Fairhurst (Ref.11) has demonstrated that although the stiffness of the applied load, and hence its' stored energy, may influence fracture propagation, it does not influence fracture initiation.

This qualitative analysis of the energetics of fracture initiation has shown how observed size effects can arise from the initial degree of micro-cracking, the nature of the fracture pattern, the distribution of stresses within the specimen and stored strain energy. In any given situation, the interaction between these various influences will be complex. Fig. 1 shows how the factors discussed may combine to produce size-strength dependency. Currently, the nett effect can be predicted for only the simplest of cases (Ref.17). Nevertheless, the energy instability approach does provide a rational explanation of phenomena not otherwise satisfactorily accounted for.

## 5 THE ROLE OF STRESS GRADIENTS

Stress gradients can be shown to play a particularly important role in producing apparent size effects. In fact, there are some situations in which stress gradients are the dominant cause of observed size effects.

It is now well established that the measured "tensile strength" of brittle materials is stress gradient dependent; higher values of "tensile strength" are obtained in tests involving inhomogeneous stresses than under uniform stress fields Refs.17,21). The influence of stress gradients on the compressive failure of brittle materials has not been studied to the same extent, although it is known that some effect does exist.

The influence of stress gradients on the energetics of fracture may be determined by reconsidering the energy supply term, $W_s$. It is reasonable to expect that under stress gradients, crack propagation will be initiated in a more highly stressed zone. When a crack propagates in a non-uniform stress field from a zone of higher to a zone of lower stress, the strain energy released ($W_s$) must be lower than that released under a uniform stress field with the same stress in the zone of fracture initiation. Accordingly, in a stress gradient situation, higher local stresses must be attained before the energy release rate corresponds to that for uniform stress. This means that a higher apparent strength will be recorded. In laboratory tests, stress gradients usually vary inversely with specimen size so that the effect is more marked in smaller specimens, for which apparently higher strengths may be observed.

It is suggested that this is the cause of many of the size effects observed in compression tests on cubes and cylinders of rock. It is universally recognized that unless very special precautions are taken, non-uniform stresses are produced at the specimen-platen contact and within the specimen. Analyses of this problem yield distributions of stress that are dependent on specimen geometry but independent of specimen size, given a constant degree of end restraint. It follows, therefore, that gradients of stress will be higher in small than in large specimens. For very small specimens in the order of 1 cm. in size, quite high stress gradients will exist whereas the stress gradients in corresponding 1 metre specimens will be very small indeed. Accepting that high stress gradients produce high "strengths", it follows that an apparent size effect will result.

A convincing illustration of the influence of non-homogeneity in the stress field on size effect behaviour in rock is given by the results of punch bearing tests. Hodgson and Cook (Ref.22) measured the uniaxial compressive strengths of cylindrical specimens of a shale and a quartzite with a length to a diameter ratio of 3 and diameters of 0.56 to 15.23 cm. The compressive strengths of these rocks were virtually independent of size under the uniform stress conditions produced in their carefully performed tests. On the other hand, punch bearing strengths determined under conditions of high stress gradient in the fracture zone, were markedly size dependent. The results were expressed in the form of Equation 15 with the length parameter being the punch diameter, and $\alpha$ taking values of 0.40 for shale and 0.56 for quartzite.

As part of the present investigation, a number of punch bearing tests were carried out on blocks of gypsum plaster, Wombeyan marble and a local granite (Ref.23). Punch diameters (D) in the range 3 to 25 mm were used, and all tests were carried out on blocks of rock with widths greater than 12D. In uniaxial compression tests carried out using the brush platen technique, no size effects were recorded. In the punch bearing tests, each of the three materials tested yielded strength-size relationships of the form

$$\sigma = KD^{-\alpha} \qquad (16)$$

with K and $\alpha$ apparently constant for each material.

A significant feature of the results is that modes of failure, shapes of punch load (F) - displacement ($\Delta$) curves and values of $\alpha$ were quite different for the three materials tested (Fig.2). The observed differences can be accounted for by determining the variations of energy supply and demand with punch size to be expected for different modes of failure.

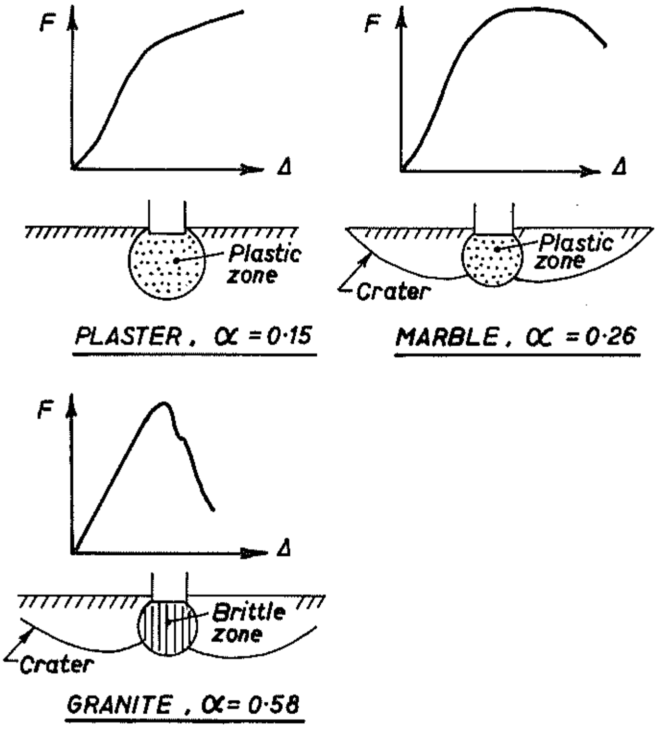


Fig. 2 Punch Bearing Test Results

The granite gave an almost linear load-displacement response with only minor cracking observed before the peak load was reached. Failure was sudden and unstable in the soft testing machine used. The crater formed as a result of fracture on a limited number of surfaces. Its' size was roughly proportional to punch diameter. This is a discrete failure plane situation for which $W_d \alpha D^2$ and $W_s \alpha D^3$ as argued in Section 4.

This explains why the experimental value of $\alpha$ is close to 0.5.

The Wombeyan marble failed in a less brittle or catastrophic manner than the granite. Post-peak behaviour was stable, and although a single surface, punch size oriented crater was produced, a bulb of crushed rock behaving as a plastic material in the manner described by Ladanyi (Ref.24) was produced under the punch. For a volume of crushed or plastic rock, both $W_s$ and $W_d$ can be expected to vary with $D^3$, so that in the perfectly plastic case, $\alpha$ should be zero. It follows that the mixed mode of failure produced in the marble should have associated with it an intermediate value of $\alpha$.

The gypsum plaster which is a porous material, failed by crushing in a zone under the punch. This zone deformed in a strain-hardening manner but no crater was formed. This failure mode is clearly closer to the perfectly plastic case than the other two, and produces a smaller size effect with a value of α closer to zero.

These punch bearing experiments show that under stress gradients, size effects may be produced in materials that show no size-strength dependency under uniform stresses. The intensity of this size effect is determined, to a large extent, by the mode of failure produced.

Gonano (Ref.23) has used model underground opening tests to further demonstrate that compressive stress gradients produce size effects. This experimental situation proved to be more complex than the punch bearing tests in that the value of α appeared to vary with stress gradient intensity. In this case, the qualitative predictions of the energy instability approach err on the side of oversimplification.

## 6 CONCLUSIONS

Statistical and empirical size effect theories are inadequate because they model the mechanics of rock fracture imperfectly in the former case and not at all in the latter. An analysis of the energetics of fracture using the principle of minimum potential energy as applied by Griffith shows that crack instability will develop only when sufficient strain energy is available, and that this critical energy level can vary with mode of fracture, initial crack or discontinuity size, and homogeneity of the stress distribution within the specimen. Stress gradients induce apparent size effects in some laboratory tests. The results of punch bearing and model underground opening tests verify theoretical conclusions that size effects may be related to mode of failure and stress gradients.

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